G-Networks with Resets

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Abstract

G-networks are product form queuing networks which, in addition to ordinary customers, contain unusual entities such as "negative customers" which eliminate normal customers and "triggers" which move other customers from some queue to another. These models have generated much intereset in the literature since the early 90's. The present paper discusses a novel model for a reliable system composed of N unreliable systems, which can hinder or enhance each other's reliability. Each of the N systems also tests other systems at random; it is able to reset another system if it is itself in working condition and discovers that the other system has failed, so that the global reliability of the system is enhanced. This paper shows how an extension of G-networks that includes novel "reset" customers can be used to model this behavior. We then show that a general G-network model with resets has product form, and prove existence and uniqueness of its solution.

1 Introduction

G-networks [2, 3] were introduced in the early 90's as a generalization of queuing networks. Currently there are several hundred references devoted to the subject, and a recent survey paper [12] cites many of these contributions. The link between these models and spiked Random Neural Networks (RNN) has been developed in [7, 8].

Traditionally, queuing networks are tools for the analysis of systems of customers which circulate among a finite set of servers, waiting for service and then obtaining service at each of the servers. Such systems are either closed (and do not receive customers from the outside world with customers cycling indefinitely in the system), or open (with customers arriving from the outside world and then leaving when they are done with their work.

These models can also be used to represent system reliability as a generalization of the machine repairman problem. In this case, arrivals are events which enhance the reliability of the system, while services correspond to events which reduce system reliability, and queue length represents how reliable the system is at a given instant of time. Thus when the queue length is zero, this represents the state where the system has failed and is not operational until the system is repaired (after an arrival occurs).

Queueing models models typically do not have provisions for some customers being used to eliminate other customers, or to redirect other customers among the queues. In other words, customers in traditional queuing networks cannot exert direct control on other customers. G-network models overcome some of the limitations of conventional queueing network models and still preserve the computationally attractive "product form" property of certain Markovian queuing networks. Generalized or "Gelenbe" networks are product form queuing networks which, in addition to ordinary customers, contain unusual customers such as "negative customers" which eliminate normal customers, and "triggers" which move other customers from some queue to another [5, 6]. Multiple class versions of these models are discussed in [9, 10], and in [11] many additional results are provided. These queuing have generated much intereset in the literature and many authors have devoted papers to various aspects and extensions of these models since the early 90's. A recent journal special issue [13] provides insight into some of the research issues and developments in this area.

In this paper we present a novel extension of G-networks: we introduce *"reset" customers* and show product form of the stationary solution of the network. Resets can travel from any queue to any other queue, and can also arrive from outside the network. When they arrive at a queue they have one of two effects:

- If the queue is non-empty the reset has no effect and is immediately lost.
- If the queue is empty then the queue length is reset to a random queue length whose distribution is identical to the stationary distribution at that queue.

Informally speaking, the "reset" cutsomer sets the queue length at the server where it arrives to the value where "it should be".

We prove that this model has product form solution under the usual assumptions (Poisson arrivals, exponential survice times for ordinary customers, Markovian customer movement). Although the balance equations for the model contain transitions which differ significantly from those of an ordinary G-network *without resets*, our result shows that this product form is identical to that of an ordinary G-network with larger arrival rates of positive customers than the corresponding model withour resets.

In Section Rely we will illustrate how this model can represent a system of N unreliable subsystems which can rest each other when any one of them has failed.

2 G-Networks with Resets

Let us first present the assumptions underlying the model considered in the present paper. This queueing network contains N queues, and each queue j has independent and identically exponentially distributed service times with rates μ_j for $j = 1, ..., N$. Each queue j receives positive customers from outside the network according to a Poisson process of rate Λ_j^+. Positive customers are the ordinary queuing network customers which receive service at the various queues.

In addition to the usual customers, the network also contains "signals" which cover three special types of customers:

- negative customers which destroy an existing customer,
- triggers which move a customer from some queue to another,
- resets – the new type of customer which was mentioned earlier, and which will be presented in detail below.

When an ordinary customer leaves a queue at the end of a service epoch, it can either leave the network, or go to some other queue as an ordinary customer, or it can go to some other queue as a signal. These transitions between queues are described by the transition probability matrices $P^+ = [P_{ij}^+]$ for a positive customer leaving a queue and joining another queue as a positive customer, and $P^- = [P_{ij}^-]$ for a positive customer leaving some queue to enter another queue as a signal. Signals can also arrive from outside the network to some queue i in a Poisson stream of rate Λ_i^-.

Departures of customers from the network are represented by tha fact that the matrix $P = P^+ + P^-$ is sub-stochastic, and the probability d_i that a positive customer leaves the network after receiving service at queue i is given by:

$$d_i = 1 - \sum_{j=1}^{N} [P_{ij}^+ + P_{ij}^-]. \tag{1}$$

The effect of signals is described as follows:

- When a signal arrives from some queue i to some queue j with probability P_{ij}^-:
  1. If queue j is *non-empty*, the signal triggers a customer to move instantaneously to some other queue with probability $\alpha_{i,j}$, and with probability $[1 - \alpha_{i,j}]$ the signal has no effect. With probability Q_{jk} the customer which has been triggered moves to some other queue k, or leaves the network with probability $Q_{j,N+1}$. Note that for any i, we have:
  $$Q_{i,N+1} = 1 - \sum_{j=1}^{N} Q_{i,j}. \tag{2}$$
  .
  2. If queue j is *empty*, the signal will create a random batch of Y_{ij} customers at queue j with probability $\beta_{i,j}$. With probability $[1 - \beta_{i,j}]$ the signal has no effect. The distribution of Y_{ij} is denoted by $Pr[Y_{ij} = x_j] = \tau_j(x_j)$.
- In all cases, after a signal has acted upon the queue to which it arrives, it vanishes (i.e., signals do not queue or receive service).

Note that Item (2) above is the novel behavior of signals introduced in this paper, while Item (1) corresponds to previously known behavior. Note also that Item (2) describes a behavior which is dependent on the stationary solution of the network. Thus signals behave in a manner as though they already "know" the stationary solution of the network.

When a signal arrives from outside the network to some queue j, we replace the probabilities $\alpha_{i,j}$, $\beta_{i,j}$ by $\alpha_{0,j}$, $\beta_{0,j}$.

3 Global Balance Equations and Product Form Solution

Denote by $(X_1(t), \ldots, X_N(t))$ the queue length vector at time t for the G-network with N servers, and let $\pi(\vec{x})$ denote the stationary probability, if it exists, that the network state is $\vec{x} = (x_1, \ldots, x_N)$:

$$\pi(\vec{x}) = lim_{\ t \to \infty} Pr[(X_1(t), \ldots, X_N(t)) = \vec{x}]. \tag{3}$$

Let $\vec{e_i}$ denote the N-vector which is 0 everywhere, except in position i where it has the value 1: $\vec{e_i} = (0, .., 1, .., 0)$. The Chapman-Kolmogorov equations in steady-state, i.e. the global balance equations, can be written as:

$$\begin{aligned}
\pi(\vec{x}) \textstyle\sum_{i=1}^{N} \Big[\lambda_i^+ + (\mu_i + \lambda_i^- \alpha_{0,i}) 1_{\{x_i>0\}} + \lambda_i^- 1_{\{x_i=0\}} \beta_{0,i}\Big] = \\
& \textstyle\sum_{i=1}^{N} \lambda_i^+ \pi(\vec{x} - \vec{e_i}) 1_{\{x_i>0\}} \\
+ & \textstyle\sum_{i=1}^{N} \mu_i \pi(\vec{x} + \vec{e_i}) d_i \\
+ & \textstyle\sum_{i=1}^{N} \lambda_i^- \alpha_{0,i} Q_{i,N+1} \pi(\vec{x} + \vec{e_i}) \\
+ & \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i^- \alpha_{0,i} Q_{i,j} \pi(\vec{x} + \vec{e_i} - \vec{e_j}) 1_{\{x_j>0\}} \\
+ & \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \pi(\vec{x} + \vec{e_i} - \vec{e_j}) P_{ij}^+ 1_{\{x_j>0\}} \\
+ & \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \pi(\vec{x} + \vec{e_i} + \vec{e_j}) P_{ij}^- Q_{j,N+1} \alpha_{i,j} \\
+ & \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \mu_i \pi(\vec{x} + \vec{e_i} + \vec{e_j} - \vec{e_k}) P_{ij}^- Q_{j,k} \alpha_{i,j} 1_{\{x_k>0\}} \\
+ & \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \pi(\vec{x} + \vec{e_i}) P_{ij}^- (1 - \alpha_{i,j}) 1_{\{x_j>0\}} \\
+ & \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \pi(\vec{x} + \vec{e_i} - x_j \vec{e_j}) P_{ij}^- \beta_{i,j} 1_{\{x_j>0\}} \tau_j(x_j) \\
+ & \textstyle\sum_{i=1}^{N} \lambda_i^- \beta_{0,i} \pi(\vec{x} - x_i \vec{e_i}) 1_{\{x_i>0\}} \tau_i(x_i) \\
+ & \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \pi(\vec{x} + \vec{e_i}) P_{ij}^- (1 - \beta_{i,j}) 1_{\{x_j=0\}}
\end{aligned} \tag{4}$$

3.1 Main Result

The traffic equations for the network describe the total rates Λ_i^+, Λ_i^- (resp.) at which positive customers and signals arrive to each queue i. Positive customers to a queue either arrive from outside the network, or they arrive from other queues. In the latter case, they are either positive customers which have been triggered (by signals) from other queues, or they are positive customers which have completed service at some other queue. Signals, on the other hand, can either arrive from outside the network, or they result from the change of a positive customer leaving a queue to become a signal at another queue. The network's traffic equations are defined as follows:

$$\begin{aligned}
\Lambda_i^+ &= \lambda_i^+ + \sum_{j=1}^{N} \mu_j \rho_j P_{ji}^+ + \sum_{j=1}^{N} \lambda_j^- \rho_j \alpha_{0,j} Q_{j,i} \\
&\quad + \sum_{j=1}^{N} \sum_{k=1}^{N} \mu_j \rho_j P_{jk}^- \rho_k \alpha_{j,k} Q_{k,i} + \lambda_i^- \beta_{0,i} + \sum_{j=1}^{N} \mu_j \rho_j P_{ji}^- \beta_{i,j}
\end{aligned} \tag{5}$$

and:

$$\Lambda_i^- = \lambda_i^- \alpha_{0,i} + \sum_{j=1}^{N} \mu_j \rho_j P_{ji}^- \alpha_{j,i} \tag{6}$$

where:

$$\rho_i = \frac{\Lambda_i^+}{\mu_i + \Lambda_i^-} \tag{7}$$

Note that we can write (5) as:

$$\Lambda_i^+ = \lambda_i^+ + \sum_{j=1}^{N} \mu_j \rho_j P_{ji}^+ + \sum_{j=1}^{N} \Lambda_j^- \rho_j Q_{j,i} + \lambda_i^- \beta_{0,i} + \sum_{j=1}^{N} \mu_j \rho_j P_{ji}^- \beta_{i,j} \tag{8}$$

Theorem 1 *Consider the G-network with:*

$$\tau_j(x_j) = (1-\rho_j)(\rho_j)^{(x_j-1)}, \; x_j > 0 \tag{9}$$
$$\tau_j(0) = 0. \tag{10}$$

If equations (5), (6), (7) have non-negative solutions such that $\rho_i < 1$, for $i = 1, \dots, N$, then the stationary distribution for the model exists and has the product form:

$$\pi(\vec{x}) = \prod_{i=1}^{N} \pi_i(x_i) \tag{11}$$

where the marginal probabilities of queue length are given by $\pi_i(x_i) = (\rho_i)^{x_i}(1 - \rho_i)$.

The proof is given in Section 5.

4 A Model of N Interacting Unreliable Subsystems

In order to illsustrate a possible area of application of G-networks with resets, consider a model for the overall reliability of N interacting subsystems which can enhance or reduce each others' level of reliability. The state of each subsystem i is represented by an integer X_i. If at some time $X_i = 0$ this means that i has failed, and the larger the value of X_i the more reliable it is (i.e., the further away it is from failure).

The reliability state of subsystem i which we denote by X_i is modified by internal and external events:

- An exponentially distributed service time of parameter μ_i is used to model *internal events* in the subsystem:
  - With probability $P^+_{i,i}$ the internal event does not affect the subsystem's reliability (i.e., X_i does not change),
  - With probability $(1-P^+_{i,i})$ the internal event reduces its reliability and reduces X_i by one.
- Internal events which improve its reliability (i.e., repair) are represented by an arrival process of normal customers of rate λ^+_i to the queue i.
- Subsystems can also enhance or reduce the reliability of each other. Thus:
  - When an internal event occurs a subsystem j can enhance subsystem i's reliability, increasing X_i by 1 with probability $P^+_{j,i}$, or reduce its reliability by reducing it by 1 with probability $P^r_{j,i}$.
  - Finally, a susbsytem j, when an internal event occurs, can also send a *reset signal* to subsystem i with probability $P^R_{j,i}$. When a reset signal is received by a subsystem, one of two things happen:
    * If subsystem i is operating normally (i.e., $X_i > 0$), then i just ignores the reset signal, or

* If subsystem i is in failed mode (i.e., $X_i = 0$), then the reset signal moves subsystem i into its *normal functioning state* by bringing it to it's *steady-state distribution* $P[X_i = x_i] = \rho_i^{x_i}(1-\rho_i)$ where the steady-state probability ρ_i that subsystem i is functioning normally, is given by:

$$\rho_i = \frac{\lambda_i^+ + \sum_{j=1}^N \rho_j \mu_j [P_{j,i}^+ + P_{j,i}^R]}{\mu_i} \tag{12}$$

By appropriate restrictions and parameter choices, this model is a special case of the G-network with resets.

First note that the μ_i representing service rates in the G-network model will represent the rate of internal events in each susbsystem i, while the λ_i^+ are the rates of "repairs" which improve the railability in each susbsystem. Note further that $\lambda_i^- = 0$ in the reliability model, and that the $P_{i,j}^+$ of the G-network model have the obvious interpretation (and identical notation) in the reliability model.

The $P_{i,j}^R$ in the reliability model are derived from the $P_{i,j}^-$ in the G-network model. We first need to set $\alpha_{i,j} = 0$, so that the trigger always has no effect when queue j is non-empty. Then we set $\beta_{i,j} = 1$ so that the trigger arriving to a non-empty queue *always* has an effect. Finally we set to $\tau_j(x_j) = \rho_j^{x_j}(1-\rho_j)$ for all $j = 1, \ldots, N$.

5 Proof of Theorem 1

We proceed with the proof of the product theorem by showing that (11) satisfies the global balance equations (4). In order to do so, we substitute (11) in (4), and divide both sides of the resulting expression by $(\rho_i)^{x_i}$ to obtain:

$$\begin{aligned}
\sum_{i=1}^N \Big[\lambda_i^+ + (\mu_i + \lambda_i^- \alpha_{0,i})1_{\{x_i>0\}} + \lambda_i^- 1_{\{x_i=0\}}\beta_{0,i}\Big] = \\
& \sum_{i=1}^N \lambda_i^+ \frac{1}{\rho_i} 1_{\{x_i>0\}} \\
+ & \sum_{i=1}^N \mu_i \rho_i d_i \\
+ & \sum_{i=1}^N \lambda_i^- \alpha_{0,i} Q_{i,N+1} \rho_i \\
+ & \sum_{i=1}^N \sum_{j=1}^N \lambda_i^- \alpha_{0,i} Q_{i,j} \frac{\rho_i}{\rho_j} 1_{\{x_j>0\}} \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \frac{\rho_i}{\rho_j} P_{ij}^+ 1_{\{x_j>0\}} \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i \rho_j P_{ij}^- Q_{j,N+1} \alpha_{i,j} \\
+ & \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \mu_i \frac{\rho_i \rho_j}{\rho_k} P_{ij}^- Q_{j,k} \alpha_{i,j} 1_{\{x_k>0\}} \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- (1-\alpha_{i,j}) 1_{\{x_j>0\}} \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \frac{\rho_i}{(\rho_j)^{x_j}} P_{ij}^- \beta_{i,j} 1_{\{x_j>0\}} \tau_j(x_j) \\
+ & \sum_{i=1}^N \frac{\lambda_i^- \beta_{0,i}}{\rho_i^{x_i}} 1_{\{x_i>0\}} \tau_i(x_i) \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- (1-\beta_{i,j}) 1_{\{x_j=0\}}
\end{aligned} \tag{13}$$

which, using the expression for $\tau_j(x_j)$ simplifies to:

$$
\begin{array}{rl}
\sum_{i=1}^{N}\Big[\lambda_i^+ + (\mu_i+\lambda_i^-\alpha_{0,i})1_{\{x_i>0\}} + \lambda_i^- 1_{\{x_i=0\}}\beta_{0,i}\Big] = & \\
& \sum_{i=1}^{N}\lambda_i^+\frac{1}{\rho_i}1_{\{x_i>0\}} \\
+ & \sum_{i=1}^{N}\mu_i\rho_i d_i \\
+ & \sum_{i=1}^{N}\lambda_i^-\alpha_{0,i}Q_{i,N+1}\rho_i \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i^-\alpha_{0,i}Q_{i,j}\frac{\rho_i}{\rho_j}1_{\{x_j>0\}} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\frac{\rho_i}{\rho_j}P_{ij}^+1_{\{x_j>0\}} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i\rho_jP_{ij}^-Q_{j,N+1}\alpha_{i,j} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\mu_i\frac{\rho_i\rho_j}{\rho_k}P_{ij}^-Q_{j,k}\alpha_{i,j}1_{\{x_k>0\}} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_iP_{ij}^-(1-\alpha_{i,j})1_{\{x_j>0\}} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\frac{\rho_i}{\rho_j}(1-\rho_j)P_{ij}^-\beta_{i,j}1_{\{x_j>0\}} \\
+ & \sum_{i=1}^{N}\lambda_i^-\beta_{0,i}[\frac{1}{\rho_i}\ -\ 1]1_{\{x_i>0\}} \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_iP_{ij}^-(1-\beta_{i,j})1_{\{x_j=0\}}
\end{array}
\tag{14}
$$

or better still, to:

$$
\begin{array}{rll}
\sum_{i=1}^{N}\Big[\lambda_i^+ + (\mu_i+\lambda_i^-\alpha_{0,i})1_{\{x_i>0\}} + \lambda_i^- 1_{\{x_i=0\}}\beta_{0,i}\Big] = & & \\
& \sum_{i=1}^{N}\lambda_i^+\frac{1}{\rho_i}1_{\{x_i>0\}} & [1] \\
+ & \sum_{i=1}^{N}\mu_i\rho_i d_i & [2] \\
+ & \sum_{i=1}^{N}\lambda_i^-\alpha_{0,i}Q_{i,N+1}\rho_i & [3] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i^-\alpha_{0,i}Q_{i,j}\frac{\rho_i}{\rho_j}1_{\{x_j>0\}} & [4] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\frac{\rho_i}{\rho_j}P_{ij}^+1_{\{x_j>0\}} & [5] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i\rho_jP_{ij}^-Q_{j,N+1}\alpha_{i,j} & [6] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\mu_i\frac{\rho_i\rho_j}{\rho_k}P_{ij}^-Q_{j,k}\alpha_{i,j}1_{\{x_k>0\}} & [7] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_iP_{ij}^-(1-\alpha_{i,j})1_{\{x_j>0\}} & [8] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\frac{\rho_i}{\rho_j}P_{ij}^-\beta_{i,j}1_{\{x_j>0\}} & [9] \\
- & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_iP_{ij}^-\beta_{i,j}1_{\{x_j>0\}} & [10] \\
+ & \sum_{i=1}^{N}\frac{\lambda_i^-\beta_{0,i}}{\rho_i}1_{\{x_i>0\}} & [11] \\
- & \sum_{i=1}^{N}\lambda_i^-\beta_{0,i}1_{\{x_i>0\}} & [12] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_iP_{ij}^-(1-\beta_{i,j})1_{\{x_j=0\}} & [13]
\end{array}
\tag{15}
$$

Now notice from (5) and (15), that $\sum_{i=1}^{N}\Lambda_i^+1_{\{x_i>0\}} \ = \ \rho_i(\ [1]\ +[4]\ +[5]\ +[7]\ +[9]\ +\ [11])$ where the numbers in brackets correspond to the terms as numbered in equation (15). Using also the relationship (7) which is $\rho_i \ = \frac{\Lambda_i^+}{\mu_i\ +\ \Lambda_i^-}$, we can replace (15) by:

$$
\begin{array}{rll}
\sum_{i=1}^{N}\Big[\lambda_i^+ + (\mu_i+\lambda_i^-\alpha_{0,i})1_{\{x_i>0\}} + \lambda_i^- 1_{\{x_i=0\}}\beta_{0,i}\Big] = & & \\
& \sum_{i=1}^{N}[\ \mu_i\ +\ \Lambda_i^-\ ]1_{\{x_i>0\}} & \\
+ & \sum_{i=1}^{N}\mu_i\rho_i d_i & [2] \\
+ & \sum_{i=1}^{N}\lambda_i^-\alpha_{0,i}Q_{i,N+1}\rho_i & [3] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i\rho_jP_{ij}^-Q_{j,N+1}\alpha_{i,j} & [6] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_iP_{ij}^-(1-\alpha_{i,j})1_{\{x_j>0\}} & [8] \\
- & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_iP_{ij}^-\beta_{i,j}1_{\{x_j>0\}} & [10] \\
- & \sum_{i=1}^{N}\lambda_i^-\beta_{0,i}1_{\{x_i>0\}} & [12] \\
+ & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_iP_{ij}^-(1-\beta_{i,j})1_{\{x_j=0\}} & [13]
\end{array}
\tag{16}
$$

and cancelling the μ_i term on the left and right we have:

$$
\begin{array}{rll}
\sum_{i=1}^N \left[\lambda_i^+ + \lambda_i^- \alpha_{0,i} 1_{\{x_i>0\}} + \lambda_i^- 1_{\{x_i=0\}} \beta_{0,i}\right] = & & \\
 & \sum_{i=1}^N \Lambda_i^- \ 1_{\{x_i>0\}} & \\
+ & \sum_{i=1}^N \mu_i \rho_i d_i & [2] \\
+ & \sum_{i=1}^N \lambda_i^- \alpha_{0,i} Q_{i,N+1} \rho_i & [3] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i \rho_j P_{ij}^- Q_{j,N+1} \alpha_{i,j} & [6] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- (1-\alpha_{i,j}) 1_{\{x_j>0\}} & [8] \\
- & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- \beta_{i,j} 1_{\{x_j>0\}} & [10] \\
- & \sum_{i=1}^N \lambda_i^- \beta_{0,i} 1_{\{x_i>0\}} & [12] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- (1-\beta_{i,j}) 1_{\{x_j=0\}} & [13]
\end{array} \tag{17}
$$

We can further simplify equation (17) by using (6), with the middle term on the left-hand-side of (17) and [8] in (17), yielding:

$$
\begin{array}{rll}
\sum_{i=1}^N \left[\lambda_i^+ + \lambda_i^- 1_{\{x_i=0\}} \beta_{0,i}\right] = & & \\
+ & \sum_{i=1}^N \mu_i \rho_i d_i & [2] \\
+ & \sum_{i=1}^N \lambda_i^- \alpha_{0,i} Q_{i,N+1} \rho_i & [3] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i \rho_j P_{ij}^- Q_{j,N+1} \alpha_{i,j} & [6] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- 1_{\{x_j>0\}} & [8*] \\
- & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- \beta_{i,j} 1_{\{x_j>0\}} & [10] \\
- & \sum_{i=1}^N \lambda_i^- \beta_{0,i} 1_{\{x_i>0\}} & [12] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- (1-\beta_{i,j}) 1_{\{x_j=0\}} & [13]
\end{array} \tag{18}
$$

Notice now that $1_{\{x_i>0\}} = \ 1 \ - 1_{\{x_i=0\}}$. Using it in the left-hand-side term, and in [10], we can group terms in the left-hand-side, eliminate [12], and modify [10], yielding:

$$
\begin{array}{rll}
\sum_{i=1}^N \left[\lambda_i^+ + \lambda_i^- \beta_{0,i}\right] = & & \\
+ & \sum_{i=1}^N \mu_i \rho_i d_i & [2] \\
+ & \sum_{i=1}^N \lambda_i^- \alpha_{0,i} Q_{i,N+1} \rho_i & [3] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i \rho_j P_{ij}^- Q_{j,N+1} \alpha_{i,j} & [6] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- 1_{\{x_j>0\}} & [8*] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- \beta_{i,j} 1_{\{x_j=0\}} & [10] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- (1-\beta_{i,j}) 1_{\{x_j=0\}} & [13]
\end{array} \tag{19}
$$

Now [10] and [13] simplify to yield:

$$
\begin{array}{rll}
\sum_{i=1}^N \left[\lambda_i^+ + \lambda_i^- \beta_{0,i}\right] = & & \\
+ & \sum_{i=1}^N \mu_i \rho_i d_i & [2] \\
+ & \sum_{i=1}^N \lambda_i^- \alpha_{0,i} Q_{i,N+1} \rho_i & [3] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i \rho_j P_{ij}^- Q_{j,N+1} \alpha_{i,j} & [6] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- 1_{\{x_j>0\}} & [8*] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- 1_{\{x_j=0\}} & [13*]
\end{array} \tag{20}
$$

while [8∗] and [13∗] simplify again, and using (1) and (2), the terms [2], [3], [6] can be re-written:

$$
\begin{array}{rll}
\sum_{i=1}^N \left[\lambda_i^+ + \lambda_i^- \beta_{0,i}\right] = & & \\
+ & \sum_{i=1}^N \mu_i \rho_i [\, 1 - \ \sum_{j=1}^N (P_{ij}^+ + P_{ij}^- \ )] & [2*] \\
+ & \sum_{i=1}^N \lambda_i^- \alpha_{0,i} \rho_i [\, 1 - \ \sum_{k=1}^N Q_{i,k}] & [3*] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i \rho_j P_{ij}^- \alpha_{i,j} [\, 1 - \ \sum_{k=1}^N Q_{i,k}] & [6*] \\
+ & \sum_{i=1}^N \sum_{j=1}^N \mu_i \rho_i P_{ij}^- & [8**]
\end{array} \tag{21}
$$

and will then simplify to:

$$\begin{array}{llr} \sum_{i=1}^{N}\left[\lambda_i^+ + \lambda_i^-\beta_{0,i}\right] = & & \\ + & \sum_{i=1}^{N}\mu_i\rho_i[\,1 - \ \sum_{j=1}^{N}P_{ij}^{+}\,] & [2**] \\ + & \sum_{i=1}^{N}\lambda_i^-\alpha_{0,i}\rho_i[\,1 - \ \sum_{k=1}^{N}Q_{i,k}] & [3*] \\ + & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i\rho_j P_{ij}^{-}\alpha_{i,j}[\,1 - \ \sum_{k=1}^{N}Q_{j,k}] & [6*] \end{array} \tag{22}$$

We can now group the negative terms on the right-hand-side with $[\ \lambda_i^+ \ + \ \lambda_i^-\beta_{0,i}\ ]$ on the left-hand-side and write:

$$\begin{array}{llr} \sum_{i=1}^{N}\Lambda_i^+ = & & \\ + & \sum_{i=1}^{N}\mu_i\rho_i & [2***] \\ + & \sum_{i=1}^{N}\lambda_i^-\alpha_{0,i}\rho_i & [3**] \\ + & \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i\rho_j P_{ij}^{-}\alpha_{i,j} & [6**] \end{array} \tag{23}$$

Then we use (6) to write:

$$\sum_{i=1}^{N}\Lambda_i^+ = \sum_{i=1}^{N}[\mu_i + \Lambda_i^-]\rho_i \tag{24}$$

which completes the proof. **Q.E.D.**

6 Existence and Uniqueness of the Solution to the Traffic Equations

Unlike BCMP or Jackson networks [1], the traffic equations (5), (6) and (7) of the model we consider are non-linear. Therefore issues of existence and uniqueness of their solutions have to be examined and the product form theorem depends on the existence of a solution to these traffic equations. If existence is established, then uniqueness follows easily for a simple reason. We are dealing with the stationary solution of a system of Chapman-Kolmogorov equations, which is known to be unique if it exists [4].

Define the following vectors:

$\vec{\Lambda}^+$ with elements $[\Lambda_i^+]$,

$\vec{\lambda}^+$ with elements $[\lambda_i^-]$,

$\vec{\Lambda}^-$ with elements $[\Lambda_i^-]$,

$\vec{\lambda}^-$ with elements $[\lambda_i^-]$.

We denote by P^+, P^-, Q, the matrices with elements $P^+[i,j]$, $P^-[i,j]$ and $Q[i,j]$, respectively. Let us also introduce the matrices U with elements $U \ = \ P^-[i,j]\alpha_{i,j}$, and V with elements $V \ = \ P^-[i,j]\beta_{i,j}$.

Also define the following $N - vectors$:

$$\begin{aligned} u &= (\lambda_1^-\alpha_{0,1},\ \ldots,\lambda_N^-\alpha_{0,N}), \\ v &= (\lambda_1^-\beta_{0,1},\ \ldots,\lambda_N^-\beta_{0,N}), \end{aligned}$$

and note that matrices P^-, P^+, Q, U, V, and vectors u, v, $\vec{\lambda}^+$, $\vec{\lambda}^-$ are known quantities which are given when the G-network is specified.

Using (7), equation (6) can be written as:

$$\Lambda_i^- = \lambda_i^- \alpha_{0,i} + \sum_{j=1}^{N} \Lambda_j^+ \frac{\mu_j}{\mu_j + \Lambda_j^-} P_{ji}^- \alpha_{j,i}, \tag{25}$$

while (8) becomes:

$$\Lambda_i^+ = \lambda_i^+ + \sum_{j=1}^{N} \Lambda_j^+ \frac{\mu_j}{\mu_j + \Lambda_j^-} P_{ji}^+ + \sum_{j=1}^{N} \Lambda_j^+ \frac{\Lambda_j^-}{\mu_j + \Lambda_j^-} Q_{j,i} + \lambda_i^- \beta_{0,i} + \sum_{j=1}^{N} \Lambda_j^+ \frac{\mu_j}{\mu_j + \Lambda_j^-} P_{ji}^- \beta_{i,j}. \tag{26}$$

Let F be an $N \times N$ matrix with all elements equal to zero, except the ones on the diagonal:

$$\begin{aligned} F_{j,j} &= \frac{\mu_j}{\mu_j + \Lambda_j^-}, \\ F_{i,j} &= 0, \ i \neq j. \end{aligned} \tag{27}$$

We can now write equations (25) and (26) in matrix notation:

$$\vec{\Lambda}^- = u + \vec{\Lambda}^+ F U, \tag{28}$$

$$\vec{\Lambda}^+ = \vec{\lambda}^+ + v + \vec{\Lambda}^+ F P^+ + \vec{\Lambda}^+ (\mathbf{I} - F) Q + \vec{\Lambda}^+ F V, \ \textit{or}$$

$$\vec{\Lambda}^+ = \vec{\lambda}^+ + v + \vec{\Lambda}^+ F (P^+ + V) + \vec{\Lambda}^+ (\mathbf{I} - F) Q, \tag{29}$$

where equation (29) results in the formal solution:

$$\vec{\Lambda}^+ = (\vec{\lambda}^+ + v)[\mathbf{I} - (F(P^+ + V) + (\mathbf{I} - F)Q]^{-1}. \tag{30}$$

We are now ready to state the following existence theorem.

Theorem 2 *If $P = P^+ + P^-$ and Q are transition probability matrices of transient Markov chains, then the traffic equations (5) and (6) always have a solution.*

Proof: For the proof, we turn to equation (30) and consider the series:

$$\textstyle\sum_{n=0}^{\infty} \ [(F(P^+ + V) + (\mathbf{I} - F)Q]^n.$$

Note that $V \leq P^-$. This series is always convergent because it is the convex sum of two transient sub-stochastic matrices Q and $P^+ + V$.

But: $[\mathbf{I} - [F(P^+ + V) + (\mathbf{I} - F)Q] \ ]^{-1} = \sum_{n=0}^{\infty} \ [F(P^+ + V) + (\mathbf{I} - F)Q]^n$, so that:

$$\vec{\Lambda}^+ = (\vec{\lambda}^+ + v) \sum_{n=0}^{\infty} \ [F(P^+ + V) + (\mathbf{I} - F)Q]^n. \tag{31}$$

Using (28) this yields:

$$\vec{\Lambda}^- - u = (\vec{\lambda}^+ + v) \sum_{n=0}^{\infty} \ [F(P^+ + V) + (\mathbf{I} - F)Q]^n F U. \tag{32}$$

Define the vector $y = \vec{\Lambda}^- - u$. We can now write (32) as the fixed-point equation $y = G(y)$ where:

$$G(y) = (\vec{\lambda}^+ + v) \sum_{n=0}^{\infty} \ [F(P^+ + V) + (\mathbf{I} - F)Q]^n F U, \tag{33}$$

with the dependence on y in the right-hand-side comes from the fact that the matrix F contains the term Λ_j^- in each row j.

Notice that G is continuous. Notice further that the mapping $G : [\mathbf{0}, G(\mathbf{0})] \rightarrow [\mathbf{0}, G(\mathbf{0})]$, where the bold-face $\mathbf{0}$ reminds us that it denotes a vector. As a consequence by *Brouwer's Fixed-Point Theorem* $y = G(y)$ has a fixed point; call it y^*. This fixed-point will in turn yield the solution of (6), (5) using (32) and (30) and the fact that F depends on y:

$$\vec{\Lambda}^- = y^* + u, \quad \vec{\Lambda}^+ = (\vec{\lambda}^+ + v)[\mathbf{I} - [\ F(y^*)(P^+ + V) + (\mathbf{I} - F(y^*))Q]\ ]^{-1}. \tag{34}$$

which completes the existence proof for the traffic equations. **Q.E.D.**

The above Theorem states that the traffic equations *always* have a solution. On the other hand, the product form (11) will only exist if the resulting network is stable. The stability condition is summarized below and the proof is identical to that of a similar result in [4].

Theorem 3 *Let z^* be a solution of $z = G(z)$ obtained by setting F as in (27). Let $\vec{\Lambda}^-(z^*)$, $\vec{\Lambda}^+(z^*)$ be the corresponding traffic values, and let the $\rho_i(z^*)$, $i = 1, \ldots, N$ be obtained from (7). Then the G-network with resets has a steady-state solution if all of the $\rho_i(z^*)$ satisfy $0 \le \rho_i(z^*) < 1$ for all $i = 1, \ldots, N$. Otherwise it does not have a steady-state solution.*

7 Conclusions

In this paper we have extended the class of queueing networks with product form solutions by introducing the concept of "reset" customers which can be used to rest the state of a queues in a network to their steady-state distribution whenever these queues may have become empty. We show that the product form solution is preserved when such customers are introduced in the framework of G-networks. Thus the class of product form networks now includes queueing networks with negative customers and triggers, as well as resets.

We have provided a proof of the product form result. Then we have illustrated this class of models by considering a set of interconnected unreliable systems which can rest each other in order to enhance their overall reliability.

Finally we have proved existence of the solution to the non-linear traffic equations which result from this model. The uniqueness of the solutions are tied to the fact that the stationary solutions of these models, if they exist, will necessarily be unique since they are solutions to Chpman-Kolmogorov equations. Future work will consider extensions of these results to models with multiple classes of customers.

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