Stability and Optimal Control of the Packet Switching Broadcast Channel

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ABSTRACT The purpose of this paper is to analyze and optimize the behavior of the broadcast channel for a packet transmission operating in the slotted mode Mathematical methods of Markov chain theory are used to prove the inherent instability of the system If no control is applied, the effective throughput of the system will tend to zero if the population of user terminals is sufficiently large Two classes of control policies are examined, the first acts on admissions to the channel from active terminals, and the second modifies the retransmission rate of packets In each case sufficient conditions for channel stability are given. In the case of retransmission controls it is shown that only policies which assure a rate of retransmission from each blocked terminal of the form of $f = 1/n$, where n is the total number of blocked terminals, will yield a stable channel It is also proved that the optimal policy which maximizes the maximum achievable throughput with a stable channel is of the form $f = (1 - \lambda)/n$ Simulations illustrating channel instability and the effect of the optimal control are provided

KEY WORDS AND PHRASES computer networks, packet switching, broadcast channel, optimal control

CR CATEGORIES 4 3, 4 6, 5 5, 6 2, 6 35

1. *Introduction*

Computer networks using packet switching techniques have been implemented [1, 5, 7, 11, 18, 19] in order to allow a large community of communicating users to share and transmit data and utilize efficiently the excess computing power which may be available at remote locations In this paper we are concerned with packet switching networks using radio channels similar to the ALOHA system [1].

We consider a large set of terminals communicating over a single radio channel in such a way that a packet is successfully transmitted only if its transmission does not overlap in time with the transmission of another packet; otherwise all packets being simultaneously transmitted are lost A terminal whose transmission is unsuccessful is said to be *blocked*; it has to repeat the transmission until it succeeds Either a terminal which is not blocked

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is *active* or it is transmitting a packet. The operation of the system is shown schematically in Figure 1, where the different state transitions of a terminal are shown. Since the only means of communication between terminals is the channel itself, it is not easy to schedule transmissions so as to avoid collisions between packets. It is also obvious that a terminal would in no case transmit more than one packet simultaneously.

Various methods for controlling the transmission of packets have been suggested. The simplest is to allow terminals to transmit packets at any instant of time. The second method, known as the *slotted ALOHA* scheme, has been shown to increase channel throughput over the first method [2]. Here, time is divided into "slots" of equal duration, each slot can accommodate the transmission time of one packet, and packets are all of the same length Packet transmission is synchronized so that it initiates at the beginning of a slot for any terminal and it terminates at the end of the same slot Other schemes have been suggested elsewhere [20, 21]

Kleinrock and Lam [13] have discussed the stability problem of the slotted ALOHA channel Their results, based on simulations and fluid approximations, indicate that the channel becomes saturated if the set of terminals is very large, independently of the arrival rate of packets to the channel, saturation being the phenomenon whereby the number of blocked terminals becomes arbitrarily large. They also compute the expected time to attain a given level of saturation In [14] policies designed to optimize the throughput of the channel, defined as the expected number of successful transmissions per slot, are presented.

The purpose of this paper is to give a theoretical treatment of some control policies which can be applied to the broadcast channel in order to stabilize it and to maximize its performance We first review the proof of instability in [8], extending it to a finite source model taken in the limit as the total number of terminals becomes very large, and showing that the channel instability implies that the equilibrium value of the throughput is zero. Two simple control policies are then presented, and necessary and sufficient conditions for stability of the controlled channel are derived. Bounds for the equilibrium value of the channel throughput with these policies are obtained. Problems related to the practical implementation of the policies we propose here are discussed in [3, 4], where an approach using stochastic approximation oriented algorithms is used and shown to converge in practice to the optimal controls

2. *Mathematical Model*

A precise definition of stability can be considered only in the context of a model of the behavior of the broadcast channel. In this section we present a model identical to the one we considered in an earlier paper [8], except that we take into account here both finite and infinite source systems.

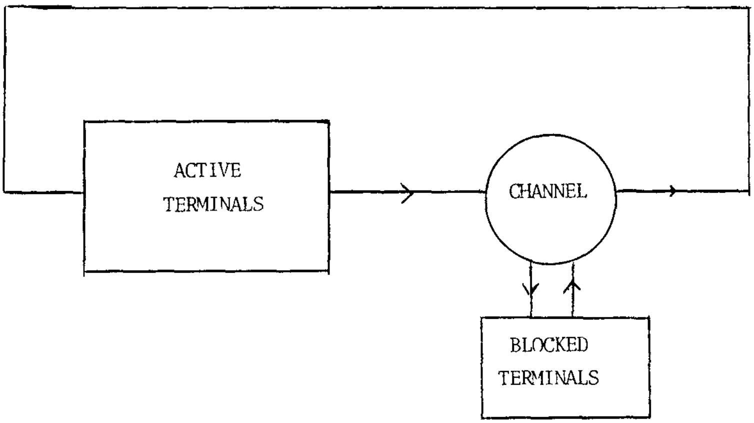

FIG 1

Assuming that the slot, and the time necessary to transmit a packet, are of unit length, we define $N(k)$ as the number of blocked terminals at the instant when the kth ($k = 0, 1, 2, \ldots$) slot begins. Let X_k be the number of packets transmitted from the set of active terminals during the kth slot, and denote by X_k the number of blocked terminals transmitting during the kth slot. In the *infinite source model*, (X_k) is the sequence of independent and identically distributed random variables with the common distribution given by

$$Pr(X_k = i) = c_i, \quad i \geq 0.$$

In the *finite source model*, we let M denote the number of terminals in the system, and we assume that the event $(X_k = i \mid N(k) = j)$ is independent of values of X_t for $t < k$; its probability is given by

$$q_j(n) = Pr(X_k = j \mid N(k) = n) = \binom{M-n}{j} b^j (1-b)^{M-n-j}$$

for $0 \leq j \leq M - n$, where b is the probability that any one active terminal transmits a packet during a slot.

For both models, we denote by f the probability that any one blocked terminal transmits a packet during a slot. We then define

$$g_i(n) = Pr(Y_k = i \mid N(k) = n), \tag{1}$$

where we assume that the event $(Y_k \mid N(k))$ is independent of Y_t for $t < k$. Therefore

$$g_i(n) = \binom{n}{i} f^i (1-f)^{n-1}, \tag{2}$$

and more particularly

$$g_0(n) = (1-f)^n, \quad g_1(n) = nf(1-f)^{n-1}. \tag{3}$$

Definition 1. The infinite source broadcast channel is *unstable* if for $k \to \infty$ the probability $Pr(N(k) < j) \to 0$ for all finite values of j, otherwise it is *stable*. For the finite source model, the system is unstable if the above condition holds as we let $M \to \infty$, $b \to 0$, $M \cdot b \to d$, where d is a constant.

The definition given here simply states that instability exists if (with probability one) the number of blocked terminals becomes infinite as time tends to infinity.

THEOREM 1. *The broadcast channel is unstable for both the finite and infinite source models.*

PROOF. Let us first consider the infinite source model. The proof given here is identical to the one we presented in [8]. Let $p_n(k)$ denote the probability that $N(k) = n$. The following balance equation may be written for the infinite source model[1]:

$$\begin{aligned} p_n(k+1) = \sum_{j=2}^{n} p_{n-j}(k)c_j &+ p_{n+1}(k)g_1(n+1)c_0 \\ &+ p_n(k)(1 - g_1(n))c_0 + p_n(k)g_0(n)c_1 + p_{n-1}(k)(1 - g_0(n-1))c_1. \end{aligned} \tag{4}$$

To illustrate the interpretation of the right-hand side of (4), we note that the first term covers the cases where two or more packets have been transmitted by the active terminals during the kth slot and the second term covers the case in which exactly one blocked terminal has transmitted while no active terminal has done so. Notice that $\{N(k); k = 0, 1, \ldots\}$ is a Markov chain and that it is aperiodic and irreducible. It is ergodic if an invariant probability measure $\{p_n : n = 0, 1, \ldots\}$ exists satisfying (4) such that $p_n > 0$ for all n and $p_n = \lim_{k\to\infty} p_n(k)$. To show that $\lim_{k\to\infty} Pr(N(k) < j) = 0$ for all finite values of j, it suffices to show that the Markov chain representing the number of blocked terminals is not ergodic. Taking the limit of (4) and using $p_n = \lim_{k\to\infty} p_n(k)$, we obtain

$$p_n = \sum_{j=0}^{n} p_{n-j}c_j + p_{n+1}g_1(n+1)c_0 + p_n(g_0(n)c_1 - g_1(n)c_0) - p_{n-1}g_0(n-1)c_1. \tag{5}$$

[1] Equation (4) is valid for all $n \geq 0$ if we adopt the rule that $P_i(k) = 0$, $i < 0$.

Letting

$$S_N = \sum_{n=0}^{N} p_n, \tag{6}$$

we then have, for any $N \gtrless 0$,

$$S_N = p_{N+1}g_1(N+1)c_0 + p_N g_0(N)c_1 + \sum_{n=0}^{N} S_{N-n}c_n \tag{7}$$

or

$$S_N(1-c_0) = \sum_{n=1}^{N} S_{N-n}c_n + p_{N+1}g_1(N+1)c_0 + p_N g_0(N)c_1$$

or equivalently

$$p_N(-c_0) \le p_{N+1}g_1(N+1)c_0 + p_N g_0(N)c_1 \tag{8}$$

But then, from (3) and (8), we have

$$p_{N+1}/p_N \ge (1 - c_0 - (1-f)^N c_1)/(N+1)f(1-f)^N c_0$$

for any nonnegative integer N. This result implies that the ratio $(p_{N+1}/p_N) \to \infty$ as $N \to \infty$; so the sum S_∞ can exist only if $p_N = 0$ for all finite values of N – otherwise S_∞ is divergent, which cannot be the case when the p_N, $N \ge 0$, define a probability distribution. Thus the Markov chain representing the number of blocked terminals is not ergodic, and the broadcast channel under the infinite source assumption is unstable

Now consider the finite source model. Using the rule that $p_i(k) = 0$ for $i < 0$, we find that the balance equation for $0 \le n < M$ is

$$\begin{aligned} p_n(k+1) = {} & \sum_{j=2}^{n} p_{n-j}(k)q_j(n-j) + p_{n+1}(k)g_1(n+1)q_0(n+1) \\ & + p_n(k)(1-g_1(n))q_0(n) + p_n(k)g_0(n)q_1(n) + p_{n-1}(k)(1-g_0(n-1))q_0(n-1) \end{aligned} \tag{9}$$

Defining, for $0 \le N < M$, the sum S_N as in (6) for the finite source model, we take the limit of (9) and obtain

$$S_N = p_{N+1}g_1(N+1)q_0(N+1) + p_N g_0(N)q_1(N) + \sum_{n=0}^{N}\sum_{j=0}^{n} p_{n-j}q_j(n-j). \tag{10}$$

Notice first that since the Markov chain is aperiodic, irreducible, and finite, it is ergodic for each $M < \infty$. Therefore $p_i > 0$, $0 \le i \le M$, $\sum_{i=1}^{M} p_i = 1$. With $N = M - 1$, (10) yields

$$S_{M-2} - \sum_{n=0}^{M-2}\sum_{j=0}^{n} p_{n-j}\,q_j(n-j) + b\,p_{M-1}[1-(1-f)^{M-1}] = p_M f(1-f)^{M-1},$$

or, as can be easily verified,

$$(p_M/p_{M-1}) \ge b\,[1-(1-f)^{M-1}]/f(1-f)^{M-1}.$$

Therefore as $M \to \infty$ this ratio tends to infinity, which can only imply that $p_{M-1} \to 0$. But the argument is valid for any p_N/p_{N-1}, $0 < N \le M$. Therefore as $M \to \infty$ we have $p_N \to 0$, $0 < N < M$, and $p_M \to 1$. This result proves the theorem for the case of the finite source □

In the context of this study another measure of interest is the throughput of the broadcast channel. Indeed this throughput may well be the primary performance measure for the system under consideration

Definition 2 The *conditional throughput* $D_n(k)$ of the broadcast channel is the conditional probability that one packet is successfully transmitted during the kth slot given that $N(k) = n$.

Clearly the conditional throughput cannot exceed 1. It can also be defined as the expected value of the number of successful transmissions during the kth slot conditional on there being n blocked terminals at the beginning of that slot

Definition 3. The *throughput* of the broadcast channel is defined as

$$D = \lim_{k \to \infty} \sum_{n=0}^{\infty} D_n(k) p_n(k)$$

The conditional throughput is $D_n(k) = c_0 g_1(n) + c_1 g_0(n)$ for the infinite source model; for the finite source model we replace c_0 and c_1 by $q_0(n)$ and $q_1(n)$, respectively. This quantity is obviously independent of k; therefore in the following we simply write D_n instead of $D_n(k)$.

THEOREM 2. *For $f > 0$, the throughput of the broadcast channel is 0 for the infinite source model, and for the finite source model, as we let $M \to \infty$, $b \to 0$, $M \cdot b \to d$.*

The proof is straightforward and is not presented here.

3. *Certain Channel Control Policies*

In [14] various control policies for the broadcast channel have been classified, roughly speaking, into three groups: policies that regulate access to the channel from the active terminals; those that regulate access from the blocked terminals; and mixed policies In this section we discuss two policies in some detail and give a definition of stability in each case. We see that this definition will be a variant of, or identical to, the definition given above. The first control policy that we describe typifies the first group of policies, although it may well be impossible to implement, the second policy is of the second group and has a better chance of being realizable.

3.1 A THRESHOLD CONTROL POLICY An input control policy as defined by Lam [14] is one that limits access to the channel from the active terminals depending on the present state and past history of the channel Borrowing the terminology of Markov decision theory [10], we say that a policy is *stationary* if it depends only on the present state of the system.

The first policy we present is described in Figure 2 If the number of blocked terminals exceeds θ, the threshold, an active terminal that wishes to initiate the transmission of a packet, is not allowed to transmit and joins the *impeded set*; if the threshold is not exceeded, the transmission takes place as in the uncontrolled channel As soon as the number of blocked terminals decreases below θ (this can only take place in steps of one), an impeded terminal joins the blocked set, thus the number of blocked terminals can be less than θ only if there are no impeded terminals The retransmission rate of blocked terminals is constant. We refer to this scheme as the *threshold control policy*.[2]

In this context stability must be defined in terms of the number of impeded or blocked terminals

Definition 4 Let $U(k)$ be the number of blocked or impeded terminals at the beginning of the kth slot for the threshold control policy. The infinite source channel, with this control scheme, is unstable if, in the limit $k \to \infty$, $Pr\{U(k) < j\}$ is zero for all finite values of j; for the finite source model the same definition is used with the stipulation that $b \to 0$ and $M \cdot b \to d$.

The following equations, which must be satisfied by the equilibrium probabilities p_n for the number of blocked or impeded terminals at the beginning of a slot, may be derived

$n \le \theta$:

$$p_n = \sum_{j=0}^{n} p_{n-j} c_j + p_{n+1} A_1(n+1) c_0 + p_n[c_1 g_0(n) - c_0 g_1(n)] - p_{n-1} g_0(n-1) c_1.$$

$n > \theta + 1$.

[2] Lam's [14] input control policy differs from ours in that he assumes that packets are lost

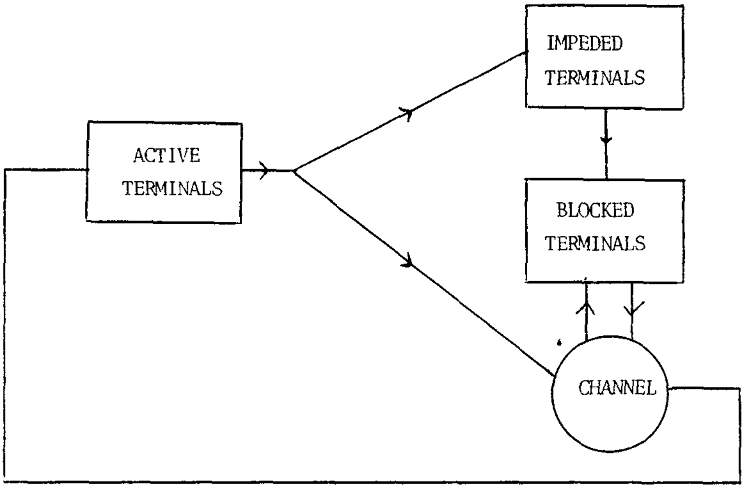

FIG 2

$$p_n = \sum_{j=0}^{\theta} p_j c_{n-j} + \sum_{j=\theta+1}^{n} p_j[c_{n-j}(1 - A) + c_{n-j+1}A] + p_{n+1}c_0A,$$

where $A = A_1(n) = g_1(\theta)$ if $n > \theta$, and $A_1(n) = g_1(n)$ if $n \leq \theta$.

$n = \theta + 1$:

$$p_{\theta+1} = p_{\theta+1}[c_1A + c_0(1 - A)] + p_{\theta+2}c_0A + p_\theta c_1(1 - A) + \sum_{j=0}^{\theta-1} p_j c_{\theta+j-1}.$$

We obtain the following result concerning the stability of the threshold control policy. For simplicity let $A = g_1(\theta)$.

THEOREM 3. *If the expected arrival rate $\lambda = \sum_{i=1}^{\infty} ic_i$ of active packets for the infinite source model is less than A, then the broadcast channel with a stationary threshold control policy is stable.*

The proof is given in Appendix 1.

The threshold control policy may be quite difficult to implement in practice, even though it is conceptually very simple It is effectively not easy to fill up the impeded set

3.2 A RETRANSMISSION CONTROL POLICY A retransmission control policy is one that regulates access to the channel from the set of blocked terminals as a function of the past and present states of the system. We consider a stationary policy that uses information concerning only the present state to regulate the retransmission rate of the ensemble of blocked terminals. The appropriate definition of stability for this case is then given in Defintion 1. The equations for the controlled system are (4) for the infinite source model and (5) for the finite source model with the following modification The parameter f that determines $g_i(n)$ (see (1) and (2)) and gives the probability that a blocked terminal retransmits a packet during a slot will be a function of n.[3] Denoting this function by $f(n)$, we have

$$g_i(n) = \binom{n}{i} = [f(n)]^i[1 - f(n)]^{n-i}$$

The following result can then be established.

THEOREM 4. *A stationary retransmission control policy yields a stable infinite source broadcast channel if $\lambda = \sum_{i=1} ic_i < d$ and an unstable one if $\lambda > d$, where $d = \lim_{n\to\infty} [c_1g_0(n) + c_0g_1(n)]$.*

The proof of this result is given in Appendix 2. We do not have a proof of instability

[3] Lam [14] considers retransmission control policies in which $f(n)$ can take one of only two values

for $\lambda = d$ except for a special case; the question is only of mathematical interest, however.

Remark. In fact Theorem 1 is a corollary of Theorem 4 since if f is independent of n we have $d = 0$, as in the case for the uncontrolled broadcast channel.

Another consequence of Theorem 4 concerns the form that the function $f(n)$ must take to ensure stability.

THEOREM 5. *For the infinite source broadcast channel under stationay retransmission control to be stable, it is necessary that*

$$\lim_{n\to\infty} f(n) = 0 \quad and \quad \lim_{n\to\infty} nf(n) > 0.$$

PROOF Clearly, if the first condition is not satisfied, we have $d = 0$, and the channel is unstable Now suppose that the second condition is not satisfied, that is, $\lim_{n\to\infty} nf(n) = 0$, but that the first condition is satisfied Then $d = c_1$, and we cannot have $\lambda < d$; therefore by Theorem 4 the system is unstable, which completes the proof. □

We see by this last result that a stationary retransmission control policy (with expected time between attempts of a blocked terminal to retransmit given by $[f(n)]^{-1}$) may stabilize the channel only if $f(n)$ decreases with n but no faster than the function n^{-1}.

3 3 AN OPTIMAL RETRANSMISSION CONTROL POLICY. It is natural to seek retransmission control policies that maximize the output rate of the channel; for a stabilizing policy, the maximum value is d of Theorem 4 since the input rate is identical to the output rate Consider

$$D_n(f) = c_1(1 - f)^n + c_0 nf(1 - f)^{n-1}.$$

By differentiating this expression with respect to f and setting the result equal to zero, we see that $D_n(f)$ is maximized by setting f equal to $f^* = (c_0 - c_1)(nc_0 - c_1)^{-1}$ for $n \geq 1$, or $f^* = (1 - \alpha)(n - \alpha)^{-1}$ if $\alpha = c_1/c_0$, where we are restricted to $\alpha < 1$ (for instance, with a Poisson arrival process $\alpha = \lambda$) The maximum value of $D_n(f)$ is then $D_n(f^*) = c_0[(n - 1)(n - \alpha)^{-1}]^{n-1}$. In the limit as $n \to \infty$, we obtain the throughput $d = \exp(\log c_0 + \alpha - 1)$ If the arrival process is Poisson, we obtain $d = e^{-1}$, as predicted by Abramson [1] and Kleinrock and Lam [12] for the maximum throughput of the channel.

In Figure 3 we present time series characterizing channel behavior obtained by Monte Carlo simulation with a Poisson arrival process of packets from active terminals. In Figure 3(a) we show the behavior of the uncontrolled broadcast channel; we see that if the number of blocked terminals is sufficiently high, the channel is unable to recover (i e it is unstable) and the total number of blocked terminals increases indefinitely while the channel throughput tends to zero In Figure 3(b) we see the channel behavior under identical conditions, except that the retransmission probability is chosen to be f^*. The channel is now able to recover from an initial state with a large number of blocked terminals, and the throughput matches the input rate. The exact form of f chosen in the simulation results of Figure 3(b) is $f^+ = (1 - \lambda)n^{-1}$, where the denominator term of f^* has been simplified There is a simple intuitive (but nonrigorous) explanation for the choice of f^+: When there are n blocked terminals and n is very large, the set of blocked terminals behaves as a Poisson source of parameter $f^+n = 1 - \lambda$; thus the total input rate of packets to the channel is $\lambda + f^+n = 1$, which is the maximum rate it can accommodate

The optimal control policy f^* can be approximately implemented by a statistical estimation of the number of blocked terminals or by an estimation of channel traffic: These practical problems are discussed and evaluated in [4], where extensive simulation results are given In fact this policy has been implemented in a prototype system [17] in which a hardware simulator replaces the satellite channel.

4 *Conclusions*

In this paper we have given a theoretical treatment of some basic problems related to the packet switching broadcast channel Its inherent instability has motivated us to look into stabilizing control policies. The first policy examined was one in which access to the channel is controlled by placing into an impeded set those active terminals wishing to transmit a packet. Necessary and sufficient conditions under which the number of impeded or blocked terminals remains bounded were derived, and it was shown that with this policy it is theoretically possible to achieve a throughput that is arbitrarily close to 1.

We then examined control schemes based only on choosing the transmission probability of any blocked terminal as a function of the total number of blocked terminals Sufficient conditions for stability and instability of the channel and necessary conditions that must be satisfied by the retransmission probability were derived for this scheme. We

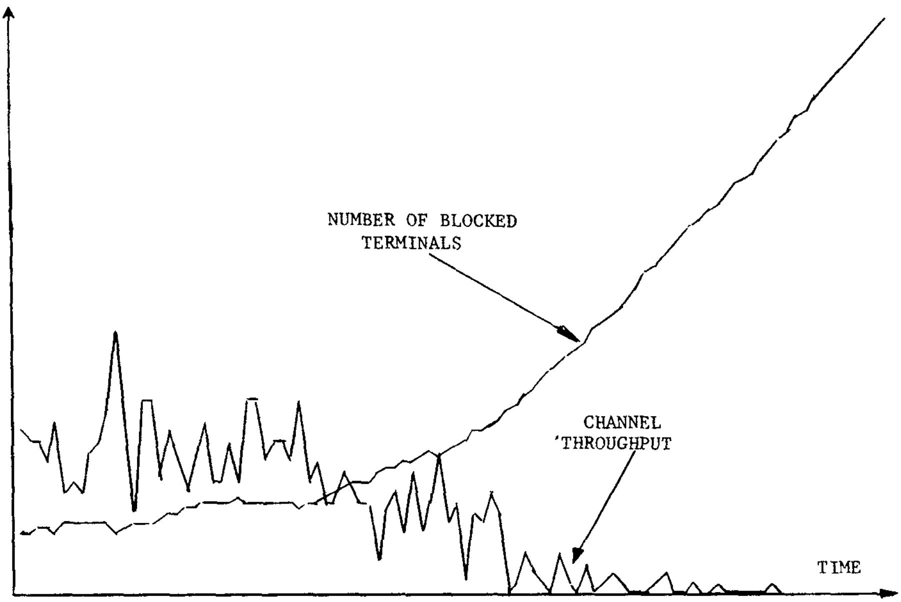

FIG 3(a)

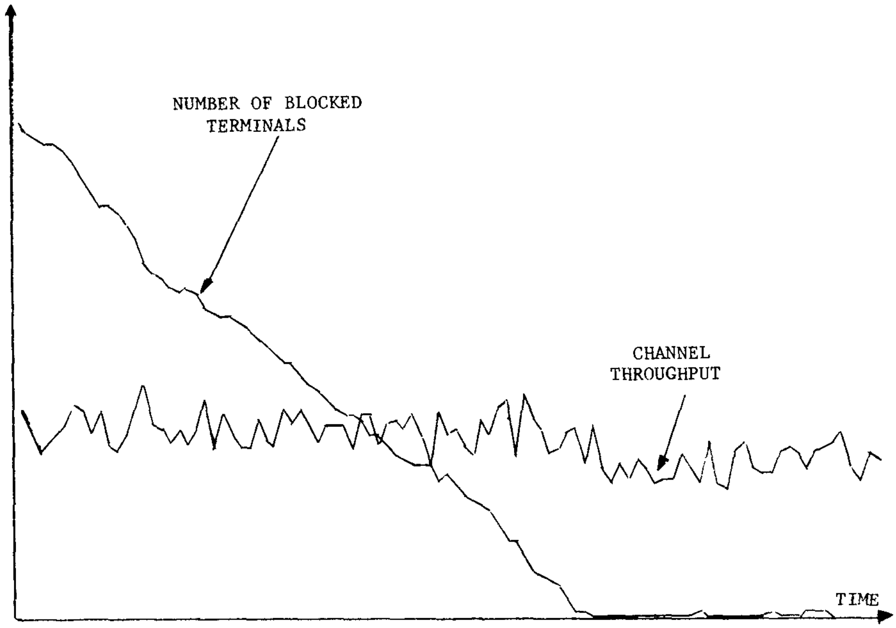

FIG 3(b)

then obtained the optimal control policy that maximizes the channel throughput. This policy appears promising as a practical means of optimizing channel performance.

Appendix 1. *Proof of Theorem* 3

The proof of Theorem 3 is easily obtained as a consequence of a theorem of Pakes [17] which we restate here.

PAKES'S LEMMA *Let $\{X_n\}_{n\geq 1}$ be an irreducible, aperiodic Markov chain whose state space is the set of nonnegative integers. The following conditions are sufficient for the Markov chain to be ergodic*:

(*a*) $|E\{X_{n+1} - X_n | X_n = i\}| < \infty$,

(*b*) $\lim_{i\to\infty} \sup E\{X_{n+1} - X_n | X_n = i\} < 0$.

We verify that these two conditions are satisfied when $\lambda = \sum_0^\infty jc_j < g_1(\theta)$ for the input control policy

Consider the case $i \geq \theta$. We then have

$$E\{X_{n+1} - X_n | X_n = i\} = \sum_0^\infty (i + j)c_j(1 - g_1(\theta)) + \sum_0^\infty (i + j - 1)c_j g_1(\theta) - i$$
$$= \lambda - g_1(\theta);$$

so both conditions are clearly satisfied. Notice that for $i < \theta$, it is only necessary to verify condition (a). If $X_n = i$, we have

$$X_{n+1} = \begin{cases} i - 1 & \text{with probability } c_0 g_1(i), \\ i & \text{with probability } c_0(1 - g_1(i)), \\ i + 1 & \text{with probability } c_1(1 - g_0(i)), \\ i + j,\ j \geq 2 & \text{with probability } c_j \end{cases}$$

Therefore $E\{X_{n+1} - X_n | X_n = i\} = \lambda - c_1 g_0(i) - c_1 g_0(i)$. □

Appendix 2. *Proof of Theorem* 4

Let us first verify that the channel is unstable if $\lambda > d$ If the limit defining d exists, then for each $\epsilon > 0$ there exists an integer n_0 such that, for all $n \geq n_0$,

$$|g_1(n) - a| \leq \epsilon \quad \text{and} \quad |g_0(n) - b| \leq \epsilon,$$

where a, b are constants such that $d = c_0 a + c_1 b$. Let $P(z) = \sum_{n=n_0}^\infty p_n z^n$, $Q(z) = \sum_{n=n_0}^\infty S_n z^n$. Then, from (7) and the discussion above, we have

$$S_n - \sum_{j=0}^n S_{n-j}c_j \leq (a + \epsilon)p_{n+1}c_0 + (b + \epsilon)p_n c_1$$

and

$$S_n - \sum_{j=0}^n S_{n-j}c_j \geq (a - \epsilon)p_{n+1}c_0 + (b - \epsilon)p_n c_1 \tag{11}$$

for all $n \geq n_0$. Thus

$$Q(z) - \sum_{n=n_0}^\infty \sum_{j=0}^n S_{n-j}c_j z^n \leq [(a + \epsilon)c_0/z][P(z) - z^{n_0}p_{n_0} + (b + \epsilon)c_1 P(z)].$$

Note that

$$\sum_{n=n_0}^\infty \sum_{j=0}^n S_{n-j}c_j z^n = \sum_{n=n_0}^\infty \sum_{j=0}^{n-n_0} S_{n-j}c_j z^n + \sum_{n=n_0}^\infty \sum_{j=n-n_0+1}^n S_{n-j}c_j z^n$$

Therefore, if we write $C(z) = \sum_{j=0}^\infty c_j z^j$,

$$Q(z)(1 - C(z)) \leq [(a + \epsilon)c_0/z][P(z) - z^{n_0}p_{n_0} + (b + \epsilon)c_1 P(z)] + \sum_{n=n_0}^\infty \sum_{j=n-n_0+1}^n S_{n-j}c_j z^n.$$

The following relationship may be verified:

$$Q(z)(1 - z) = s^{n_0}S_{n_0} + P(z) - z^{n_0}P_{n_0} = P(z) + z^{n_0}S_{n_0-1},$$

yielding, after substitution and combining of terms,

$$P(z)\left[\frac{1 - C(z)}{1 - z} - \frac{(a + \epsilon)c_0}{z} - (b + \epsilon)c_1\right]$$
$$\leq -z^{n_0}\left(\frac{1 - C(z)}{1 - z}\right)S_{n_0-1} - z^{n_0-1}(a + \epsilon)c_0p_{n_0} + \sum_{n=n_0}^{\infty}\sum_{j=n-n_0+1}^{n} S_{n-j}c_jz^n.$$

However,

$$\sum_{n=n_0}^{\infty}\sum_{j=n-n_0+1}^{n} S_{n-j}c_jz^n \leq S_{n_0-1}\sum_{n=n_0}^{\infty}\sum_{j=n-n_0+1}^{n} c_jz^n,$$

and

$$\sum_{n=n_0}^{\infty}\sum_{j=n-n_0+1}^{n} c_jz^n \leq \sum_{n=1}^{\infty}\sum_{j=n}^{\infty} c_jz^n = F(z),$$

where $\lim_{z\to 1} F(z) = \lambda$. Therefore,

$$P(z)\left[\frac{1 - C(z)}{1 - z} - \frac{(a + \epsilon)c_0}{z} - (b + \epsilon)c_1\right]$$
$$\leq S_{n_0-1}\left[-z^{n_0}\frac{1 - C(z)}{1 - z} + \sum_{n=1}^{\infty}\sum_{j=n}^{\infty} c_jz^n\right] - z^{n_0-1}(a + \epsilon)c_0p_{n_0}.$$

Now take the limit as $z \to 1$ of both sides to obtain

$$P(1)\,[\lambda - (a + \epsilon)c_0 - (b + \epsilon)\,c_1] \leq -c_0p_{n_0}(a + \epsilon)$$

Therefore, if $\lambda > d$, choosing n_0 sufficiently large that $\lambda - d > \epsilon(c_0 + c_1)$, we have that either $p_{n_0} = 0$ and $P(1) \leq 0$ or $p_{n_0} > 0$ and $P(1) < 0$; both cases imply that the balance equations satisfied by the equilibrium probability distribution $\{p_n\}$ do not possess a positive solution. Thus the Markov chain representing the number of blocked terminals at the beginning of each slot is not ergodic, and the channel is unstable when $\lambda > d$.

For $\lambda < d$ we start with (11) and proceed by arguments similar to the ones used above to obtain

$$P(1)\,[\lambda - (a - \epsilon)c_0 - (b - \epsilon)c_1] \geq -\lambda S_{n_0-1} - (a - \epsilon)c_0p_{n_0} + \sum_{n=n_0}^{\infty}\sum_{j=n-n_0+1}^{n} S_{n-j}c_j. \tag{12}$$

The last term on the right-hand side of (12) cannot exceed λS_{n_0-1}; therefore, assuming p_{n_0} is positive, we may write

$$P(1)\,[\lambda - (a - \epsilon)c_0 - (b - \epsilon)c_1] \geq -\alpha(n_0),$$

where

$$0 < \alpha(n_0) = \lambda S_{n_0-1} + (a - \epsilon)c_0p_{n_0} - \sum_{n=n_0}^{\infty}\sum_{j=n-n_0+1}^{n} S_{n-j}c_j,$$

since by choosing n_0 sufficiently large we know that $a > \epsilon$ Therefore, if $\lambda < d$,

$$P(1) \leq \alpha(n_0)/(d - \lambda - \epsilon(c_0 + c_1))$$

if n_0 is large enough that $d - \lambda > \epsilon(c_0 + c_1)$. From (5) we notice that we may write for any $n \geq 0$

$$p_n = k(n)p_0, \tag{13}$$

where $k(n) > 0$; thus

$$P(1) \le \left(\lambda \sum_{j=0}^{n_0-1} k(j) + (a - \epsilon)c_0 k(n_0)\right) \Big/ (d - \lambda - \epsilon(c_0 + c_1)) \tag{14}$$

Note that p_{n_0} is positive if and only if p_0 is positive We can now invoke Foster's theorem [6], which implies that the Markov chain is ergodic if there exists a positive solution to the equilibrium equation (5) such that $P(1) < \infty$. With p_0 set to 1 (or any positive constant), (13) represents a positive solution of (5); by (14) we have $P(1) < \infty$, and therefore we have satisfied Foster's condition, thus completing the proof that the channel is stable if $\lambda < d$. We now have to consider the case $\lambda = d$.

For $n \ge n_0$ we may write

$$g_1(n) = a + u_n, \quad g_0(n) = b + v_n;$$

so from (7) we obtain

$$Q(z)[1 - C(z)] = (ac_0/z)[P(z) - z^{n_0}p_{n_0}] + bc_1P(z) + (c_0/z)[U(z) - z^{n_0}u_{n_0}p_{n_0}] + c_1V(z) + \sum_{n=n_0}^{\infty} \sum_{j=n-n_0+1}^{n} S_{n-j}c_jz^n,$$

where

$$U(z) = \sum_{n=n_0}^{\infty} u_np_nz^n, \quad V(z) = \sum_{n=n_0}^{\infty} v_np_nz^n,$$

hence

$$P(z) = \frac{\left(-z^{n_0}S_{n_0-1}\dfrac{1 - C(z)}{1 - z} - C_0(a + u_{n_0})p_{n_0}z^{n_0-1} + \dfrac{C_0}{z}U(z) + C_1V(z) + \displaystyle\sum_{n=n_0}^{\infty}\sum_{j=n-n_0+1}^{n} S_{n-j}c_jz^n\right)}{\left(\dfrac{1 - C(z)}{1 - z} - \dfrac{aC_0}{z} - bc_1\right)}.$$

For $\lambda = d$, the denominator of $P(1)$ vanishes. Instability for $\lambda = d$ will hold if we can show that the numerator of $P(1)$ does not vanish, or that as $z \to 1$ the numerator of $P(z)$ tends to zero more slowly than the denominator If $c_0g_1(n) + c_1g_0(n) < d$ for all $n \ge n_0$ (i e. if D_n tends to d from below), then $c_0U(1) + c_1V(1) < 0$ and clearly the numerator of $P(1)$ is negative for $p_{n_0} > 0$ and $P(1)$ does not exist Under this condition, the channel is unstable for $\lambda = d$ In general, however, even though we conjecture that the channel is unstable when $\lambda = d$, we have no proof □

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