# 2: Three Different Fourier Transforms

Three different Fourier Transforms:

- CTFT (Continuous-Time Fourier Transform): $x(t) \to X(j\Omega)$
- DTFT (Discrete-Time Fourier Transform): $x[n] \to X(e^{j\omega})$
- DFT a.k.a. FFT (Discrete Fourier Transform): $x[n] \to X[k]$

| | Forward Transform | Inverse Transform |
|---|---|---|
| CTFT | $X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ | $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$ |
| DTFT | $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ | $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$ |
| DFT | $X[k] = \sum_{0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$ | $x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$ |

We use $\Omega$ for "real" and $\omega = \Omega T$ for "normalized" angular frequency.
Nyquist frequency is at $\Omega_{\text{Nyq}} = 2\pi\frac{f_s}{2} = \frac{\pi}{T}$ and $\omega_{\text{Nyq}} = \pi$.

For "power signals" (energy $\propto$ duration), CTFT & DTFT are unbounded.
Fix this by normalizing:

$$X(j\Omega) = \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} x(t)e^{-j\Omega t}dt$$

$$X(e^{j\omega}) = \lim_{A\to\infty} \frac{1}{2A+1}\sum_{-A}^{A} x[n]e^{-j\omega n}$$

# Convergence of DTFT

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DTFT: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ does not converge for all $x[n]$.

Consider the finite sum: $X_K(e^{j\omega}) = \sum_{-K}^{K} x[n]e^{-j\omega n}$

Strong Convergence:

$x[n]$ absolutely summable $\Rightarrow X(e^{j\omega})$ converges uniformly

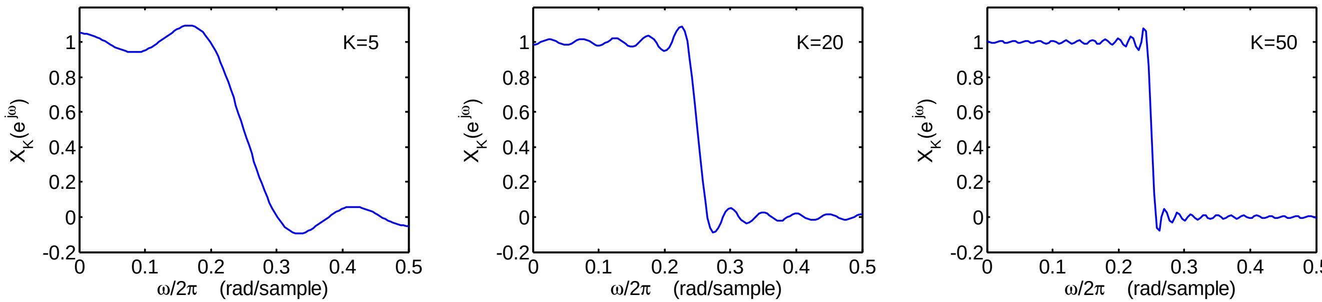
$$\sum_{-\infty}^{\infty} |x[n]| < \infty \Rightarrow \sup_\omega \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right| \xrightarrow[K\to\infty]{} 0$$

Weaker convergence:

$x[n]$ finite energy $\Rightarrow X(e^{j\omega})$ converges in mean square

$$\sum_{-\infty}^{\infty} |x[n]|^2 < \infty \Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right|^2 d\omega \xrightarrow[K\to\infty]{} 0$$

Example: $x[n] = \frac{\sin 0.5\pi n}{\pi n}$




Gibbs phenomenon:

Converges at each $\omega$ as $K \to \infty$ but peak error does not get smaller.

# [DTFT Convergence Proofs]

(1) Strong Convergence: [these proofs are not examinable]

We are given that $\sum_{-\infty}^{\infty} |x[n]| < \infty \Rightarrow \forall\epsilon > 0,\ \exists\, N \text{ such that } \sum_{|n|>N} |x[n]| < \epsilon$

For $K \geq N$, $\sup_\omega \left|X(e^{j\omega}) - X_K(e^{j\omega})\right| = \sup_\omega \left|\sum_{|n|>K} x[n]e^{-j\omega n}\right|$

$$\leq \sup_\omega \left(\sum_{|n|>K} \left|x[n]e^{-j\omega n}\right|\right) = \sum_{|n|>K} |x[n]| < \epsilon$$

(2) Weak Convergence:

We are given that $\sum_{-\infty}^{\infty} |x[n]|^2 < \infty \Rightarrow \forall\epsilon > 0,\ \exists\, N \text{ such that } \sum_{|n|>N} |x[n]|^2 < \epsilon$

Define $y^{[K]}[n] = \begin{cases} 0 & |n| \leq K \\ x[n] & |n| > K \end{cases}$ so that its DTFT is, $Y^{[K]}(e^{j\omega}) = \sum_{-\infty}^{\infty} y^{[K]}[n]e^{-j\omega n}$

We see that $X(e^{j\omega}) - X_K(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n} - \sum_{-K}^{K} x[n]e^{-j\omega n}$

$$= \sum_{|n|>K} x[n]e^{-j\omega n} = \sum_{-\infty}^{\infty} y^{[K]}[n]e^{-j\omega n} = Y^{[K]}(e^{j\omega})$$

From Parseval's theorem, $\sum_{-\infty}^{\infty} \left|y^{[K]}[n]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Y^{[K]}(e^{j\omega})\right|^2 d\omega$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega}) - X_K(e^{j\omega})\right|^2 d\omega$$

Hence for $K \geq N$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega}) - X_K(e^{j\omega})\right|^2 d\omega = \sum_{-\infty}^{\infty} \left|y^{[K]}[n]\right|^2 = \sum_{|n|>N} |x[n]|^2 < \epsilon$

DTFT: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$

- DTFT is periodic in $\omega$: $X(e^{j(\omega+2m\pi)}) = X(e^{j\omega})$ for integer $m$.
- DTFT is the $z$-Transform evaluated at the point $e^{j\omega}$:
  $$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$$
  DTFT converges iff the ROC includes $|z| = 1$.
- DTFT is the same as the CTFT of a signal comprising impulses at the sample times (Dirac $\delta$ functions) of appropriate heights:
  $$x_\delta(t) = \sum x[n]\delta(t-nT) = x(t) \times \sum_{-\infty}^{\infty} \delta(t-nT)$$
  Equivalent to multiplying a continuous $x(t)$ by an impulse train.

Proof: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$

$$\sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t-nT)e^{-j\omega \frac{t}{T}}\,dt$$

$$\overset{(i)}{=} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)e^{-j\omega \frac{t}{T}}\,dt$$

$$\overset{(ii)}{=} \int_{-\infty}^{\infty} x_\delta(t)e^{-j\Omega t}dt$$

(i) OK if $\sum_{-\infty}^{\infty} |x[n]| < \infty$. (ii) use $\omega = \Omega T$.

# DFT Properties



DFT: $X[k] = \sum_0^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$

DTFT: $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$

Case 1: $x[n] = 0$ for $n \notin [0, N-1]$

DFT is the same as DTFT at $\omega_k = \frac{2\pi}{N}k$.

The $\{\omega_k\}$ are uniformly spaced from $\omega = 0$ to $\omega = 2\pi\frac{N-1}{N}$.
DFT is the $z$-Transform evaluated at $N$ equally spaced points around the unit circle beginning at $z = 1$.

Case 2: $x[n]$ is periodic with period $N$

DFT equals the normalized DTFT

$$X[k] = \lim_{K\to\infty} \frac{N}{2K+1} \times X_K(e^{j\omega_k})$$

where $X_K(e^{j\omega}) = \sum_{-K}^{K} x[n]e^{-j\omega n}$

# [Proof of Case 2]

We want to show that if $x[n] = x[n+N]$ (i.e. $x[n]$ is periodic with period $N$) then

$$\lim_{K\to\infty} \frac{N}{2K+1} \times X_K(e^{j\omega_k}) \triangleq \lim_{K\to\infty} \frac{N}{2K+1} \times \sum_{-K}^{K} x[n]e^{-j\omega_k n} = X[k]$$

where $\omega_k = \frac{2\pi}{N}k$. We assume that $x[n]$ is bounded with $|x[n]| < B$.

We first note that the summand is periodic:

$$x[n+N]e^{-j\omega_k(n+N)} = x[n]e^{-j\omega_k n}e^{-jk\frac{2\pi}{N}N} = x[n]e^{-j\omega_k n}e^{-j2\pi k} = x[n]e^{-j\omega_k n}.$$

We now define $M$ and $R$ so that $2K+1 = MN + R$ where $0 \le R < N$ (i.e. $MN$ is the largest multiple of $N$ that is $\le 2K+1$). We can now write

$$\frac{N}{2K+1} \times \sum_{-K}^{K} x[n]e^{-j\omega_k n} = \frac{N}{MN+R} \times \sum_{-K}^{K-R} x[n]e^{-j\omega_k n} + \frac{N}{MN+R} \times \sum_{K-R+1}^{K} x[n]e^{-j\omega n}$$

The first sum contains $MN$ consecutive terms of a periodic summand and so equals $M$ times the sum over one period. The second sum contains $R$ bounded terms and so its magnitude is $< RB < NB$.

So $\frac{N}{2K+1} \times \sum_{-K}^{K} x[n]e^{-j\omega_k n} = \frac{MN}{MN+R} \times \sum_{0}^{N-1} x[n]e^{-j\omega_k n} + P = \frac{1}{1+\frac{R}{MN}} \times X[k] + P$

where $|P| < \frac{N}{MN+R} \times NB \le \frac{N}{MN+0} \times NB = \frac{NB}{M}$.

As $M \to \infty$, $|P| \to 0$ and $\frac{1}{1+\frac{R}{MN}} \to 1$ so the whole expression tends to $X[k]$.

# Symmetries



If $x[n]$ has a special property then $X(e^{j\omega})$and $X[k]$ will have corresponding properties as shown in the table (and vice versa):

| One domain | Other domain |
|---|---|
| Discrete | Periodic |
| Symmetric | Symmetric |
| Antisymmetric | Antisymmetric |
| Real | Conjugate Symmetric |
| Imaginary | Conjugate Antisymmetric |
| Real + Symmetric | Real + Symmetric |
| Real + Antisymmetric | Imaginary + Antisymmetric |

Symmetric: $x[n] = x[-n]$

$X(e^{j\omega}) = X(e^{-j\omega})$

$X[k] = X[(-k)_{\text{mod } N}] = X[N-k]$ for $k > 0$

Conjugate Symmetric: $x[n] = x^*[-n]$

Conjugate Antisymmetric: $x[n] = -x^*[-n]$

# Parseval's Theorem



Fourier transforms preserve "energy"

CTFT $\int |x(t)|^2 \, dt = \frac{1}{2\pi} \int |X(j\Omega)|^2 \, d\Omega$

DTFT $\sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$

DFT $\sum_0^{N-1} |x[n]|^2 = \frac{1}{N} \sum_0^{N-1} |X[k]|^2$

More generally, they actually preserve complex inner products:

$$\sum_0^{N-1} x[n]y^*[n] = \frac{1}{N} \sum_0^{N-1} X[k]Y^*[k]$$

Unitary matrix viewpoint for DFT:

If we regard $\mathbf{x}$ and $\mathbf{X}$ as vectors, then $\mathbf{X} = \mathbf{F}\mathbf{x}$ where $\mathbf{F}$ is a symmetric matrix defined by $f_{k+1,n+1} = e^{-j2\pi\frac{kn}{N}}$.

The inverse DFT matrix is $\mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}^H$
equivalently, $\mathbf{G} = \frac{1}{\sqrt{N}}\mathbf{F}$ is a unitary matrix with $\mathbf{G}^H\mathbf{G} = \mathbf{I}$.

# Convolution



DTFT: Convolution → Product

$$x[n] = g[n] * h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k]$$
$$\Rightarrow X(e^{j\omega}) = G(e^{j\omega})H(e^{j\omega})$$

DFT: Circular convolution→ Product

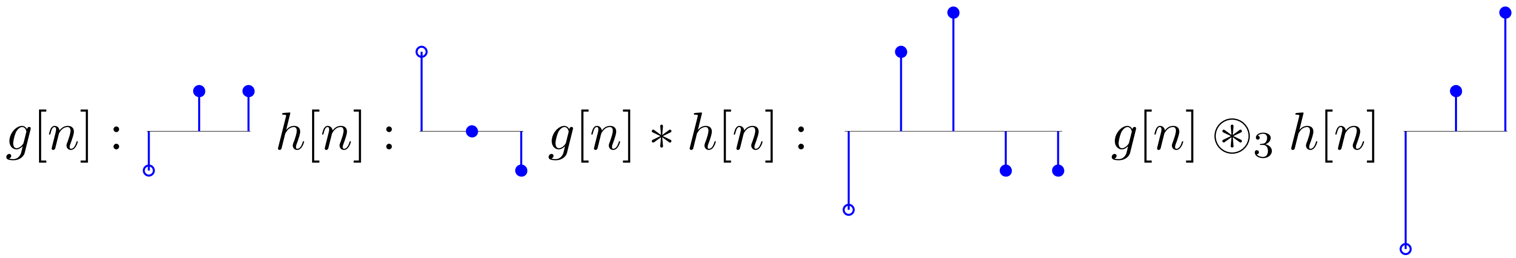
$$x[n] = g[n] \circledast_N h[n] = \sum_{k=0}^{N-1} g[k]h[(n-k)_{\text{mod} N}]$$
$$\Rightarrow X[k] = G[k]H[k]$$

DTFT: Product→ Circular Convolution $\div 2\pi$

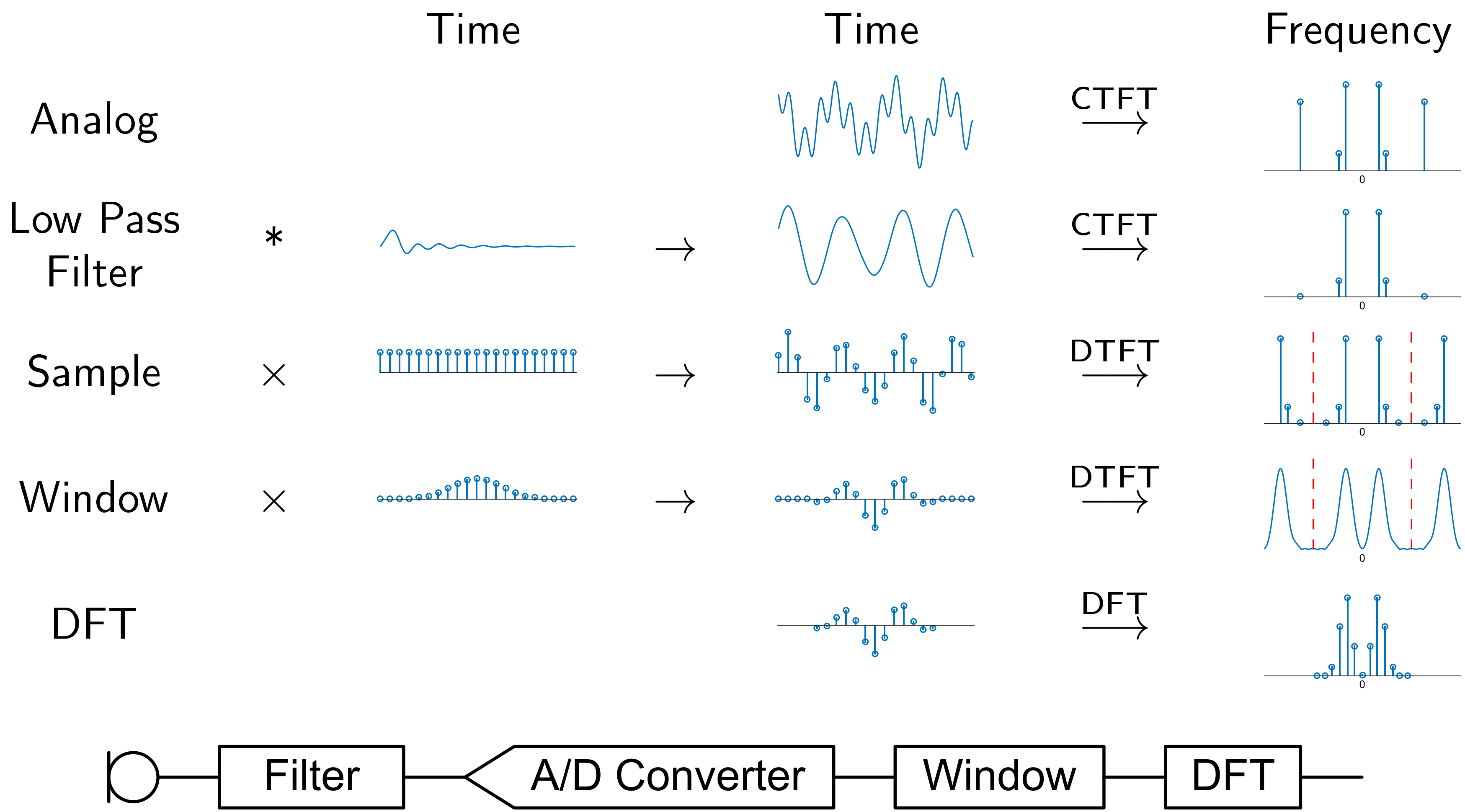
$$y[n] = g[n]h[n]$$
$$\Rightarrow Y(e^{j\omega}) = \frac{1}{2\pi}G(e^{j\omega}) \circledast_\pi H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$$

DFT: Product→ Circular Convolution $\div N$

$$y[n] = g[n]h[n]$$
$$\Rightarrow Y[k] = \frac{1}{N}G[k] \circledast_N H[k]$$



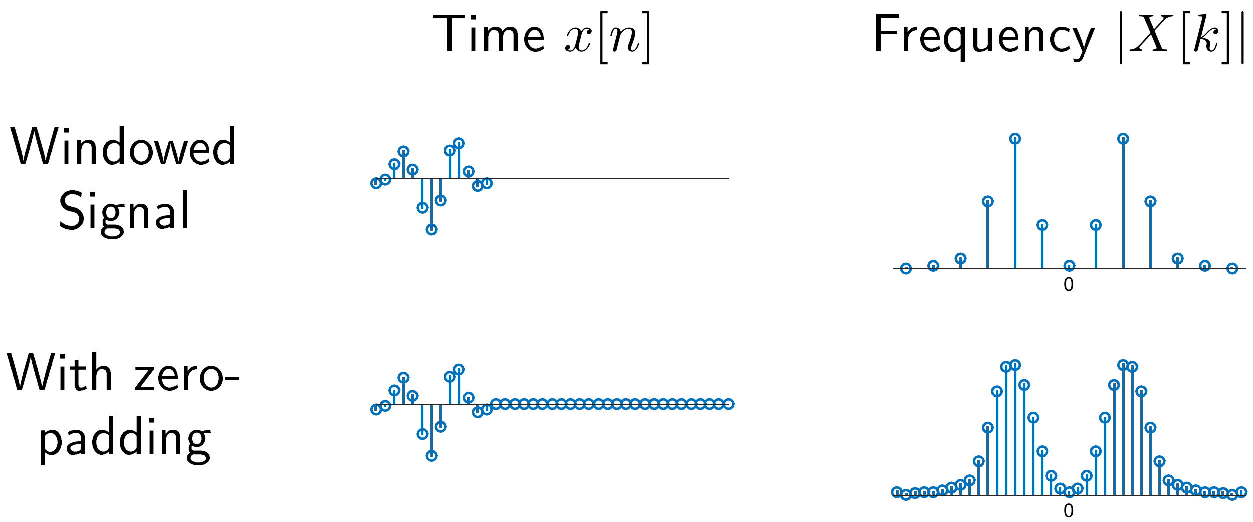
# Sampling Process



# Zero-Padding



Zero padding means added extra zeros onto the end of $x[n]$ before performing the DFT.

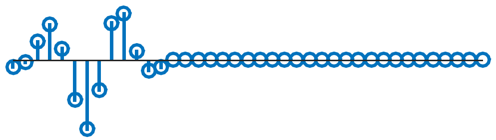



- Zero-padding causes the DFT to evaluate the DTFT at more values of $\omega_k$. Denser frequency samples.
- Width of the peaks remains constant: determined by the length and shape of the window.
- Smoother graph but increased frequency resolution is an illusion.

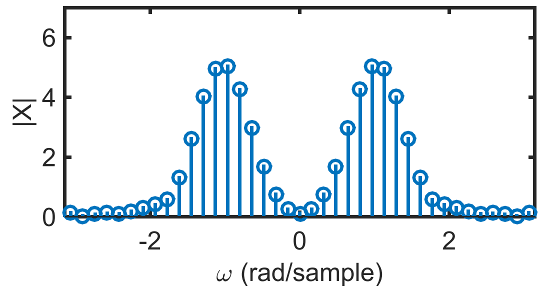
# Phase Unwrapping



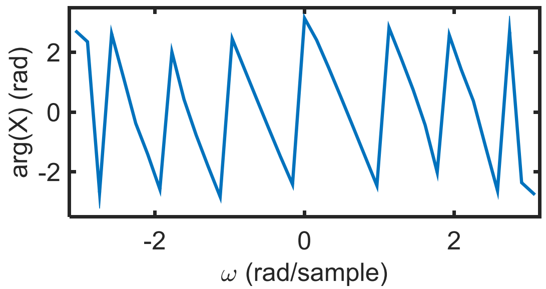
Phase of a DTFT is only defined to within an integer multiple of $2\pi$.



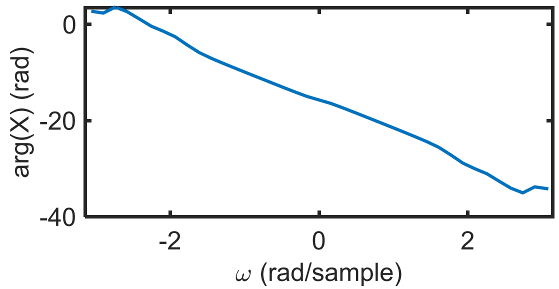
$x[n]$




$|X[k]|$




$\angle X[k]$




$\angle X[k]$ unwrapped

Phase unwrapping adds multiples of $2\pi$ onto each $\angle X[k]$ to make the phase as continuous as possible.

CTFT uncertainty principle: $\left(\frac{\int t^2|x(t)|^2 dt}{\int |x(t)|^2 dt}\right)^{\frac{1}{2}} \left(\frac{\int \omega^2|X(j\omega)|^2 d\omega}{\int |X(j\omega)|^2 d\omega}\right)^{\frac{1}{2}} \geq \frac{1}{2}$

The first term measures the "width" of $x(t)$ around $t = 0$.

It is like $\sigma$ if $|x(t)|^2$ was a zero-mean probability distribution.

The second term is similarly the "width" of $X(j\omega)$ in frequency.

A signal cannot be concentrated in both time and frequency.

Proof Outline:

Assume $\int |x(t)|^2 dt = 1 \Rightarrow \int |X(j\omega)|^2 d\omega = 2\pi$ [Parseval]

Set $v(t) = \frac{dx}{dt} \Rightarrow V(j\omega) = j\omega X(j\omega)$ [by parts]

Now $\int tx\frac{dx}{dt}dt = \frac{1}{2}tx^2(t)\Big|_{t=-\infty}^{\infty} - \int \frac{1}{2}x^2 dt = 0 - \frac{1}{2}$ [by parts]

So $\frac{1}{4} = \left|\int tx\frac{dx}{dt}dt\right|^2 \leq \left(\int t^2x^2 dt\right)\left(\int \left|\frac{dx}{dt}\right|^2 dt\right)$ [Schwartz]

$= \left(\int t^2x^2 dt\right)\left(\int |v(t)|^2 dt\right) = \left(\int t^2x^2 dt\right)\left(\frac{1}{2\pi}\int |V(j\omega)|^2 d\omega\right)$

$= \left(\int t^2x^2 dt\right)\left(\frac{1}{2\pi}\int \omega^2 |X(j\omega)|^2 d\omega\right)$

No exact equivalent for DTFT/DFT but a similar effect is true

# [Uncertainty Principle Proof Steps]

(1) Suppose $v(t) = \frac{dx}{dt}$. Then integrating the CTFT definition by parts w.r.t. $t$ gives
$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt = \left[\frac{-1}{j\Omega}x(t)e^{-j\Omega t}\right]_{-\infty}^{\infty} + \frac{1}{j\Omega}\int_{-\infty}^{\infty} \frac{dx(t)}{dt}e^{-j\Omega t}dt = 0 + \frac{1}{j\Omega}V(j\Omega)$

(2) Since $\frac{d}{dt}\left(\frac{1}{2}x^2\right) = x\frac{dx}{dt}$, we can apply integration by parts to get
$\int_{-\infty}^{\infty} tx\frac{dx}{dt}dt = \left[t \times \frac{1}{2}x^2\right]_{t=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dt}{dt} \times \frac{1}{2}x^2 dt = -\frac{1}{2}\int_{-\infty}^{\infty} x^2 dt = -\frac{1}{2} \times 1 = -\frac{1}{2}$
It follows that $\left|\int_{-\infty}^{\infty} tx\frac{dx}{dt}dt\right|^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ which we will use below.

(3) The Cauchy-Schwarz inequality is that in a complex inner product space $|\mathbf{u}\cdot\mathbf{v}|^2 \le (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v})$. For the inner-product space of real-valued square-integrable functions, this becomes $\left|\int_{-\infty}^{\infty} u(t)v(t)dt\right|^2 \le \int_{-\infty}^{\infty} u^2(t)dt \times \int_{-\infty}^{\infty} v^2(t)dt$. We apply this with $u(t) = tx(t)$ and $v(t) = \frac{dx(t)}{dt}$ to get

$$\frac{1}{4} = \left|\int_{-\infty}^{\infty} tx\frac{dx}{dt}dt\right|^2 \le \left(\int t^2x^2dt\right)\left(\int \left(\frac{dx}{dt}\right)^2 dt\right) = \left(\int t^2x^2dt\right)\left(\int v^2(t)dt\right)$$

(4) From Parseval's theorem for the CTFT, $\int v^2(t)dt = \frac{1}{2\pi}\int |V(j\Omega|^2 d\Omega$. From step (1), we can substitute $V(j\Omega) = j\Omega X(j\Omega)$ to obtain $\int v^2(t)dt = \frac{1}{2\pi}\int \Omega^2 |X(j\Omega|^2 d\Omega$. Making this substitution in (3) gives

$$\frac{1}{4} \le \left(\int t^2x^2dt\right)\left(\int v^2(t)dt\right) = \left(\int t^2x^2dt\right)\left(\frac{1}{2\pi}\int \omega^2 |X(j\Omega|^2 d\Omega\right)$$

# Summary



- Three types: CTFT, DTFT, DFT
  - DTFT = CTFT of continuous signal × impulse train
  - DFT = DTFT of periodic or finite support signal
    - ▷ DFT is a scaled unitary transform
- DTFT: Convolution → Product; Product → Circular Convolution
- DFT: Product ↔ Circular Convolution
- DFT: Zero Padding → Denser freq sampling but same resolution
- Phase is only defined to within a multiple of $2\pi$.
- Whenever you integrate over frequency you need a scale factor
  - $\frac{1}{2\pi}$ for CTFT and DTFT or $\frac{1}{N}$ for DFT
  - e.g. Inverse transform, Parseval, frequency domain convolution

For further details see Mitra: 3 & 5.

# MATLAB routines



| | |
|---|---|
| fft, ifft | DFT with optional zero-padding |
| fftshift | swap the two halves of a vector |
| conv | convolution or polynomial multiplication (not circular) |
| $x[n] \circledast y[n]$ | real(ifft(fft(x).*fft(y))) |
| unwrap | remove $2\pi$ jumps from phase spectrum |