2: Three Different Fourier Transforms

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We use Ω for "real" and $\omega = \Omega T$ for "normalized" angular frequency.
Nyquist frequency is at $\Omega_{\text{Nyq}} = 2\pi\frac{f_s}{2} = \frac{\pi}{T}$ and $\omega_{\text{Nyq}} = \pi$.

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For "power signals" (energy $\propto$ duration), CTFT & DTFT are unbounded.
Fix this by normalizing:

$$X(j\Omega) = \lim_{A\rightarrow\infty} \frac{1}{2A}\int_{-A}^{A} x(t)e^{-j\Omega t}dt$$

$$X(e^{j\omega}) = \lim_{A\rightarrow\infty} \frac{1}{2A+1}\sum_{-A}^{A} x[n]e^{-j\omega n}$$

Convergence of DTFT

DTFT: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ does not converge for all $x[n]$.

Consider the finite sum: $X_K(e^{j\omega}) = \sum_{-K}^{K} x[n]e^{-j\omega n}$

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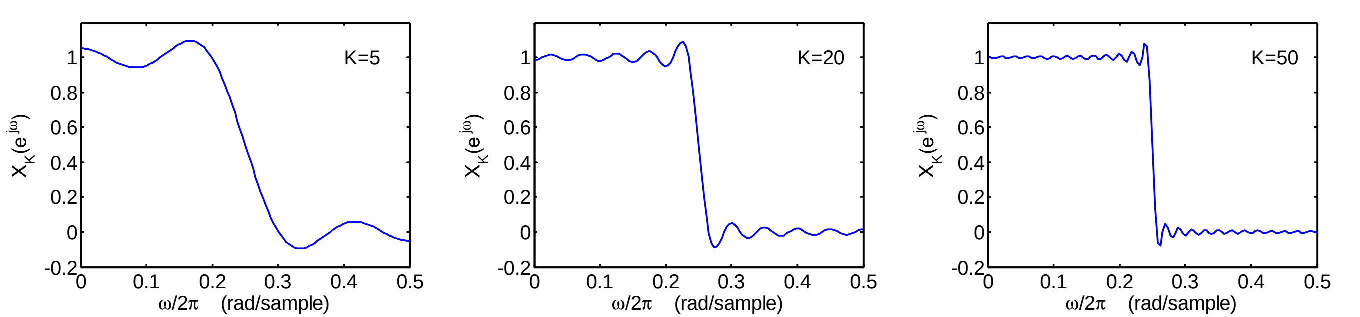
Example: $x[n] = \frac{\sin 0.5\pi n}{\pi n}$



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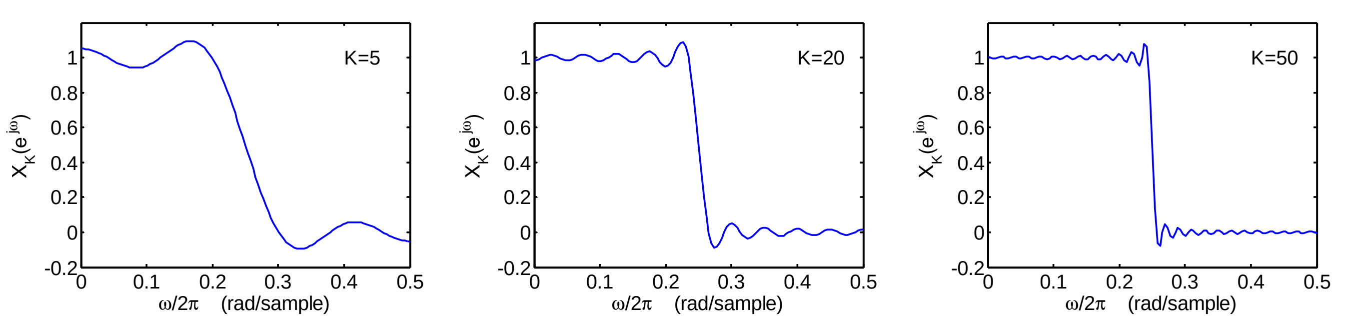

Gibbs phenomenon:

Converges at each ω as $K \to \infty$ but peak error does not get smaller.

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Symmetries

If $x[n]$ has a special property then $X(e^{j\omega})$ and $X[k]$ will have corresponding properties as shown in the table (and vice versa):

One domain	Other domain
Discrete	Periodic
Symmetric	Symmetric
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Conjugate Symmetric: $x[n] = x^*[-n]$

Symmetries

If $x[n]$ has a special property then $X(e^{j\omega})$and $X[k]$ will have corresponding properties as shown in the table (and vice versa):

One domain	Other domain
Discrete	Periodic
Symmetric	Symmetric
Antisymmetric	Antisymmetric
Real	Conjugate Symmetric
Imaginary	Conjugate Antisymmetric
Real + Symmetric	Real + Symmetric
Real + Antisymmetric	Imaginary + Antisymmetric

Symmetric: $x[n] = x[-n]$

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Fourier transforms preserve "energy"

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If we regard $\mathbf{x}$ and $\mathbf{X}$ as vectors, then $\mathbf{X} = \mathbf{F}\mathbf{x}$ where $\mathbf{F}$ is a symmetric matrix defined by $f_{k+1,n+1} = e^{-j2\pi\frac{kn}{N}}$.

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The inverse DFT matrix is $\mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}^H$
equivalently, $\mathbf{G} = \frac{1}{\sqrt{N}}\mathbf{F}$ is a unitary matrix with $\mathbf{G}^H\mathbf{G} = \mathbf{I}$.

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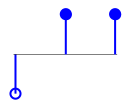
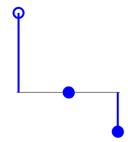
$g[n]$:  $h[n]$: 

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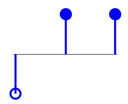
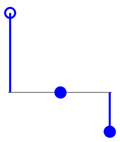
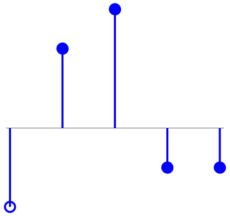
$g[n]$:  $h[n]$:  $g[n] * h[n]$: 

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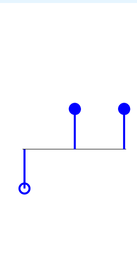
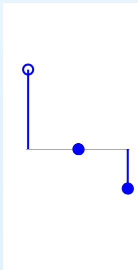
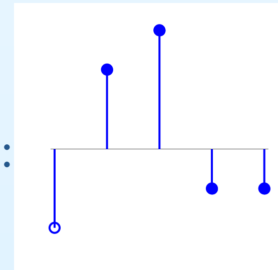
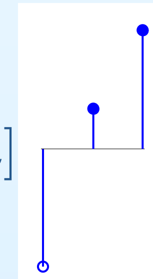
$g[n]:$  $h[n]:$  $g[n] * h[n]:$  $g[n] \circledast_3 h[n]$ 

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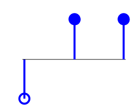
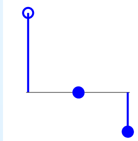
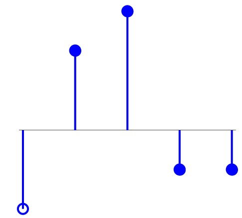
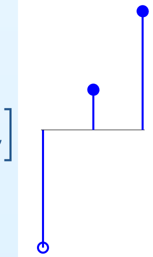
$g[n]$:  $h[n]$:  $g[n] * h[n]$:  $g[n] \circledast_3 h[n]$ 

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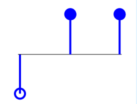
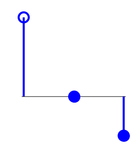
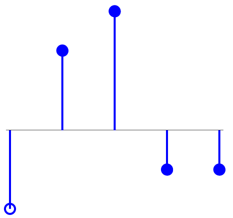
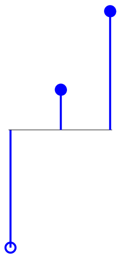
$g[n]:$  $h[n]:$  $g[n] * h[n]:$  $g[n] \circledast_3 h[n]$ 

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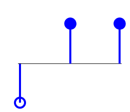
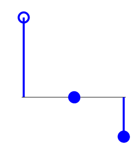
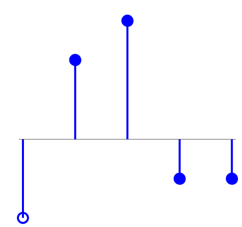
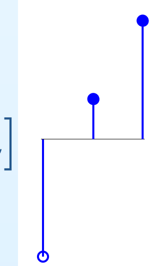
$g[n]$:  $h[n]$:  $g[n] * h[n]$:  $g[n] \circledast_3 h[n]$ 

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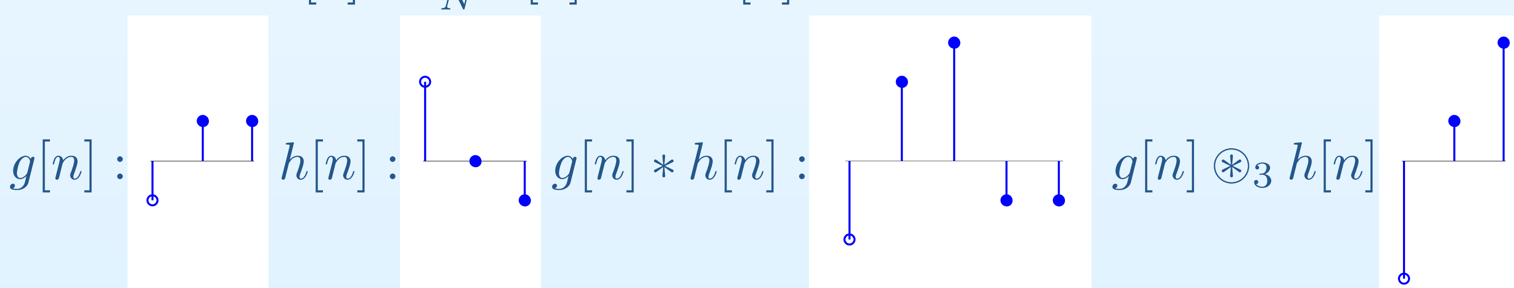


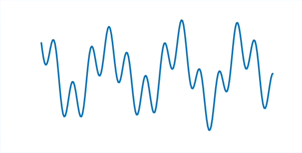
Sampling Process

Time

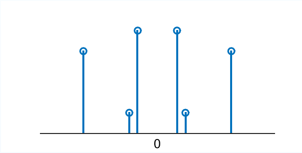
Time

Frequency

Analog



CTFT →



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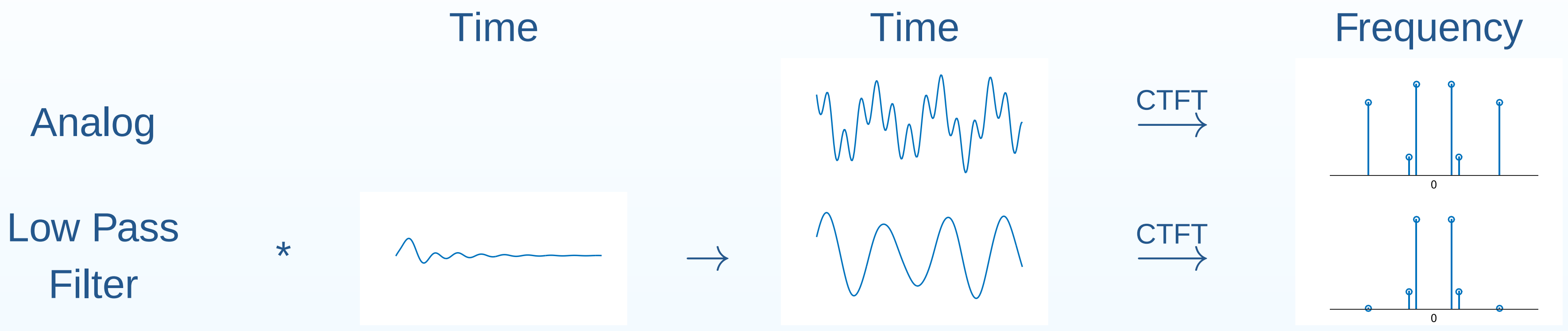

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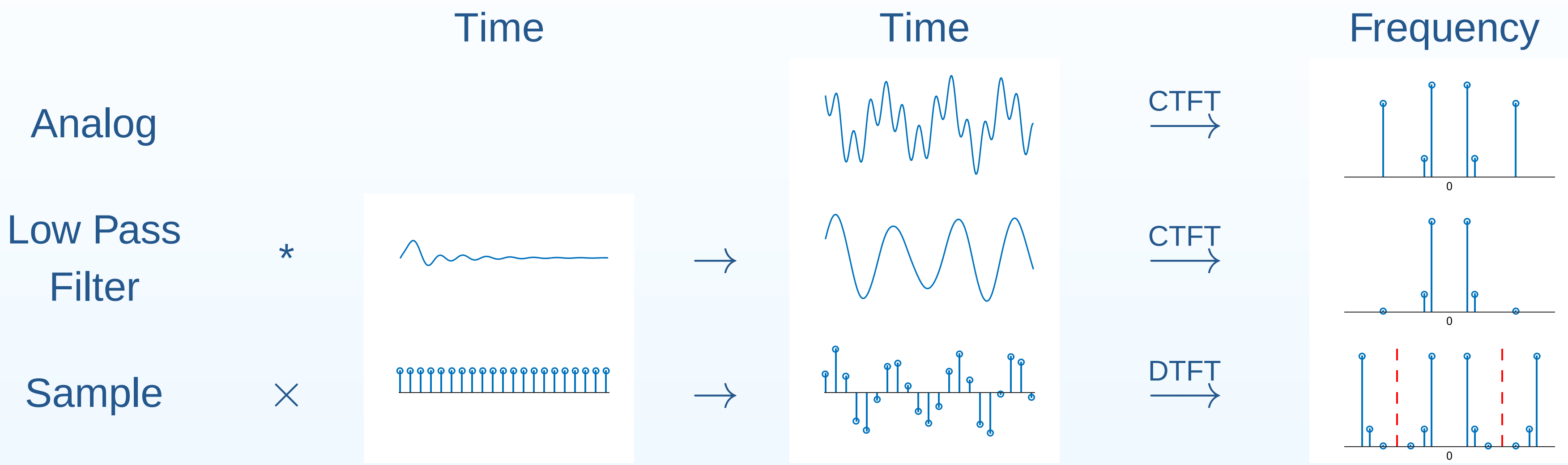

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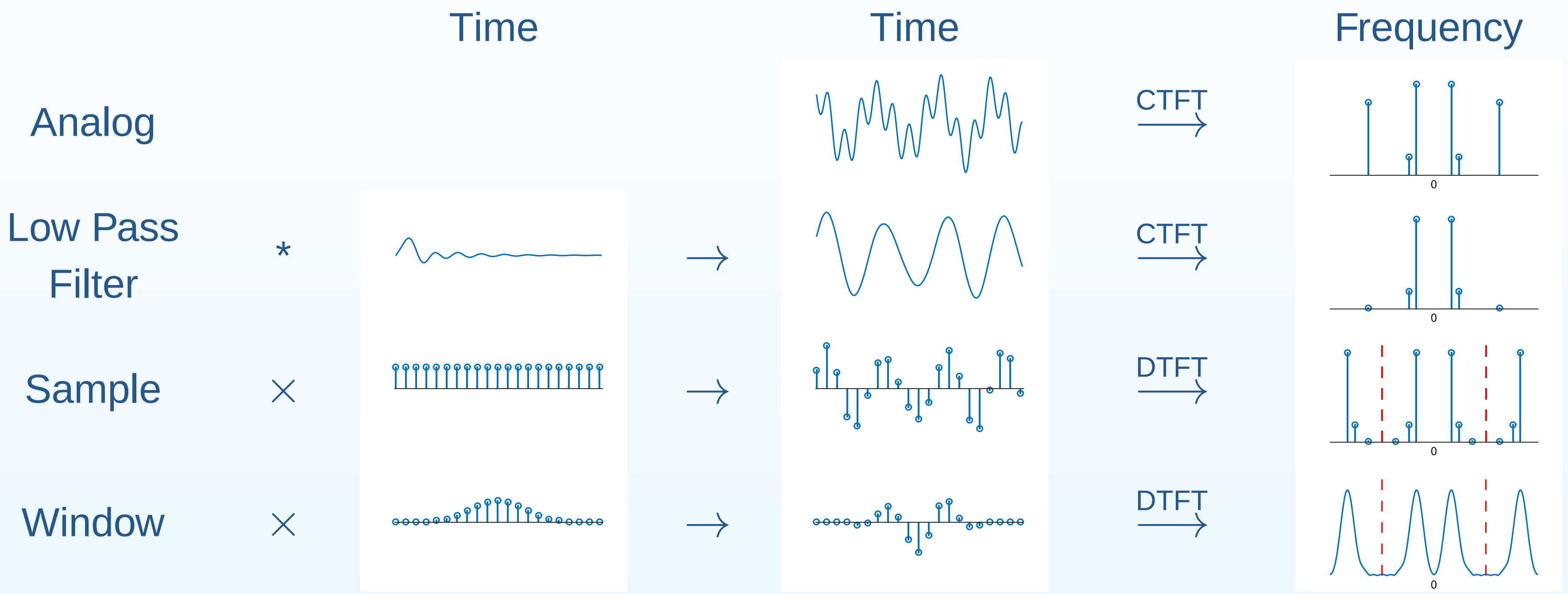

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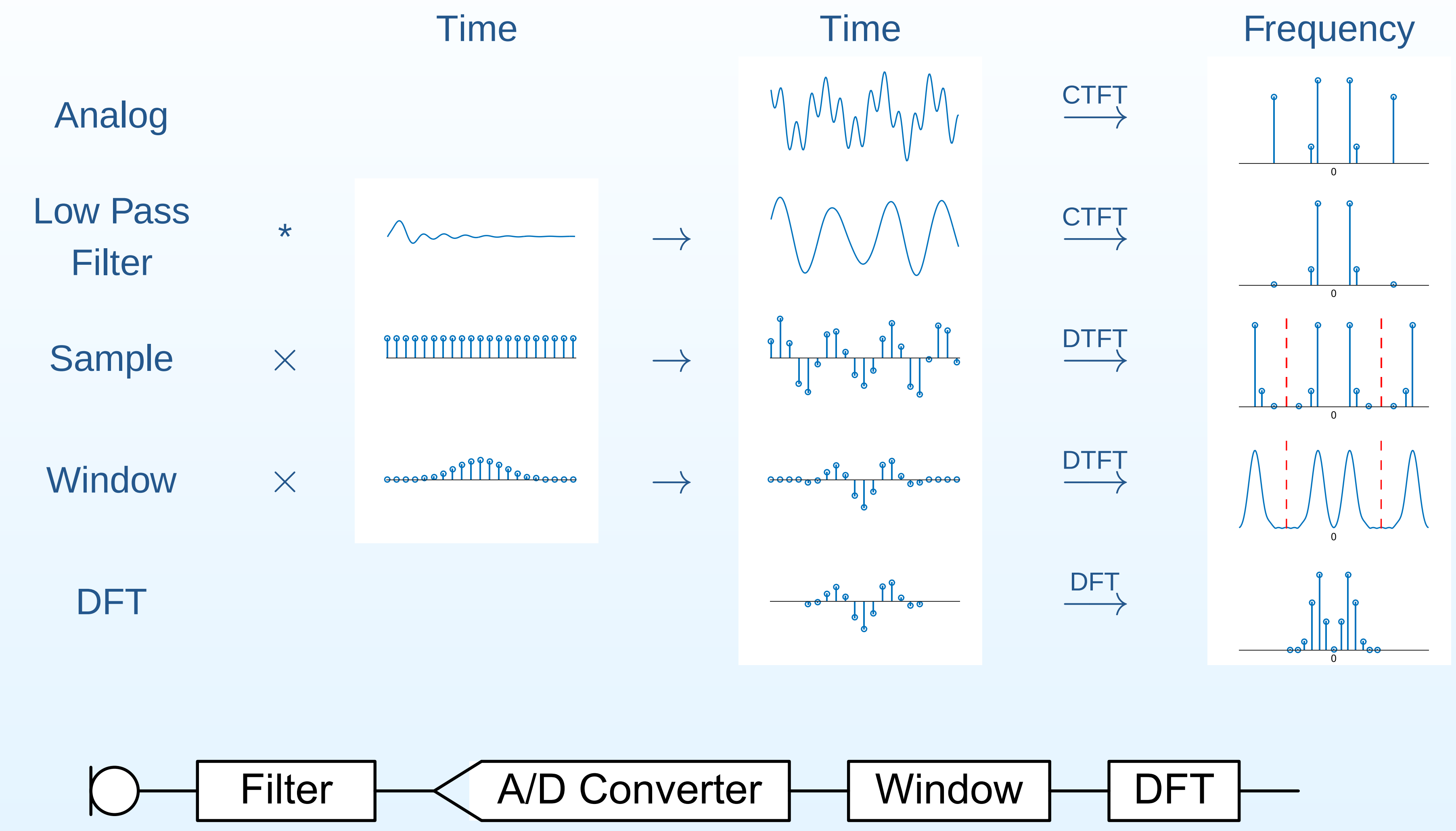
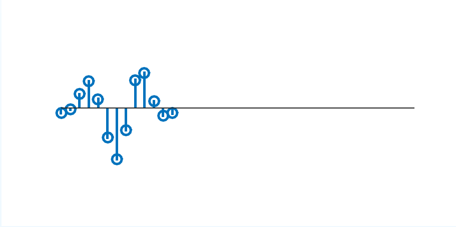


Zero-Padding

Zero padding means added extra zeros onto the end of $x[n]$ before performing the DFT.

Time $x[n]$

Windowed Signal



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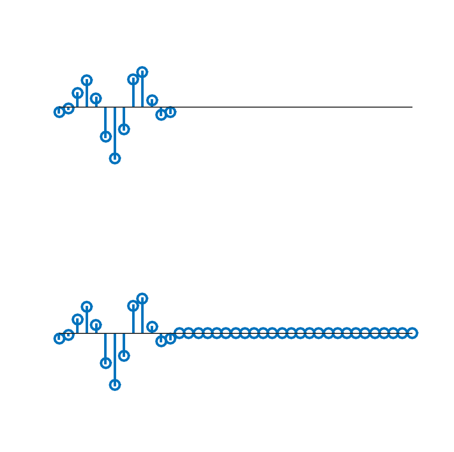
With zero-padding



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Windowed Signal

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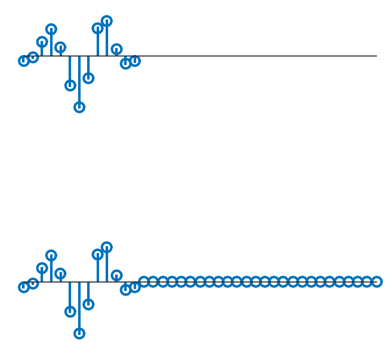

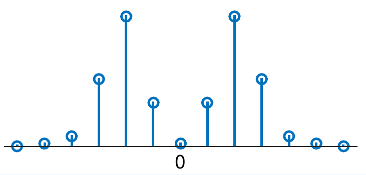

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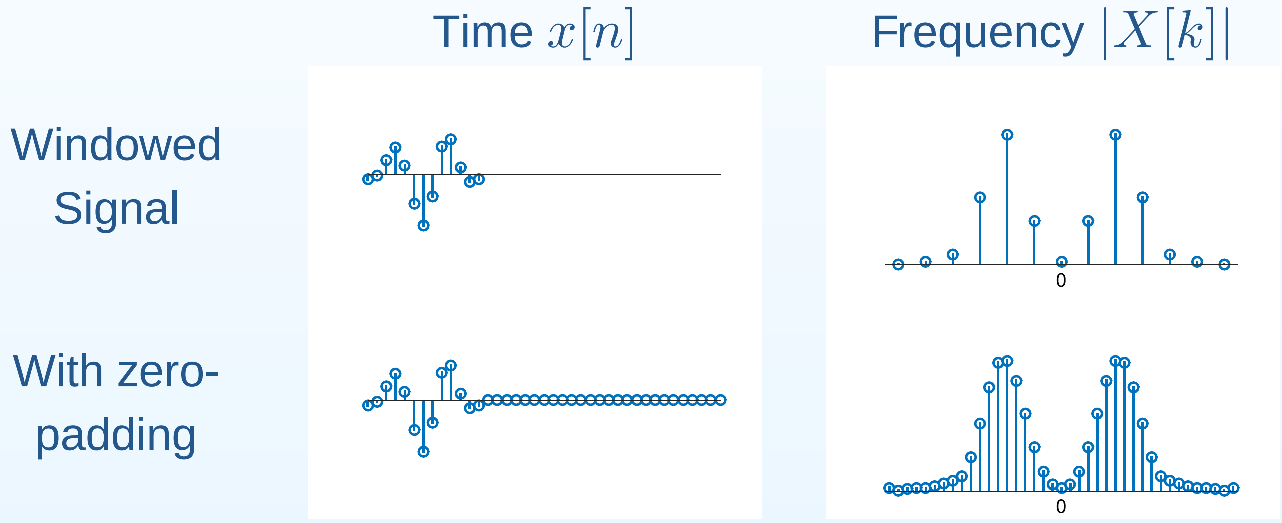

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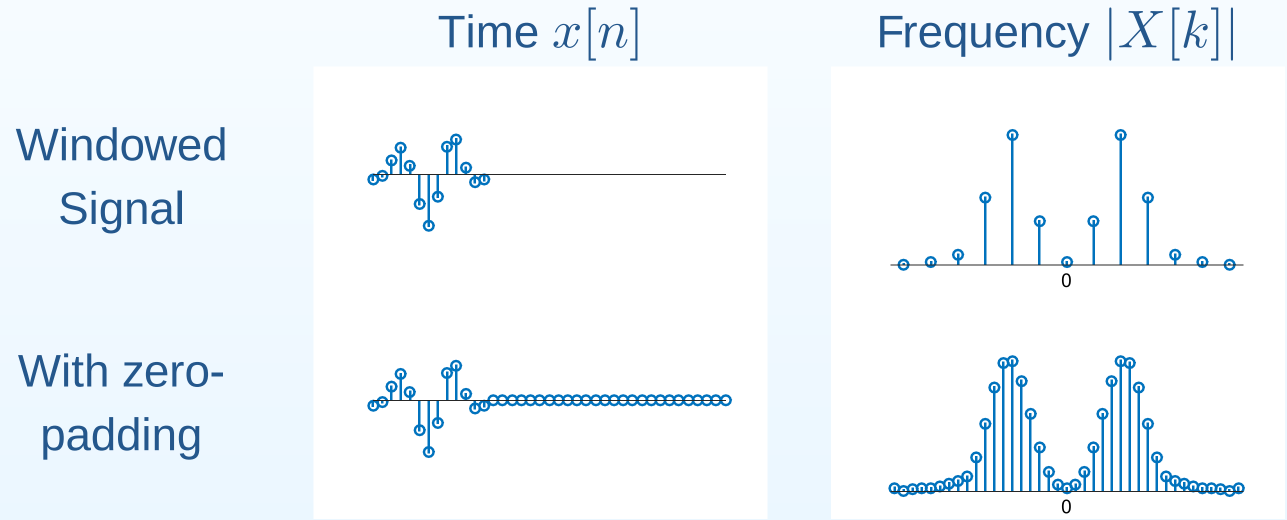

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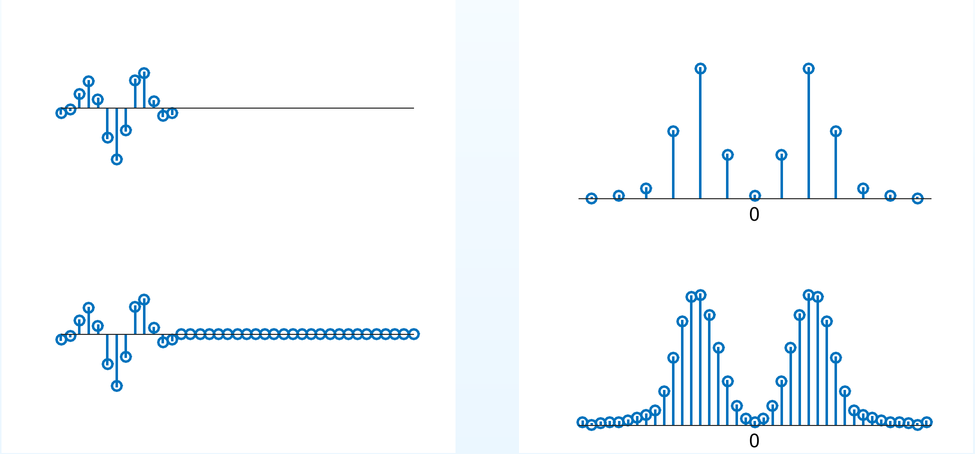

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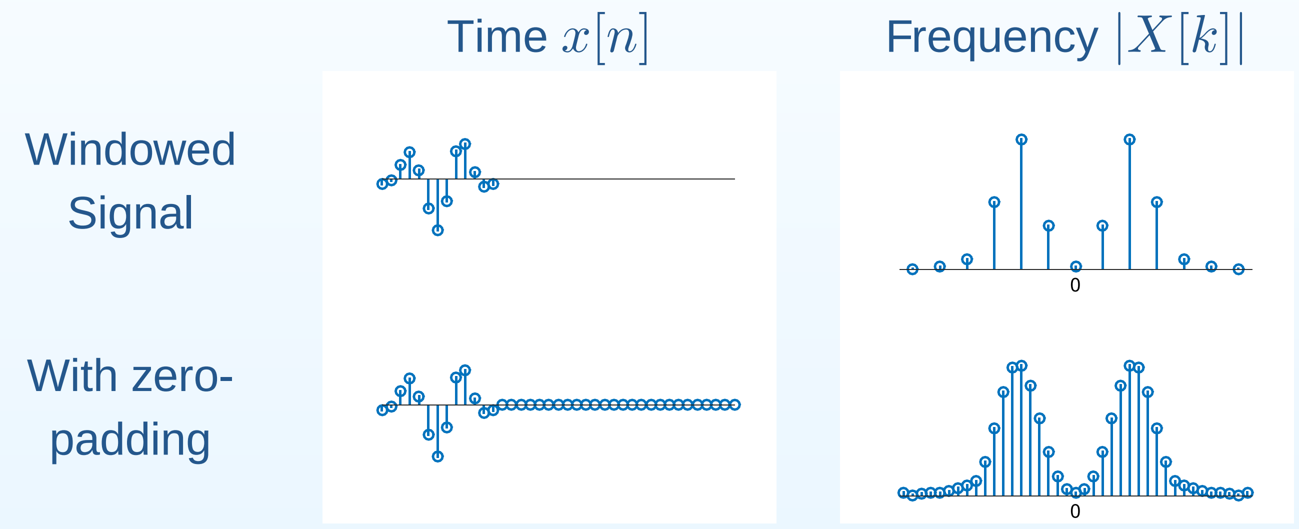

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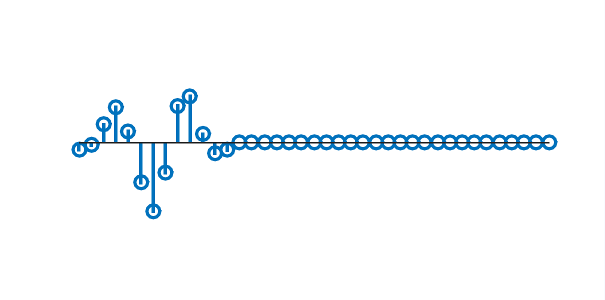
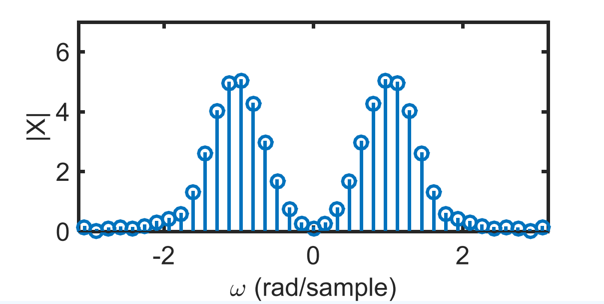
- Smoother graph but increased frequency resolution is an illusion.

Phase Unwrapping

Phase of a DTFT is only defined to within an integer multiple of 2π.



$x[n]$



$|X[k]|$

Phase Unwrapping

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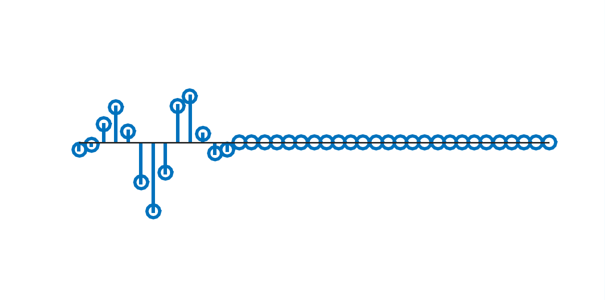
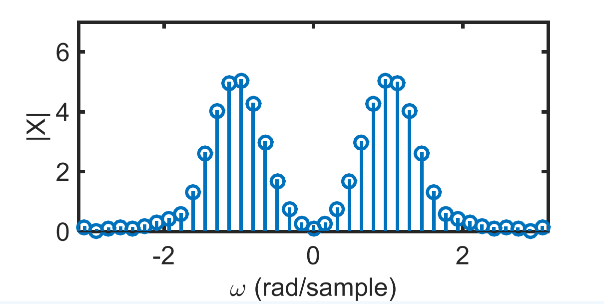


$x[n]$



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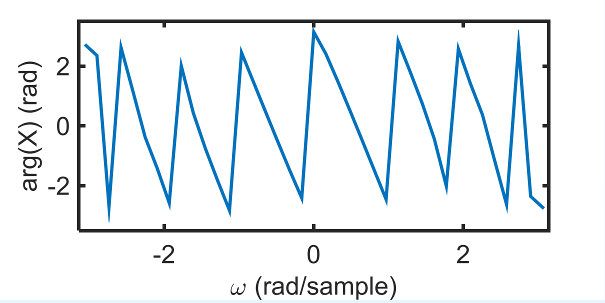

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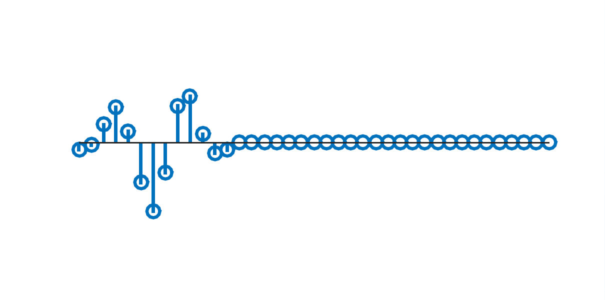

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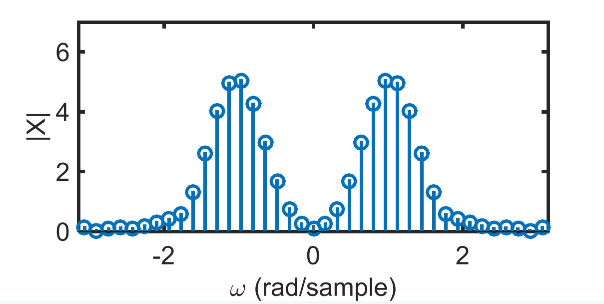

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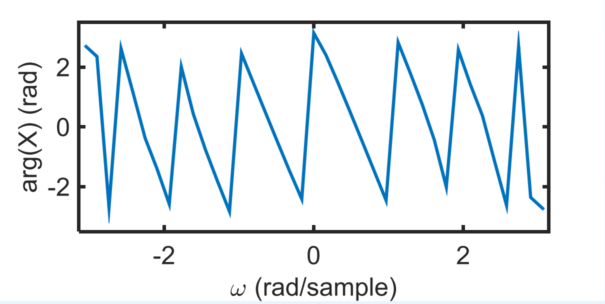

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Phase unwrapping adds multiples of 2π onto each $\angle X[k]$ to make the phase as continuous as possible.

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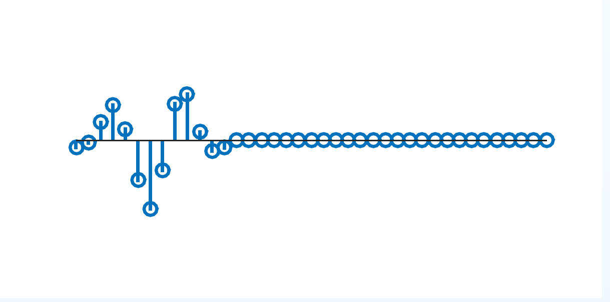

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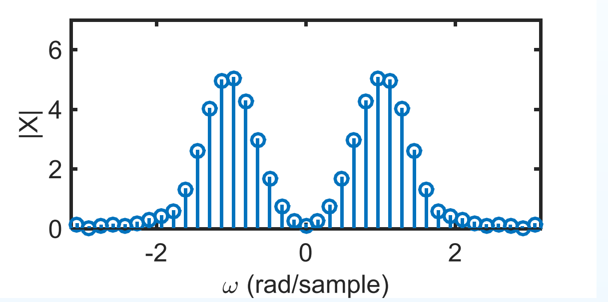

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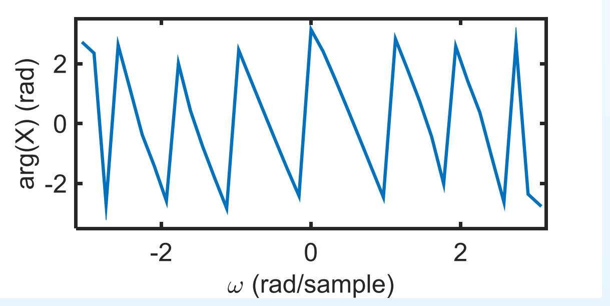

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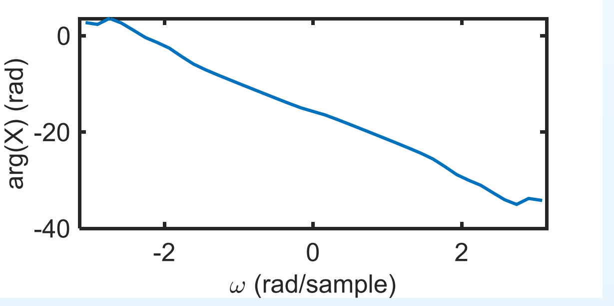

$\angle X[k]$ unwrapped

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No exact equivalent for DTFT/DFT but a similar effect is true

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For further details see Mitra: 3 & 5.

MATLAB routines

fft, ifft	DFT with optional zero-padding
fftshift	swap the two halves of a vector
conv	convolution or polynomial multiplication (not circular)
$x[n] \circledast y[n]$	real(ifft(fft(x).*fft(y)))
unwrap	remove 2π jumps from phase spectrum