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G-NETWORKS WITH SIGNALS AND BATCH REMOVAL

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We consider queueing networks containing customers and signals that were recently introduced in Gelenbe [4]. Both customers and signals can be exogenous or can be obtained by a Markovian transition of a customer after service. A signal entering a queue forces a customer to move on to another queue according to a Markovian routing rule or to leave the network in batch mode. This synchronized or triggered motion is useful in representing the effect of tokens in Petri-nets, for systems in which customers and work can be instantaneously moved from one queue to the other on the arrival of a signal as well as for other network behaviors that are encountered in parallel computer system modelling. We show that this network has product form stationary solution and establish the non-linear customer flow equations that govern it. Network stability is discussed in this new context.

1. INTRODUCTION

We consider an open network of queues with n servers that have mutually independent i.i.d. exponential service times of rates $r(1),\ldots,r(n)$. Two types of entities circulate in the network: customers and signals. External arrivals to the ith queue of the network can be either customers that arrive according to a Poisson process of rate $\Lambda(i)$ or signals that constitute a Poisson arrival process of

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rate $\lambda(i)$. An arriving customer adds 1 to the queue length. The queue length is constituted only by customers, and their service is carried out in the usual manner. A customer that leaves queue i (after finishing service) heads for queue j with probability $p^+(i,j)$ as a customer or with probability $p^-(i,j)$ as a signal, or it will depart from the network with probability $d(i) = 1 - \sum_j p(i,j)$, $1 \le i \le n$. The matrix P, with elements $p(i,j) = p^+(i,j) + p^-(i,j)$, is the transition matrix of a Markov chain representing the movement of customers and signals. A signal arriving to an empty queue will have no effect and will just disappear; if queue i is non-empty, then one of the following two events occur:

> The arriving signal triggers the instantaneous passage of a customer from queue i to some other queue j with probability $q(i,j)$; Q denotes the corresponding n-by-n transition probability matrix.
>
> With probability $D(i) = 1 - \sum_j q(i,j)$, it forces a batch of customers of random size to leave the network. Let the length of queue i be k_i at the instant of arrival of the trigger; if $k_i \ge B_i$, then its length is reduced by B_i; if $k_i < B_i$, then the queue length becomes zero. The distribution of batch size B_i is general and given by $P[B_i = s] = \pi_{is}$, $s \ge 1$.

Thus, a signal acts as an external trigger that instantaneously moves a customer from one queue to the other or a batch to the outside world. We assume that $p(i,j)$ and $q(i,j)$ are the transition probabilities of transient Markov chains, so that each customer is guaranteed to leave the network with probability one. Let $k = (k_1, \ldots, k_n)$ be any n-vector of non-negative integers. Define the following vectors:

$$k_i^{+s} = (k_1, \ldots, k_i + s, \ldots, k_n), \qquad s \ge 1$$

$$k_i^{-} = (k_1, \ldots, k_i - 1, \ldots, k_n)$$

$$k_{ij}^{+-} = (k_1, \ldots, k_i + 1, \ldots, k_j - 1, \ldots, k_n)$$

$$k_{ij}^{++s} = (k_1, \ldots, k_i + 1, \ldots, k_j + s, \ldots, k_n), \qquad s \ge 1$$

$$k_{ijm}^{++-} = (k_1, \ldots, k_i + 1, \ldots, k_j + 1, \ldots, k_m - 1, \ldots, k_n).$$

We denote by $\{k(t) : t \ge 0\}$ the continuous time Markov chain representing the state of the network, where $k(t) = (k_1(t), \ldots, k_n(t))$ and $k_i(t)$ is the number of customers in queue i at time t. $\{k(t) : t \ge 0\}$ satisfies a system of Chapman-Kolmogorov equations, and its steady-state distribution $p(k) \equiv \lim_{t\to\infty} P[k(t) = k]$, if it exists, satisfies the following system of balance equations:

$$
\begin{aligned}
p(k) \sum_i \, &[\Lambda(i) + (\lambda(i) + r(i))\mathbf{1}[k_i > 0]] \\
&= \sum_i \Bigg[p(k_i^+) r(i) d(i) + p(k_i^-)\Lambda(i)\mathbf{1}[k_i > 0] + \lambda(i) D(i) \sum_{s=1}^{\infty} \pi_{is} p(k_i^{+s}) \\
&\quad + \lambda(i) D(i) \sum_{s=1}^{\infty} \pi_{is} \sum_{v=0}^{s-1} p(k_i^{+v})\mathbf{1}[k_i = 0] \\
&\quad + \sum_j \Bigg\{ p(k_{ij}^{+-})\, [r(i)p^+(i,j) + \lambda(i) q(i,j)]\,\mathbf{1}[k_j > 0] \\
&\qquad + \sum_{s=1}^{\infty} \pi_{is} p(k_{ij}^{++s}) r(i) p^-(i,j) D(j) \\
&\qquad + \sum_{s=1}^{\infty} \pi_{is} \sum_{v=1}^{s-1} p(k_{ij}^{++v}) r(i) p^-(i,j) D(j) \mathbf{1}[k_j = 0] \\
&\quad + p(k_i^+) r(i) p^-(i,j)\mathbf{1}[k_j = 0] \\
&\quad + \sum_m p(k_{ijm}^{++-}) r(i) p^-(i,j) q(j,m) \mathbf{1}[k_m > 0] \Bigg\} \Bigg], \qquad (1)
\end{aligned}
$$

where $\mathbf{1}[X] = 1$ if X is true and 0 otherwise. The last four terms on the right-hand side of Eq. (1) cover the effect of signals that originate within the network: the first two of these cover the case where the arriving signal provokes the departure toward the outside of the network of a batch of customers, the next covers the case where the signal arrives to an empty queue, and the fourth deals with the case where the signal arriving from queue i to queue j causes the instantaneous motion of a positive customer from queue j to queue m, whose queue length therefore increases by 1.

The model of signals was introduced in Gelenbe [4]. Here we generalize it to the case of batch customer removal that was first discussed in Gelenbe [2] in the context of positive and negative customer networks. These models are generalizations of the queueing networks introduced in Gelenbe [3] and examined in Gelenbe, Glynn, and Sigman [5] for the case of the single server queue under more general assumptions concerning interarrival and service times. The "negative customer" of Gelenbe [2,3] corresponds to the case of a triggered external departure of a single normal customer (in the present paper with probability $p^-(i,j)D(j)$).

2. AN APPLICATION

In what follows we present an application of the model presented in this paper to flow control in communication networks. We wish to underline the fact that the application we discuss is a very special case of the general model and that much more sophisticated applications can be developed. Nevertheless, the example discussed below is clearly outside the possibilities of solution of Baskett-Chandy-Muntz and Palacias (BCMP) networks [1,6].

Consider the computer communication network shown below, which is composed of an input queue I and of S subnetworks $N_1, \ldots, N_S$, which offer disjoint alternate routes for the packets. Packets enter I from the outside world at some rate Λ, receive service in FIFO order at I, and are then routed into subnetwork N_i.

They may enter subnetwork N_i for one of two reasons:

They do so with some fixed probability P_i, $i = 1, \ldots, S$; this is predetermined so as to assure a minimal equitable traffic into each subnetwork.

Packets from I enter subnetwork N_i under the effect of the flow control decision: if a packet leaves subnetwork i, this indicates that the subnetwork's capacity can now accommodate a new packet. This information is used to trigger the entrance of one new packet from I into N_i. It is transformed into a flow control packet that is sent back to the input queue I. The effect of the flow control packet at the input queue is to instantaneously force the passage of a new packet from I into N_i.

3. THE PRODUCT FORM

For $x \in \mathcal{R}$ let $f_i(x) = [1 - \sum_{s=1}^{\infty} \pi_{is} x^s]/[1 - x]$, and consider the following system of non-linear equations:

$$\lambda^-(i) = \sum_j r(j) q_j p^-(j,i) + \lambda(i)$$

$$\lambda^+(i) = \sum_j r(j) q_j \left[p^+(j,i) + \sum_m p^-(j,m) q_m q(m,i) \right]$$

$$+ \sum_j \lambda(j) q_j q(j,i) + \Lambda(i), \tag{2}$$

where $q_i \equiv \lambda^+(i)/[r(j) + \lambda^-(i) D(i) f_i(q_i)]$, and $i,j,m = 1, \ldots, n$. Notice that $\lambda^+(i)$ may be written as

$$\lambda^+(i) = \sum_j r(j) q_j p^+(j,i) + \Lambda(i) + \sum_j \lambda^-(j) q_j q(j,i).$$

THEOREM 1: *If a non-negative solution* $\{\lambda^+(i), \lambda^-(i)\}$ *exists to Eq.* (2) *such that each* $\lambda^+(i) < r(i) + \lambda^-(i) D(i) f_i(q_i)$ *for* $i = 1, \ldots, n$, *then*

$$p(k) = \prod_{i=1}^{n} [1 - q_i] q_i^{ki}. \tag{3}$$

PROOF: We will only prove the result for the case where $P[B_i = 1] = 1$ for all i, i.e., when only eliminations of a *single* customer can occur. We know that if we can exhibit a solution to Eq. (1) which is a probability distribution, then it is unique because $\{k(t) : t \geq 0\}$ is an irreducible continuous time Markov chain.

Let us therefore verify that Eq. (3) satisfies Eq. (1). After this substitution Eq. (1) becomes

$$\sum_i [\Lambda(i) + (\lambda(i) + r(i))\mathbf{1}[k_i > 0]]$$

$$= \sum_i \Big[q_i r(i) d(i) + \Lambda(i)\mathbf{1}[k_i > 0]/q_i + q_i \lambda(i) D(i)$$

$$+ \sum_j \Big\{ [r(i)p^+(i,j) + \lambda(i)q(i,j)]\mathbf{1}[k_j > 0] q_i/q_j$$

$$+ q_i q_j r(i) p^-(i,j) D(j) + q_i r(i) p^-(i,j)\mathbf{1}[k_j = 0]$$

$$+ \sum_m q_i q_j r(i) p^-(i,j) q(j,m)\mathbf{1}[k_m > 0]/q_m \Big\} \Big],$$

which, by interchanging the summations on i and j in some terms on the right-hand side, then grouping terms and using Eq. (2), becomes

$$\sum_i [\Lambda(i) + (\lambda(i) + r(i))\mathbf{1}[k_i > 0]]$$

$$= \sum_i \Big[q_i r(i) d(i) + \Lambda(i)\mathbf{1}[k_i > 0]/q_i + q_i \lambda(i) D(i)$$

$$+ \sum_j \{ q_i q_j r(i) p^-(i,j) D(j) + q_i r(i) p^-(i,j)\mathbf{1}[k_j = 0] \} \Big]$$

$$+ \sum_m (\lambda^+(m) - \Lambda(m))\mathbf{1}[k_m > 0]/q_m$$

$$= \sum_i \Big[q_i r(i) d(i) + q_i \lambda(i) D(i)$$

$$+ \sum_j \{ q_i q_j r(i) p^-(i,j) + q_i r(i) p^-(i,j)\mathbf{1}[k_j = 0] \} \Big]$$

$$+ \sum_m (\lambda^-(m) + r(m))\mathbf{1}[k_m > 0].$$

Simplifying terms and using Eq. (2) again,

$$\sum_i [\Lambda(i) + \lambda(i)\mathbf{1}[k_i > 0]]$$

$$= \sum_i \Big[q_i r(i) d(i) + q_i \lambda(i) D(i) + \sum_j q_i r(i) p^-(i,j)\mathbf{1}[k_j = 0] \Big]$$

$$+ \sum_j q_j (\lambda^-(j) - \lambda(j)) D(j) + \sum_m \lambda^-(m)\mathbf{1}[k_m > 0]$$

$$= \sum_i q_i r(i) d(i) + \sum_i (\lambda^-(i) - \lambda(i))\mathbf{1}[k_i = 0]$$

$$+ \sum_i q_i \lambda^-(i) D(i) + \sum_i \lambda^-(i)\mathbf{1}[k_i > 0]$$

or

$$\sum_i [\Lambda(i) + \lambda(i)]$$

$$= \sum_i q_i r(i) d(i) + \sum_i \lambda^-(i) + \sum_i q_i \lambda^-(i) D(i)$$

$$- \sum_i \sum_j \lambda(j) q(j,i)/q_i) 1[k_i > 0]$$

$$= \sum_i \left[\lambda^+(i) + \lambda^-(i) - \sum_j q_i r(i) p^-(i,j) - \sum_j q_i r(i) p^+(i,j) \right.$$

$$\left. - \sum_j q_i \lambda^-(i) q(i,j) \right],$$

which yields

$$\sum_i \Lambda(i) = \sum_i \left[\lambda^+(i) - \sum_j q_i r(i) p^+(i,j) - \sum_j q_i \lambda^-(i) q(i,j) \right]$$

and is obviously verified. ■

4. STABILITY

The existence of a solution to the non-linear Eq. (2) and, finally, the stability of the network, i.e., the existence of the stationary probability distribution, are the last remaining questions concerning this network. They can be treated by extending the approach taken in Gelenbe and Schassberger [7], though the matter is somewhat more complicated for the present model.

The main difference with the stability proof given below with respect to the stability proof given in Gelenbe and Schassberger [7] is due to the effect of the signals which move traffic from one queue to the other and in the effect of batch customer removal.

THEOREM 2: *If the matrix* $[P^+ + Q]$ *is substochastic and transient (does not contain any ergodic classes), the solution to Eq.* (2), $\{\lambda^+(i), \lambda^-(i)\}$, $i = 1, \ldots, n$, *always exists.*

PROOF: The proof is based on Brouwer's theorem. Let us write Eq. (2) as

$$\lambda^+(i) = \sum_j \lambda^+(j) g(j) p^+(j,i) + \Lambda(i) + \sum_j \lambda^+(j) h(j) q(j,i),$$

$$\lambda^-(i) = \sum_j \lambda^+(i) g(j) p^-(j,i) + \lambda(i),$$

where we have used the following:

$$g(i) \equiv r(i)/[r(i) + \lambda^-(i) D(i) f_i(q_i)],$$

$$h(i) \equiv \lambda^-(i) D(i) f_i(q_i)/[r(i) + \lambda^-(i) D(i) f_i(q_i)], \qquad i = 1, \ldots, n.$$

Notice that $g(i)$ and $h(i) = 1 - g(i)$ are continuous functions of $r(i) \geq 0$ and of $\lambda^-(i) \geq 0$, with $0 \leq g(i)$, $h(i) \leq 1$ for all i. Define the following vectors:

$\boldsymbol{\lambda}^+$ with elements $\lambda^+(i)$,

$\boldsymbol{\lambda}^-$ with elements $\lambda^-(i)$,

$\boldsymbol{\Lambda}$ with elements $\Lambda(i)$, and

$\boldsymbol{\lambda}$ with elements $\lambda(i)$.

Let G be the diagonal matrix with elements $g(i)$. Then Eq. (4) may be written as

$$\boldsymbol{\lambda}^+ = \boldsymbol{\lambda}^+[GP^+ + (I - G)Q] + \boldsymbol{\Lambda}, \qquad \boldsymbol{\lambda}^- = \boldsymbol{\lambda}^+ GP^- + \boldsymbol{\lambda}$$

or

$$\boldsymbol{\lambda}^+(\mathbf{I} - [GP^+ + (I - G)Q]) = \boldsymbol{\Lambda}, \qquad \boldsymbol{\lambda}^- = \lambda^+ GP^- + \boldsymbol{\lambda}$$

The series $\sum_{n=0}^{\infty} [GP^+ + (I - G)Q]^n$ is geometrically convergent, because $G \leq \mathbf{I}$ and $[P^+ + Q]$ is substochastic and does not contain any ergodic classes [8, p. 43ff.]. Therefore,

$$\boldsymbol{\lambda}^+ = \boldsymbol{\Lambda} \sum_{n=0}^{\infty} [GP^+ + (I - G)Q]^n$$

so that

$$\boldsymbol{\lambda}^- - \boldsymbol{\lambda} = \boldsymbol{\Lambda} \sum_{n=0}^{\infty} [GP^+ + (I - G)Q]^n GP^-.$$

Let $\mathbf{y} = \boldsymbol{\lambda}^- - \boldsymbol{\lambda}$, and call the vector function $F(\mathbf{y}) = \boldsymbol{\Lambda} \sum_{n=0}^{\infty} [GP^+ + (I - G)Q]^n GP^-$, where the dependency of F on $\mathbf{y}$ comes from G, which depends on $\boldsymbol{\lambda}^-$. Notice that F: $[\mathbf{0}, F(\mathbf{0})] \rightarrow [\mathbf{0}, F(\mathbf{0})]$ and that it is continuous. Therefore, by Brouwer's fixed-point theorem, $\mathbf{y} = G(\mathbf{y})$ has a fixed-point $\mathbf{y}^*$. This fixed point will in turn yield the solution of Eq. (4),

$$\boldsymbol{\lambda}^-(\mathbf{y}^*) = \boldsymbol{\lambda} + \mathbf{y}^*, \qquad \boldsymbol{\lambda}^+(\mathbf{y}^*) = \boldsymbol{\Lambda} \sum_{n=0}^{\infty} (F(\mathbf{y}^*)P^+)^n,$$

completing the proof. ∎

The result concerning existence of product form solutions for this class of networks is now as follows. Clearly, by setting $\mathbf{y}^*$ in the values of $\lambda^+(i)$ and $\lambda^-(i)$, we obtain the quantity

$$q_i(\mathbf{y}^*) = [\lambda^+(i)/[r(i) + \lambda^-(i)D(i)f_i(q_i)]]\, \mathbf{y} = \mathbf{y}^*, \qquad i = 1, \ldots, n.$$

THEOREM 3: *Consider the G-network considered in this paper. As a consequence of Theorems* 1 *and* 2, *its stationary solution* $p(\mathbf{k}) \equiv \lim_{t \to \infty} P[\mathbf{k}(t) = \mathbf{k}]$ *exists if* $q_i(\mathbf{y}^*) < 1$ *and does not exist if* $q_i(\mathbf{y}^*) \geq 1$.

References

1. Baskett, F., Chandy, K.M., Muntz, R.R., & Palacios, F.G. (1975). Open, closed and mixed networks of queues with product form. *Journal of the Association for Computing Machines* 22(2): 248-260.
2. Gelenbe, E. (to appear). Non-linear resolvents for very large linear systems. In *Proceedings of the Institute for Mathematics and Its Applications Workshop on Numerical Methods in Markov Chains*, January 1991, University of Minnesota, Minneapolis.
3. Gelenbe, E. (1991). Product form queueing networks with negative and positive customers. *Journal of Applied Probability* 28: 656-663.
4. Gelenbe, E. (to appear). G-networks with triggered customer movement. *Journal of Applied Probability*, September 1993.
5. Gelenbe, E., Glynn, P., & Sigman, K. (1991). Queues with negative arrivals. *Journal of Applied Probability* 28: 245-250.
6. Gelenbe, E. & Mitrani, I. (1980). *Analysis and synthesis of computer systems.* London and New York: Academic Press.
7. Gelenbe, E. & Schassberger, R. (1992). Stability of G-networks. *Probability in the Engineering and Informational Sciences* 6: 271-276.
8. Kemeny, J.G. & Snell, J.L. (1965). *Finite Markov chains.* Princeton, NJ: Van Nostrand.