DSP & Digital Filters

Mike Brookes

1: Introduction

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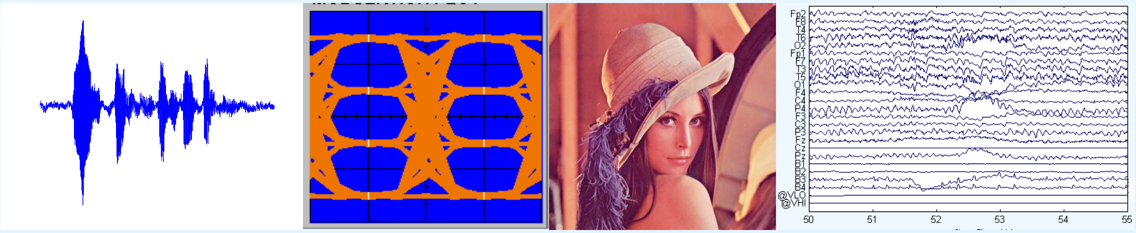
- Problems: Mitra textbook contains many problems at the end of each chapter and also MATLAB exercises

Signals

- A signal is a numerical quantity that is a function of one or more independent variables such as time or position.

Examples:



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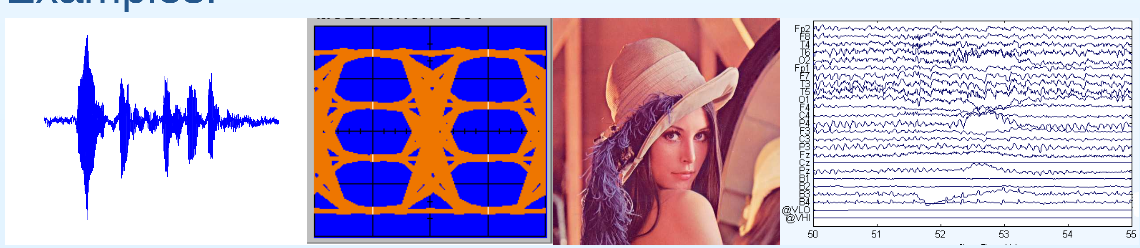

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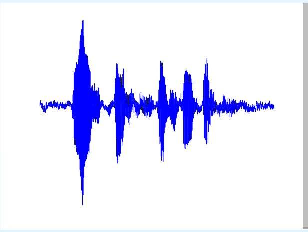
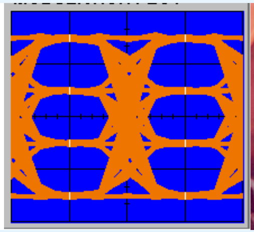

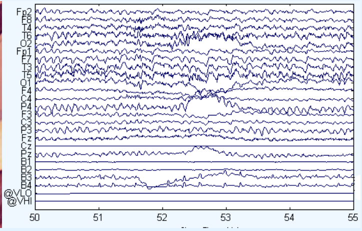

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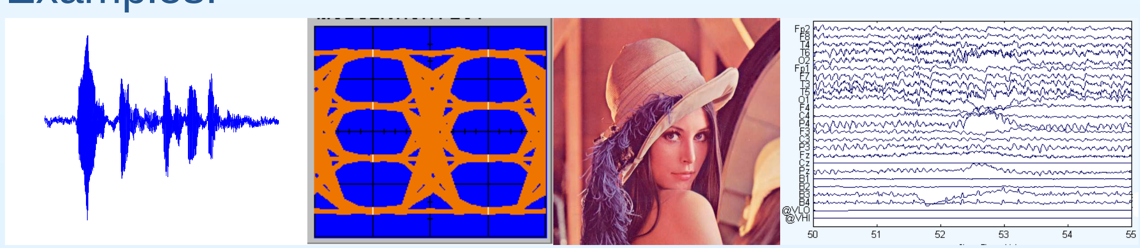

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  - Extension to multiple dimensions and complex-valued signals is straighforward in many cases.

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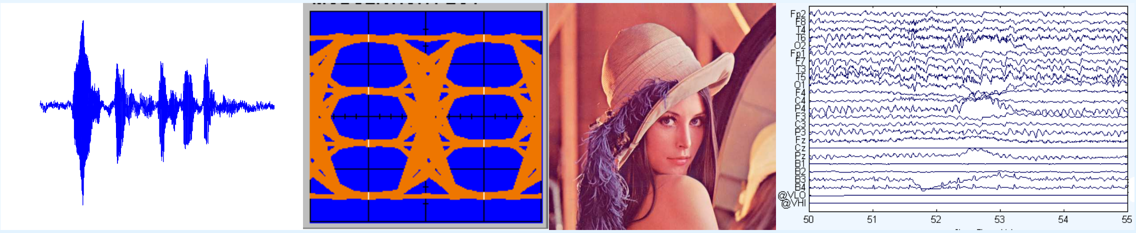

Processing

- Aims to "improve" a signal in some way or extract some information from it
- Examples:
  - Modulation/demodulation
  - Coding and decoding
  - Interference rejection and noise suppression
  - Signal detection, feature extraction
- We are concerned with linear, time-invariant processing

Syllabus

Main topics:

- Introduction/Revision
- Transforms
- Discrete Time Systems
- Filter Design
  - FIR Filter Design
  - IIR Filter Design
- Multirate systems
  - Multirate Fundamentals
  - Multirate Filters
  - Subband processing

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Warning: Several MATLAB routines scale time so that $f_s = 2$ Hz. Weird, non-standard and irritating.

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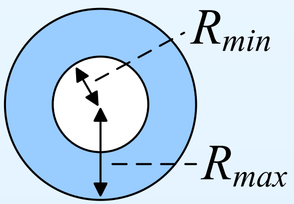

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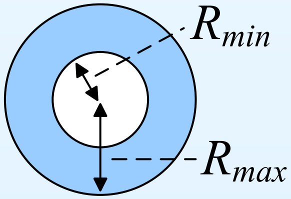

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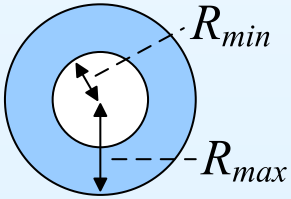

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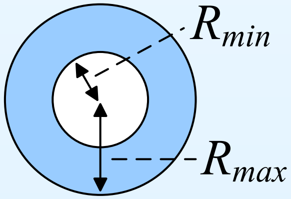

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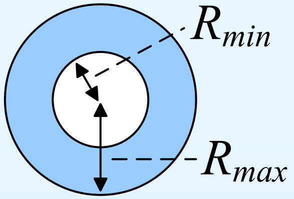

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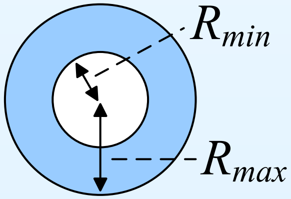

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The set of z for which $X(z)$ converges is its *Region of Convergence* (ROC).

Complex analysis $\Rightarrow$: the ROC of a power series (if it exists at all) is always an annular region of the form $0 \le R_{min} < |z| < R_{max} \le \infty$.

$X(z)$ will always converge absolutely inside the ROC and may converge on some, all, or none of the boundary.

- ◦ "converge absolutely" $\Leftrightarrow \sum_{n=-\infty}^{+\infty} |x[n]z^{-n}| < \infty$

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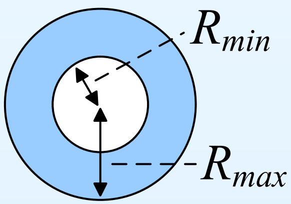

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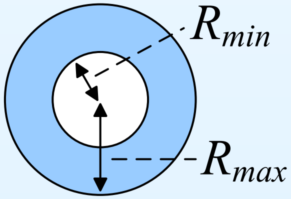

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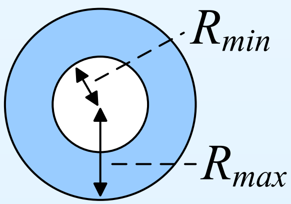

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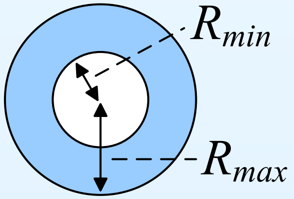
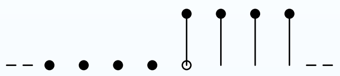


z-Transform examples

The sample at $n = 0$ is indicated by an open circle.

$u[n]$



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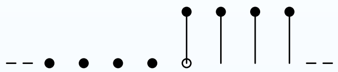
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \|z\| \le \infty$

Geometric Progression: $\sum_{n=q}^{r} \alpha^n z^{-n} = \frac{\alpha^q z^{-q} - \alpha^{r+1} z^{-r-1}}{1-\alpha z^{-1}}$

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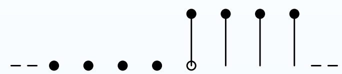
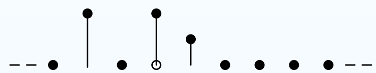
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$
$x[n]$			

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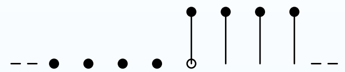
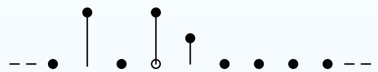
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$

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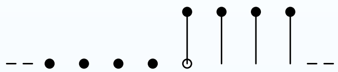
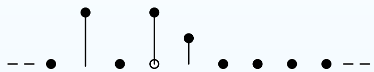
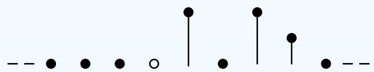
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$			

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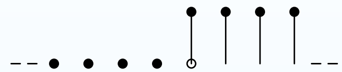
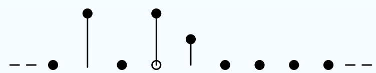
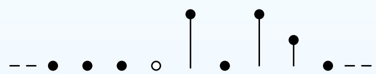
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \|z\| \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \|z\| < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \|z\| \leq \infty$

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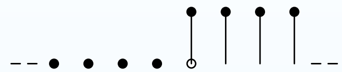
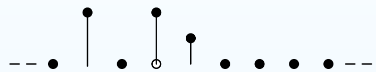
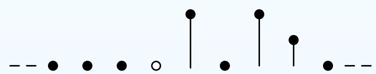
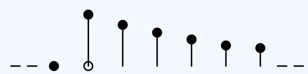
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$			

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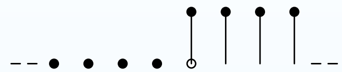
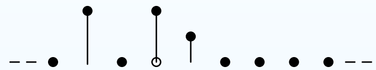
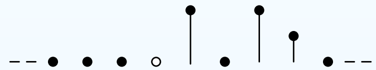
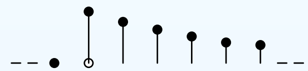
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \leq \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \leq \infty$

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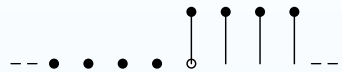
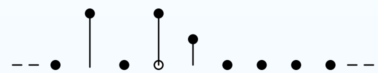
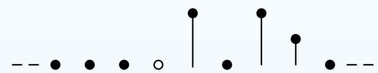
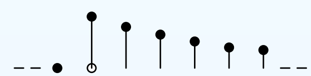
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \le \infty$
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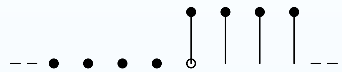
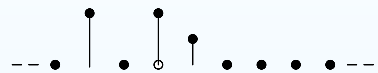
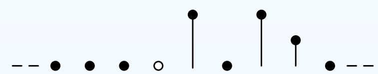
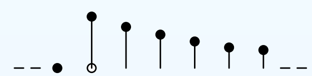
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \|z\| \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \|z\| < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \|z\| \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \|z\| \le \infty$
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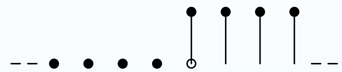
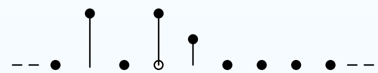
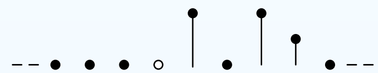
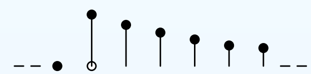
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \leq \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \leq \infty$
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Note: Examples 4 and 5 have the same z-transform but different ROCs.

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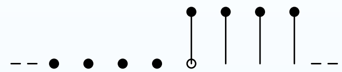
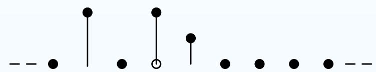
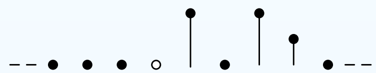
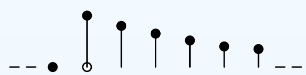
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \|z\| \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \|z\| < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \|z\| \leq \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \|z\| \leq \infty$
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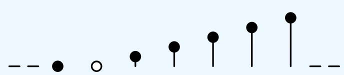
$nu[n]$			

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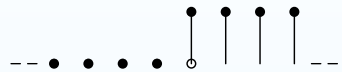
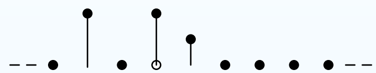
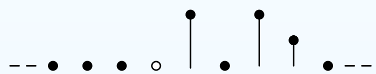
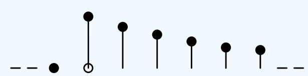
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \le \infty$
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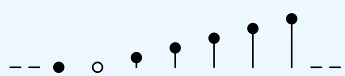
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \lvert z \rvert \le \infty$

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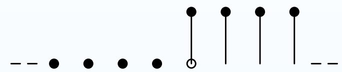
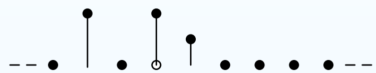
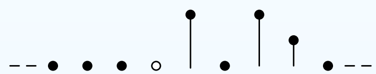
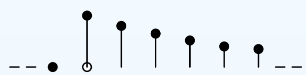
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \|z\| \leq \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \|z\| < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \|z\| \leq \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \|z\| \leq \infty$
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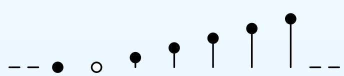
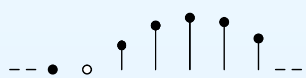
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \|z\| \leq \infty$
$\sin(\omega n)u[n]_{\omega=0.5}$			

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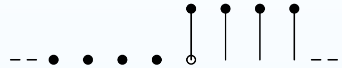
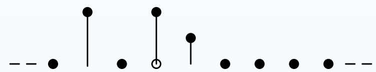
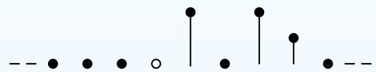
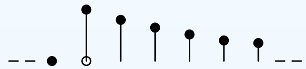
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \le \infty$
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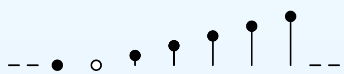
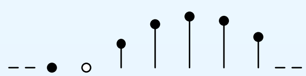
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \lvert z \rvert \le \infty$
$\sin(\omega n)u[n]_{\omega=0.5}$		$\frac{z^{-1}\sin(\omega)}{1-2z^{-1}\cos(\omega)+z^{-2}}$	$1 < \lvert z \rvert \le \infty$

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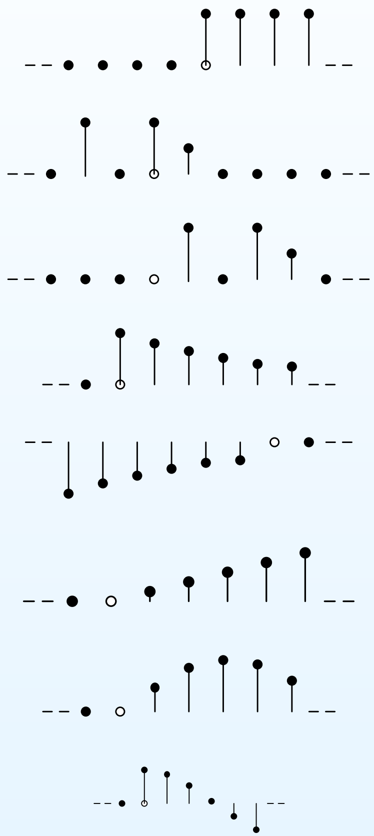

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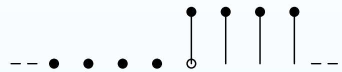
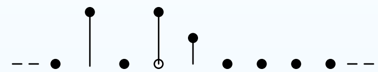
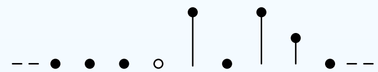
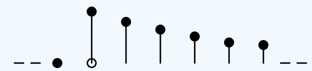
$u[n]$		$\frac{1}{1-z^{-1}}$	$1 < \lvert z \rvert \le \infty$
$x[n]$		$2z^2 + 2 + z^{-1}$	$0 < \lvert z \rvert < \infty$
$x[n-3]$		$z^{-3}\left(2z^2 + 2 + z^{-1}\right)$	$0 < \lvert z \rvert \le \infty$
$\alpha^n u[n]_{\alpha=0.8}$		$\frac{1}{1-\alpha z^{-1}}$	$\alpha < \lvert z \rvert \le \infty$
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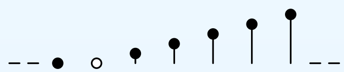
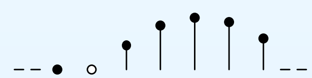
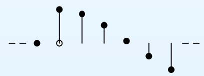
$nu[n]$		$\frac{z^{-1}}{1-2z^{-1}+z^{-2}}$	$1 < \lvert z \rvert \le \infty$
$\sin(\omega n)u[n]_{\omega=0.5}$		$\frac{z^{-1}\sin(\omega)}{1-2z^{-1}\cos(\omega)+z^{-2}}$	$1 < \lvert z \rvert \le \infty$
$\cos(\omega n)u[n]_{\omega=0.5}$		$\frac{1-z^{-1}\cos(\omega)}{1-2z^{-1}\cos(\omega)+z^{-2}}$	$1 < \lvert z \rvert \le \infty$

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Note: There are $K-M$ zeros or $M-K$ poles at $z=0$ (easy to overlook)

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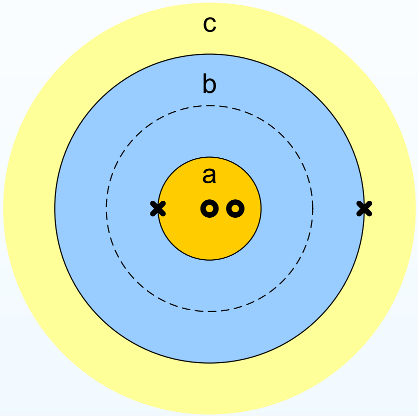

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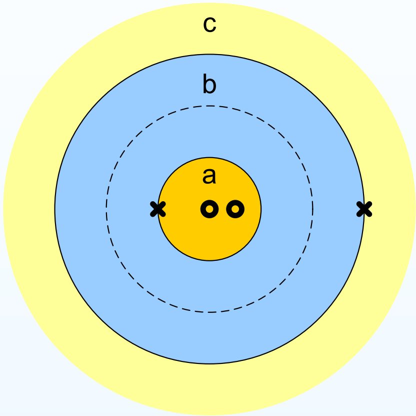

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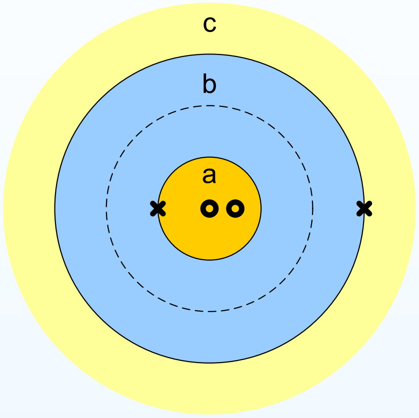

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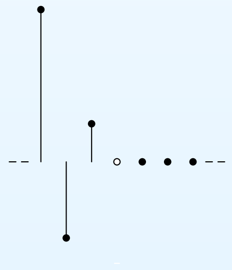
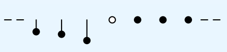
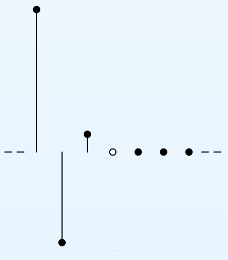
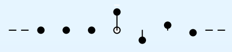
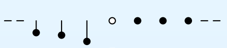
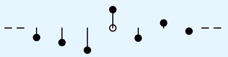
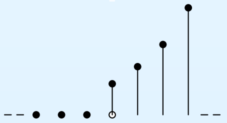
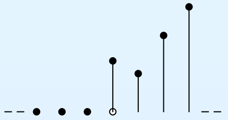
a	$0 \le \lvert z \rvert < 0.5$			
b	$0.5 < \lvert z \rvert < 1.5$			
c	$1.5 < \lvert z \rvert \le \infty$			

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$$\frac{dz}{d\theta} = jRe^{j\theta} \Rightarrow \frac{1}{2\pi j}\oint z^{k-1}dz = \frac{1}{2\pi j}\int_{\theta=0}^{2\pi} R^{k-1}e^{j(k-1)\theta} \times jRe^{j\theta}d\theta$$

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In practice use a combination of partial fractions and table of z-transforms.

MATLAB routines

tf2zp,zp2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \{z_m, p_k, g\}$
residuez	$\frac{b(z^{-1})}{a(z^{-1})} \rightarrow \sum_k \frac{r_k}{1-p_k z^{-1}}$
tf2sos,sos2tf	$\frac{b(z^{-1})}{a(z^{-1})} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2sos,sos2zp	$\{z_m, p_k, g\} \leftrightarrow \prod_l \frac{b_{0,l}+b_{1,l}z^{-1}+b_{2,l}z^{-2}}{1+a_{\in 1,l}z^{-1}+a_{2,l}z^{-2}}$
zp2ss,ss2zp	$\{z_m, p_k, g\} \leftrightarrow \begin{cases} x' = Ax + Bu \\ y = Cx + Du \end{cases}$
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For further details see Mitra:1 & 6.