3: Discrete Cosine Transform

DFT Problems

For processing 1-D or 2-D signals (especially coding), a common method is to divide the signal into "frames" and then apply an invertible transform to each frame that compresses the information into few coefficients.

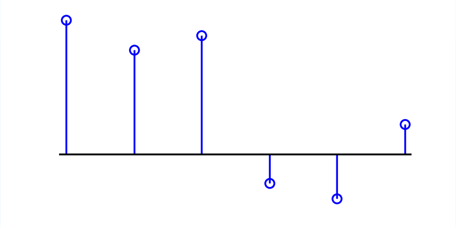
The DFT has some problems when used for this purpose:

DFT Problems

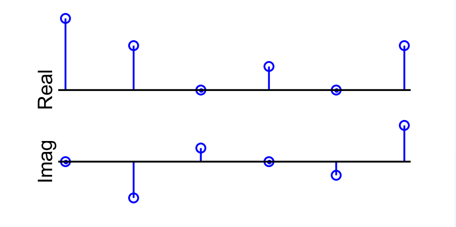
For processing 1-D or 2-D signals (especially coding), a common method is to divide the signal into "frames" and then apply an invertible transform to each frame that compresses the information into few coefficients.

The DFT has some problems when used for this purpose:

- N real $x[n] \leftrightarrow N$ complex $X[k]$: 2 real, $\frac{N}{2} - 1$ conjugate pairs



$\rightarrow$

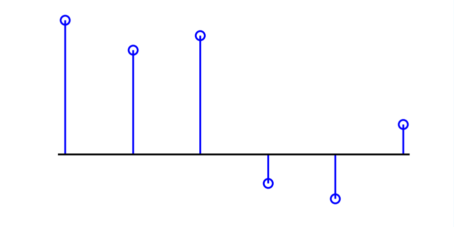


DFT Problems

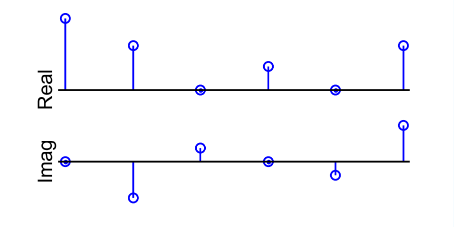
For processing 1-D or 2-D signals (especially coding), a common method is to divide the signal into "frames" and then apply an invertible transform to each frame that compresses the information into few coefficients.

The DFT has some problems when used for this purpose:

- N real $x[n] \leftrightarrow N$ complex $X[k]$: 2 real, $\frac{N}{2} - 1$ conjugate pairs



$\rightarrow$



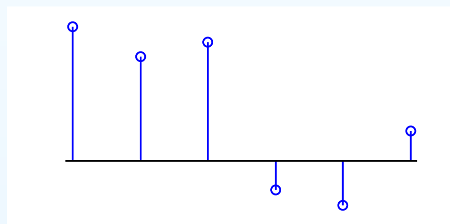
- DFT $\propto$ the DTFT of a periodic signal formed by replicating $x[n]$.

DFT Problems

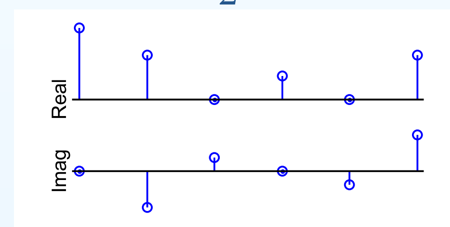
For processing 1-D or 2-D signals (especially coding), a common method is to divide the signal into “frames” and then apply an invertible transform to each frame that compresses the information into few coefficients.

The DFT has some problems when used for this purpose:

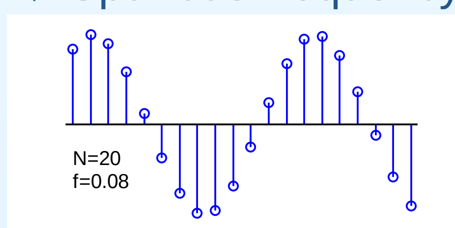
- N real $x[n] \leftrightarrow N$ complex $X[k]$: 2 real, $\frac{N}{2} - 1$ conjugate pairs



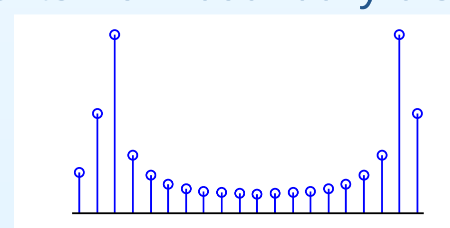
$\rightarrow$



- DFT $\propto$ the DTFT of a periodic signal formed by replicating $x[n]$.
  $\Rightarrow$ Spurious frequency components from boundary discontinuity.



$\rightarrow$

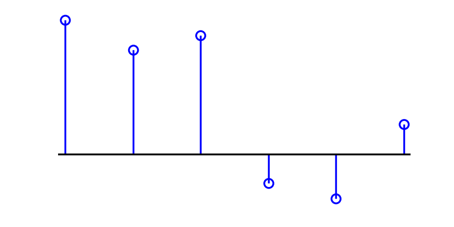


DFT Problems

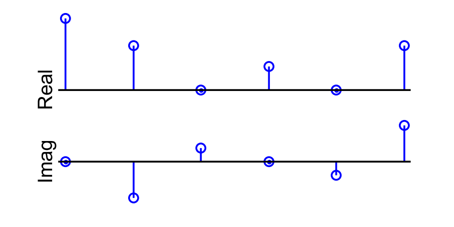
For processing 1-D or 2-D signals (especially coding), a common method is to divide the signal into "frames" and then apply an invertible transform to each frame that compresses the information into few coefficients.

The DFT has some problems when used for this purpose:

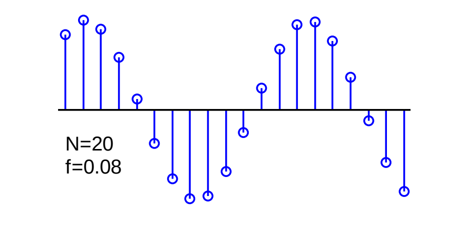
- N real $x[n] \leftrightarrow N$ complex $X[k]$: 2 real, $\frac{N}{2} - 1$ conjugate pairs



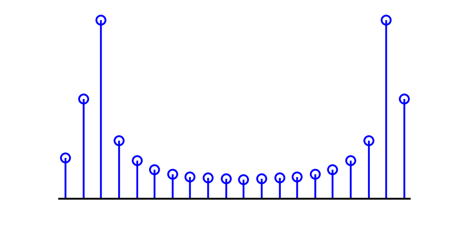
$\rightarrow$



- DFT $\propto$ the DTFT of a periodic signal formed by replicating $x[n]$.
  $\Rightarrow$ Spurious frequency components from boundary discontinuity.



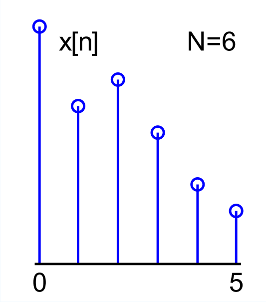
$\rightarrow$



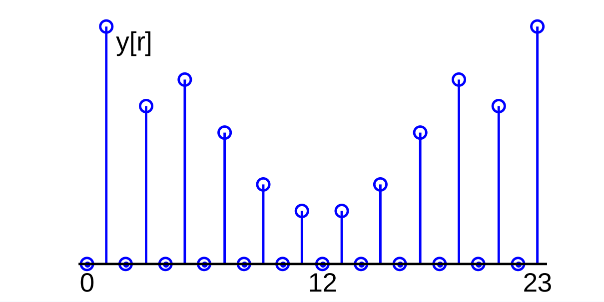
The Discrete Cosine Transform (DCT) overcomes these problems.

DCT

To form the Discrete Cosine Transform (DCT), replicate $x[0:N-1]$ but in reverse order and insert a zero between each pair of samples:



$\rightarrow$

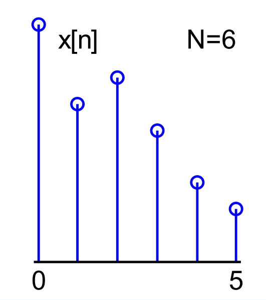


Take the DFT of length $4N$ real, symmetric, odd-sample-only sequence.

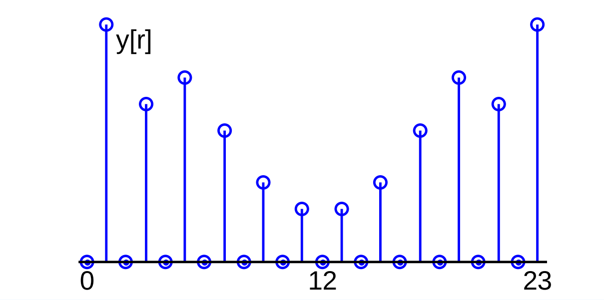
DCT

+

To form the Discrete Cosine Transform (DCT), replicate $x[0 : N-1]$ but in reverse order and insert a zero between each pair of samples:

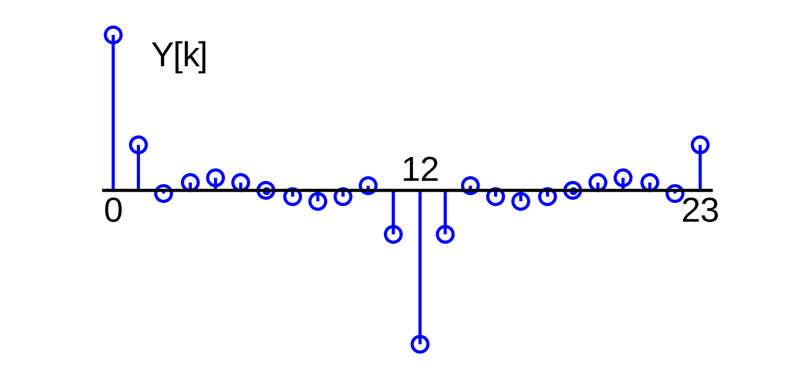


$\rightarrow$



Take the DFT of length $4N$ real, symmetric, odd-sample-only sequence.

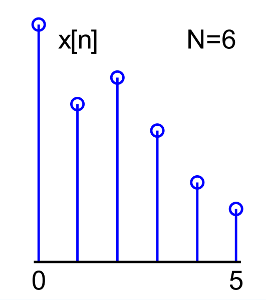
Result is real, symmetric and anti-periodic:



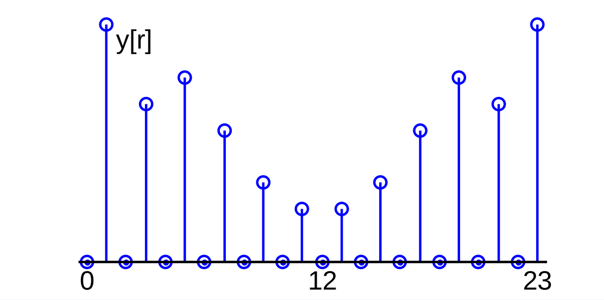
DCT

+

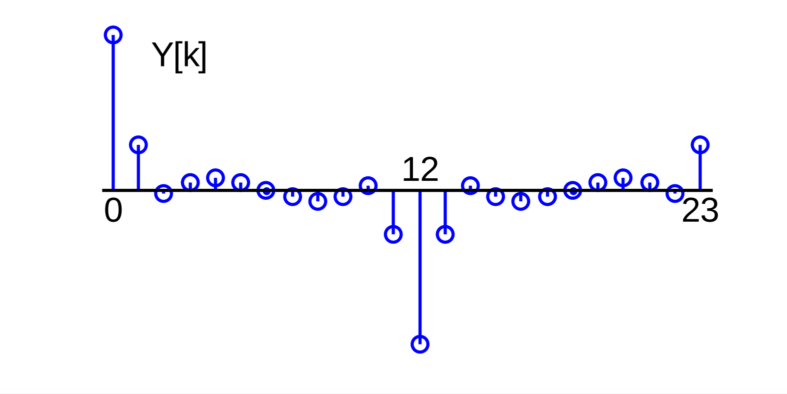
To form the Discrete Cosine Transform (DCT), replicate $x[0 : N-1]$ but in reverse order and insert a zero between each pair of samples:



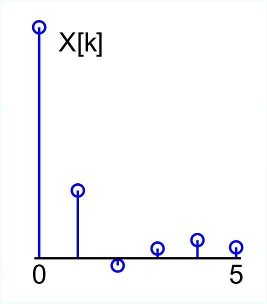
$\rightarrow$



Take the DFT of length $4N$ real, symmetric, odd-sample-only sequence.
Result is real, symmetric and anti-periodic: only need first N values

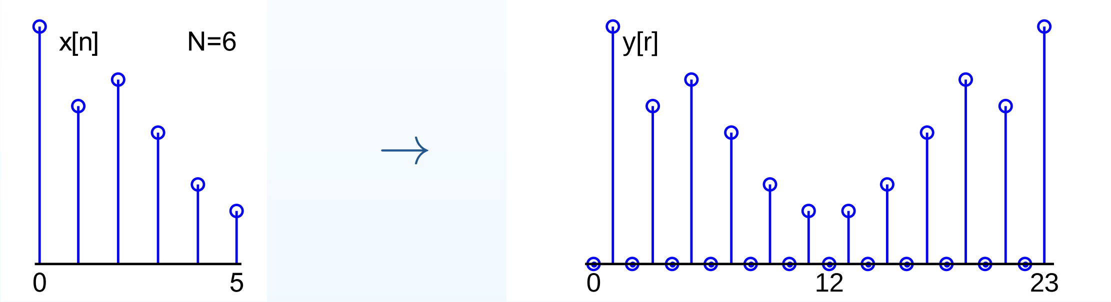


$\xrightarrow{\div 2}$

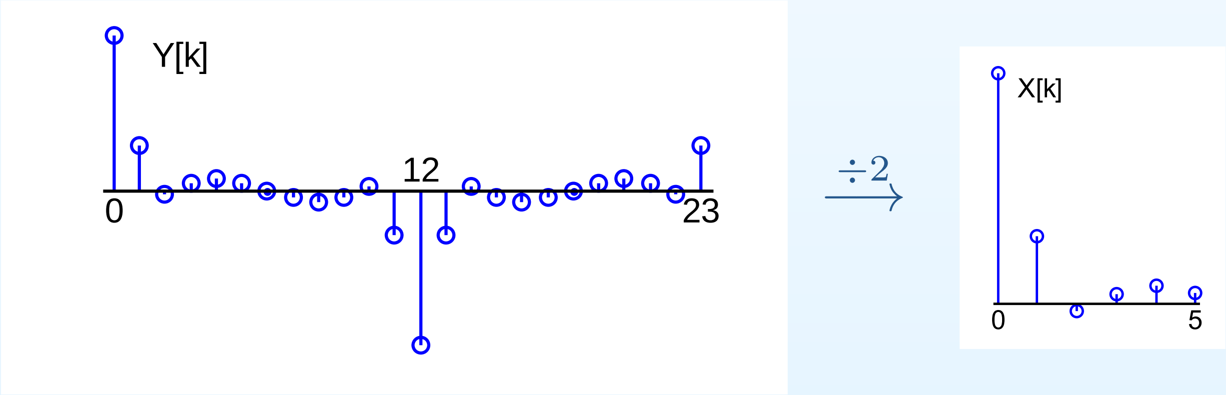


DCT

To form the Discrete Cosine Transform (DCT), replicate $x[0:N-1]$ but in reverse order and insert a zero between each pair of samples:



Take the DFT of length $4N$ real, symmetric, odd-sample-only sequence.
Result is real, symmetric and anti-periodic: only need first N values

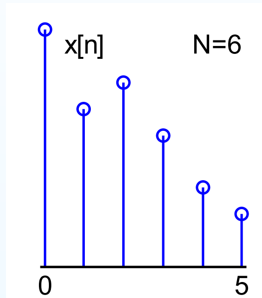


Forward DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$ for $k = 0 : N-1$

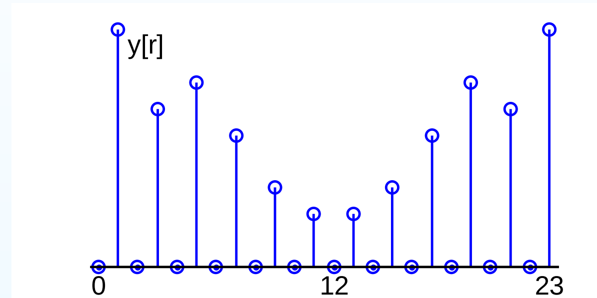
DCT

+

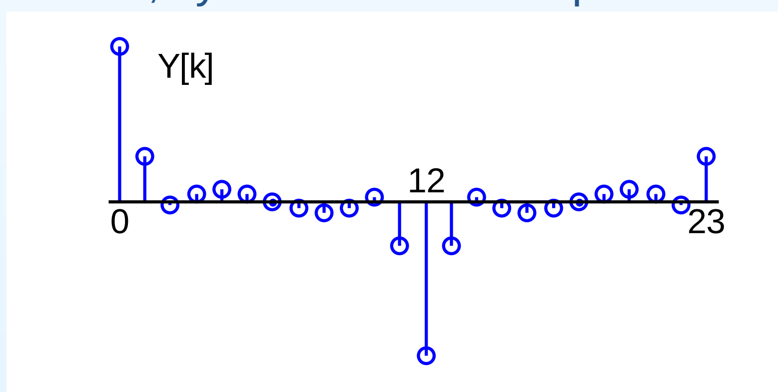
To form the Discrete Cosine Transform (DCT), replicate $x[0:N-1]$ but in reverse order and insert a zero between each pair of samples:



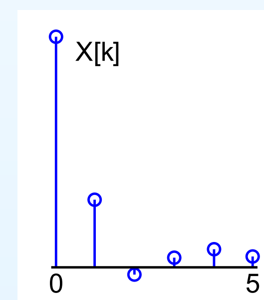
$\rightarrow$



Take the DFT of length $4N$ real, symmetric, odd-sample-only sequence.
Result is real, symmetric and anti-periodic: only need first N values



$\xrightarrow{\div 2}$

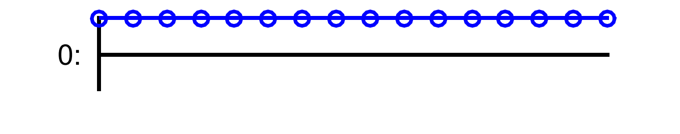


Forward DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$ for $k = 0 : N-1$

Inverse DCT: $x[n] = \frac{1}{N} X[0] + \frac{2}{N} \sum_{k=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$

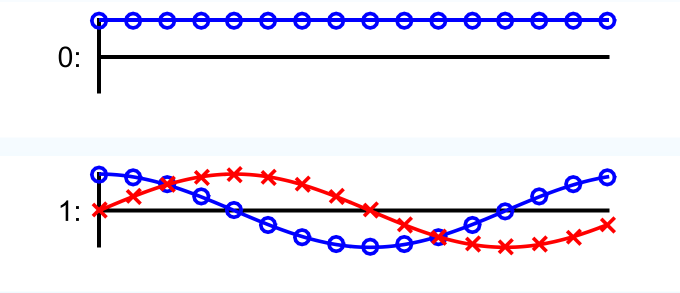
Basis Functions

DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$



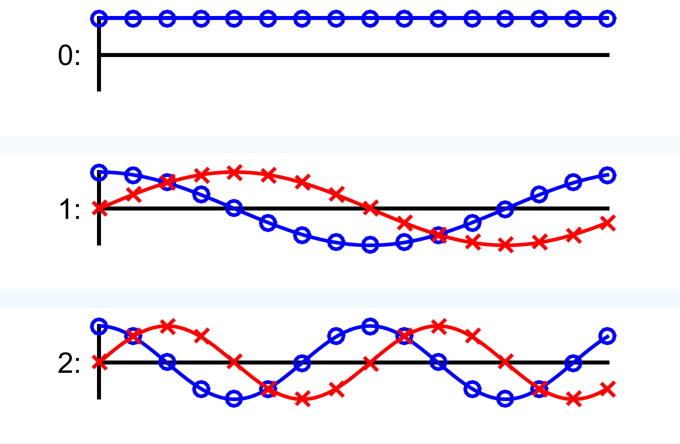
Basis Functions

DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$



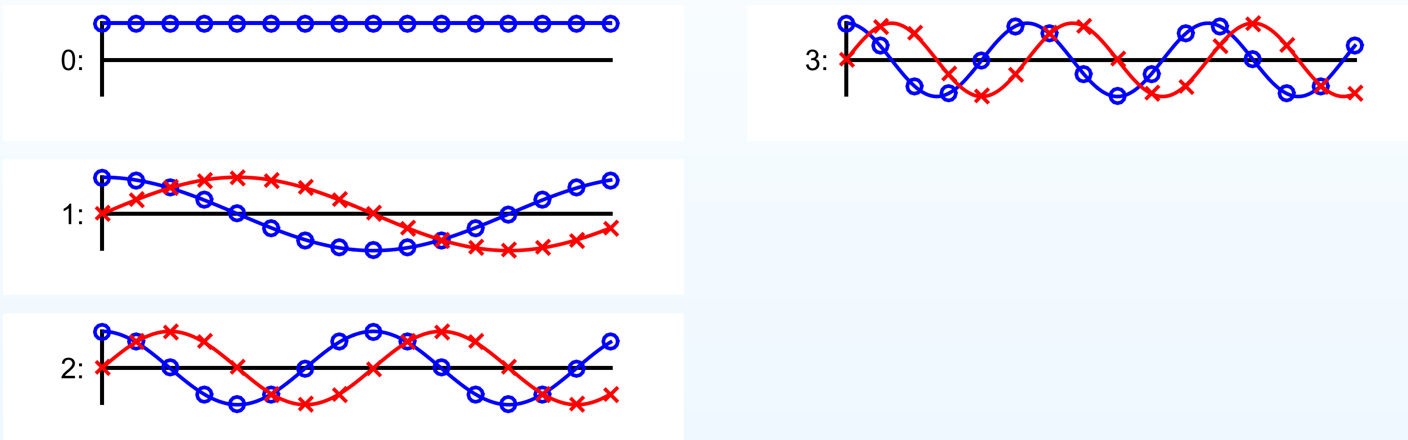
Basis Functions

DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$



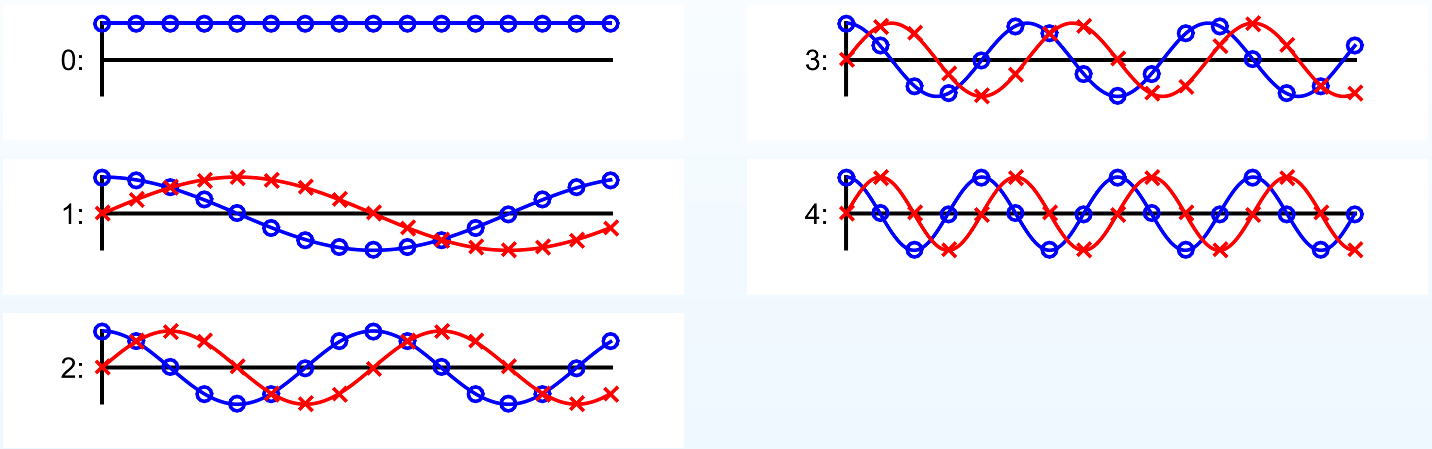
Basis Functions

DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$



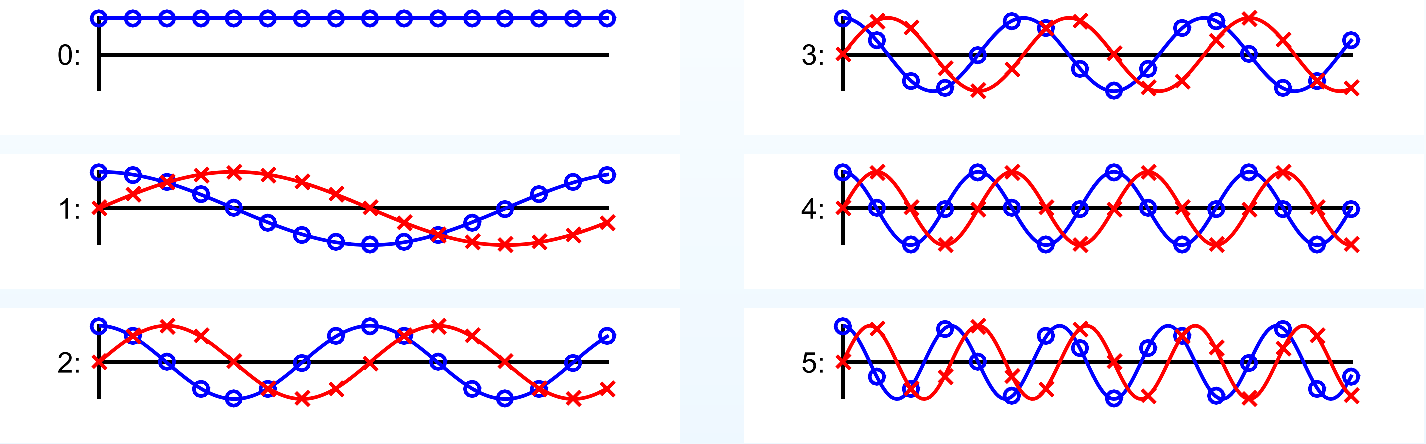
Basis Functions

DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$



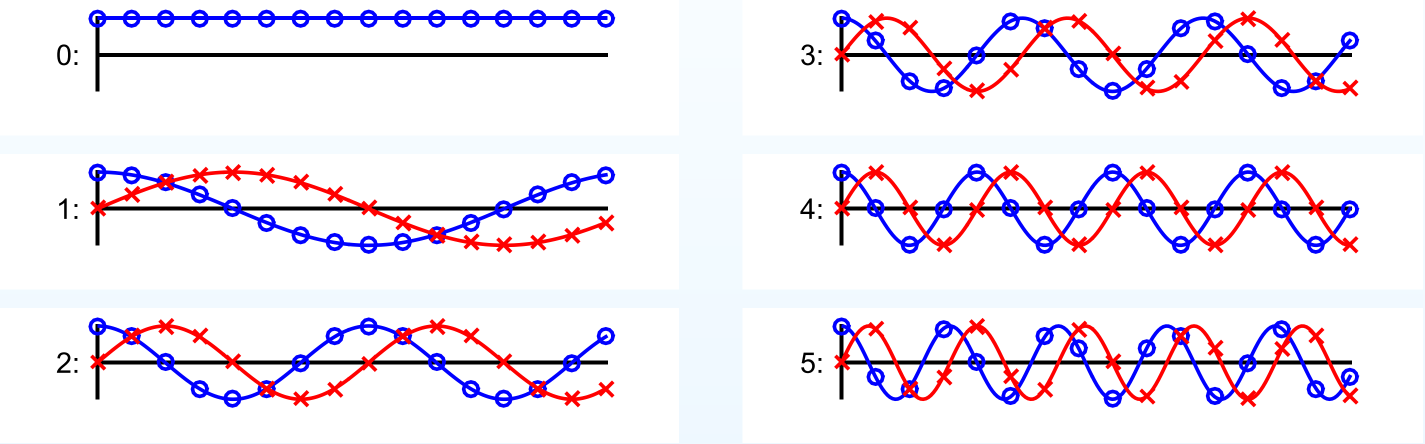
Basis Functions

DFT basis functions: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$



Basis Functions

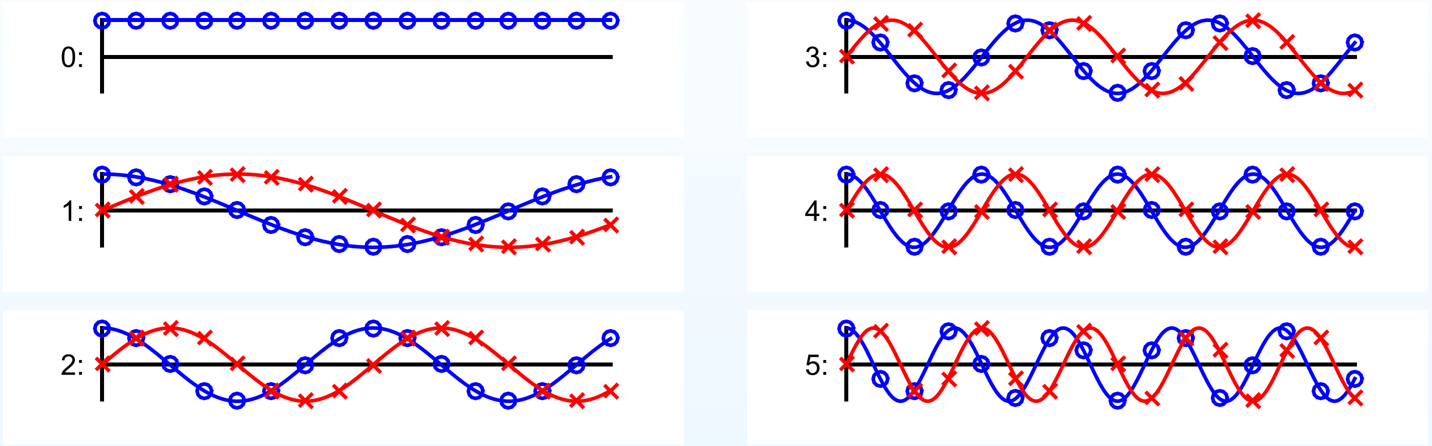
DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$



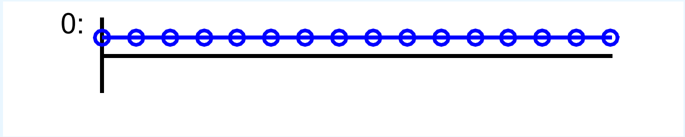
DCT basis functions: $x[n] = \frac{1}{N}X[0] + \frac{2}{N}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$

Basis Functions

DFT basis functions: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$

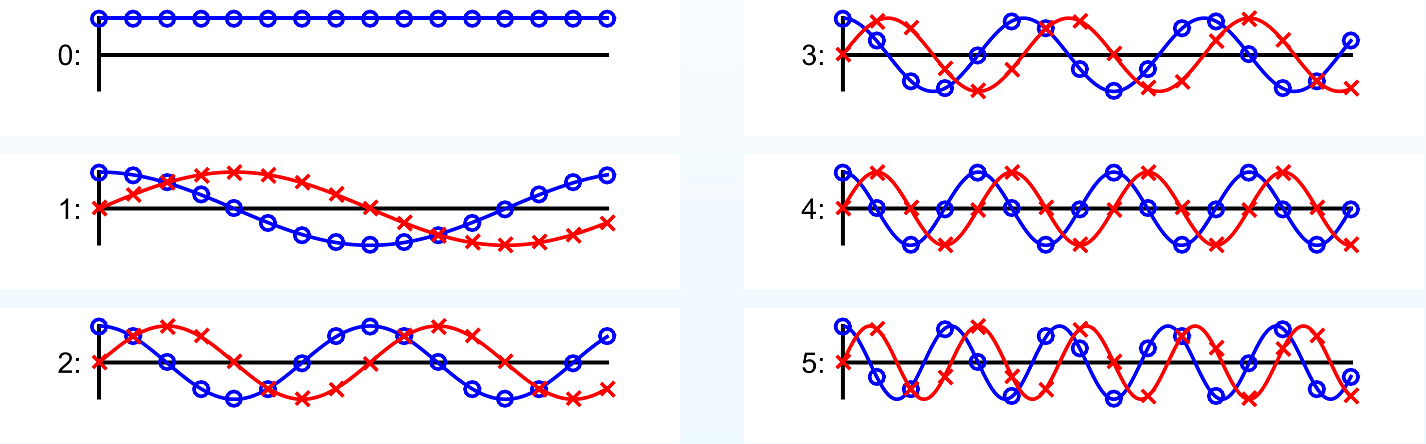


DCT basis functions: $x[n] = \frac{1}{N} X[0] + \frac{2}{N} \sum_{k=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$

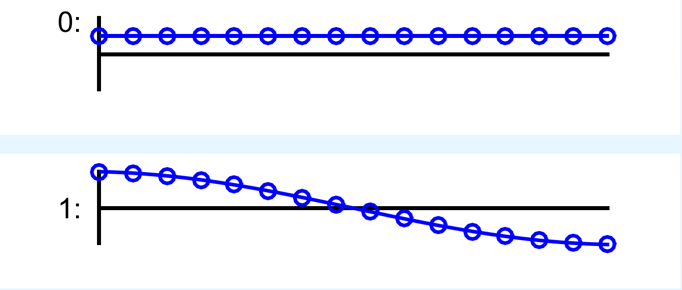


Basis Functions

DFT basis functions: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$

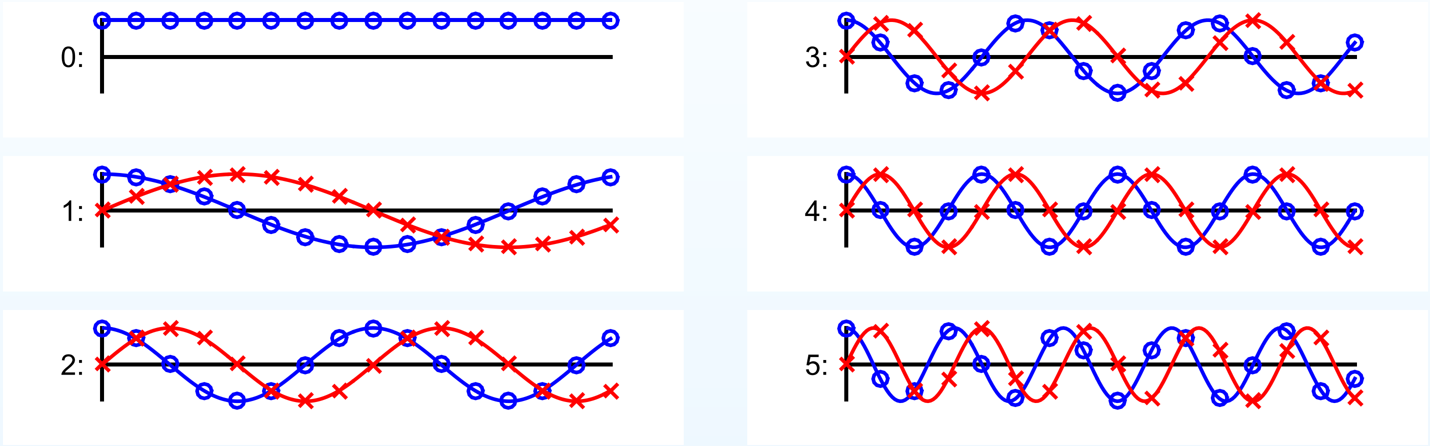


DCT basis functions: $x[n] = \frac{1}{N} X[0] + \frac{2}{N} \sum_{k=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$

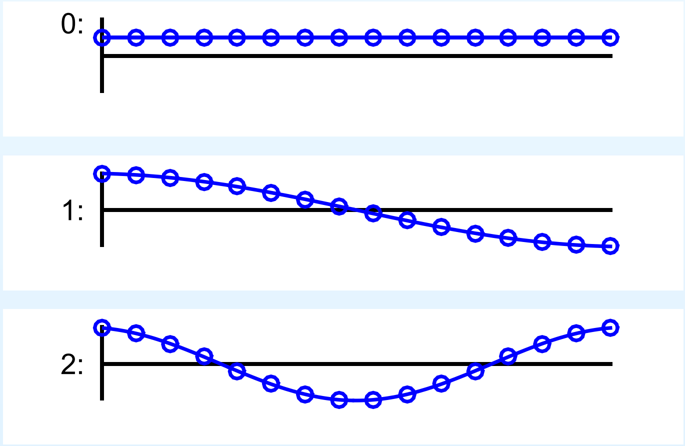


Basis Functions

DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$

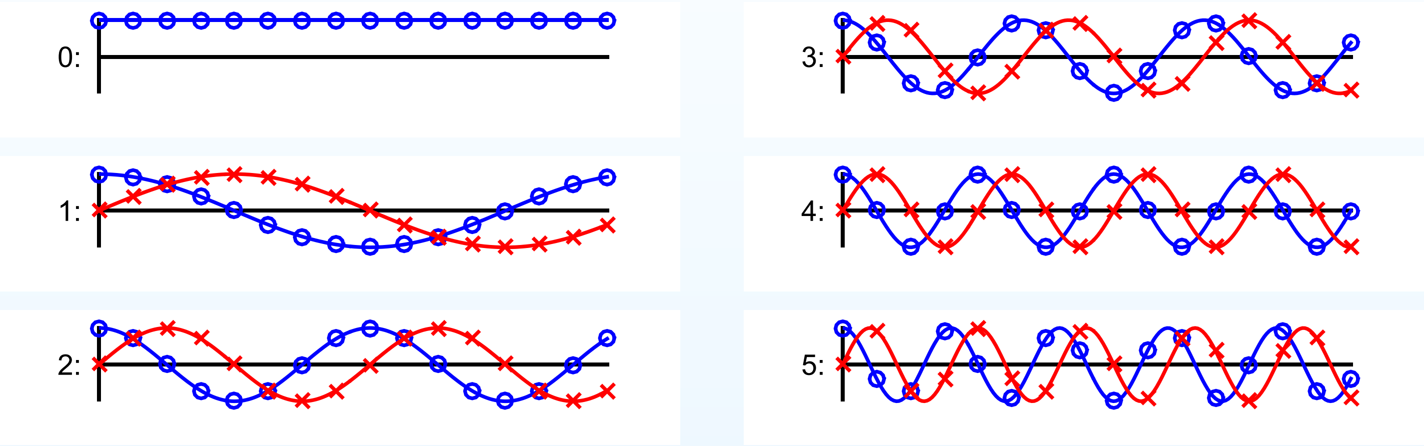


DCT basis functions: $x[n] = \frac{1}{N}X[0] + \frac{2}{N}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$

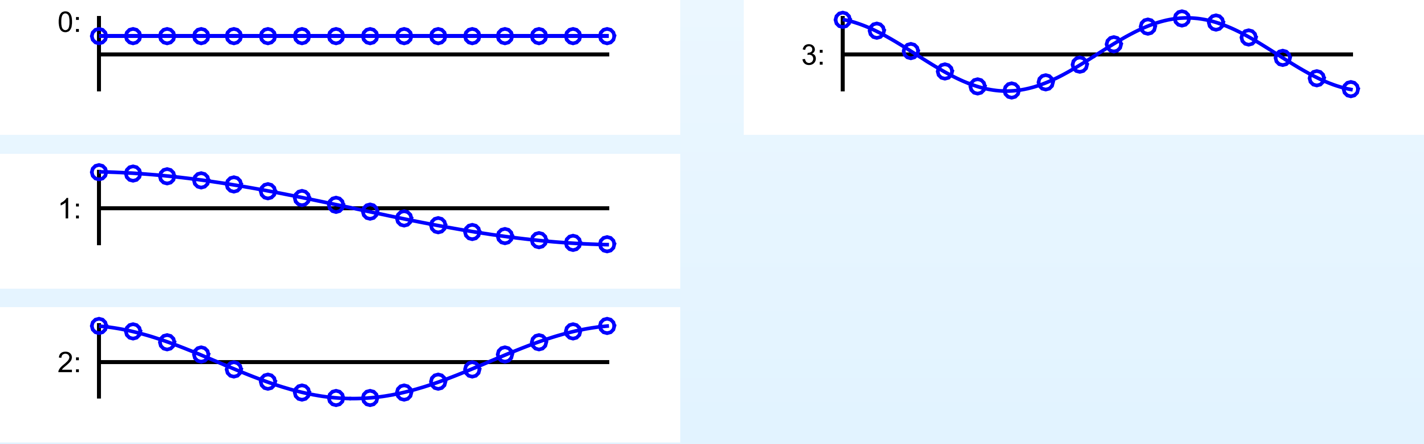


Basis Functions

DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$

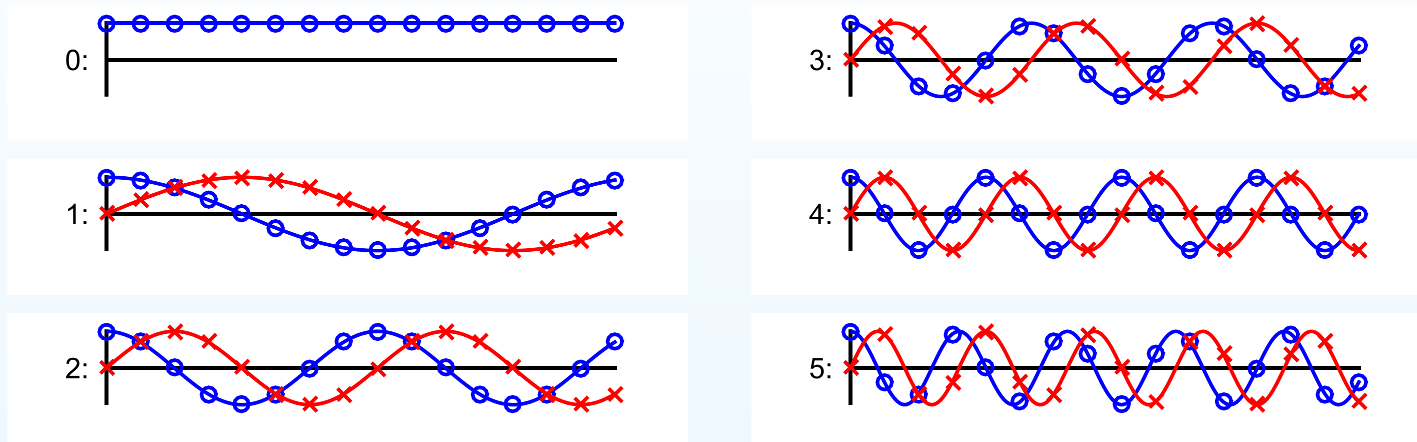


DCT basis functions: $x[n] = \frac{1}{N}X[0] + \frac{2}{N}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$

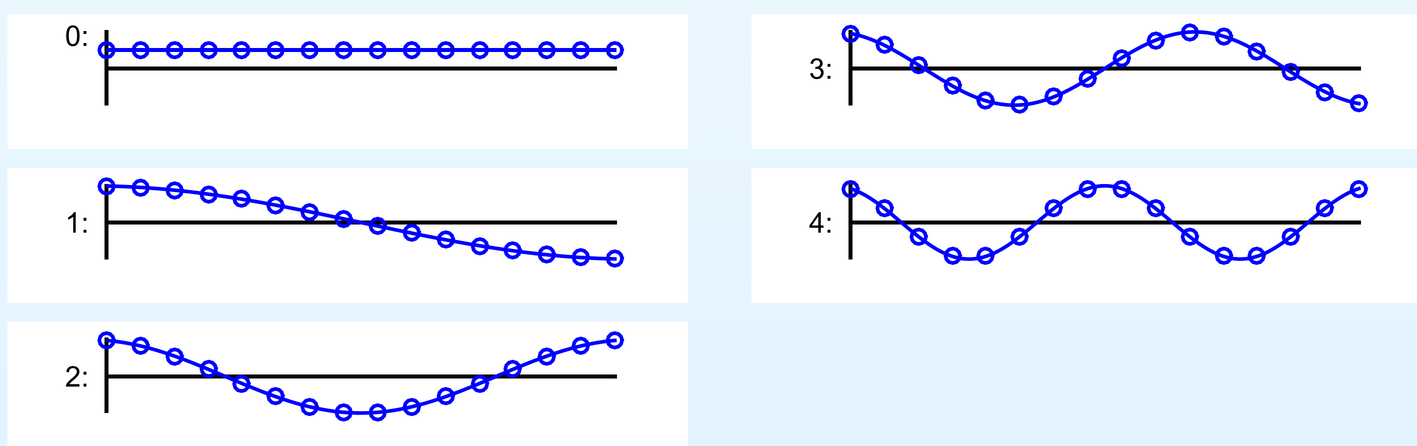


Basis Functions

DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$

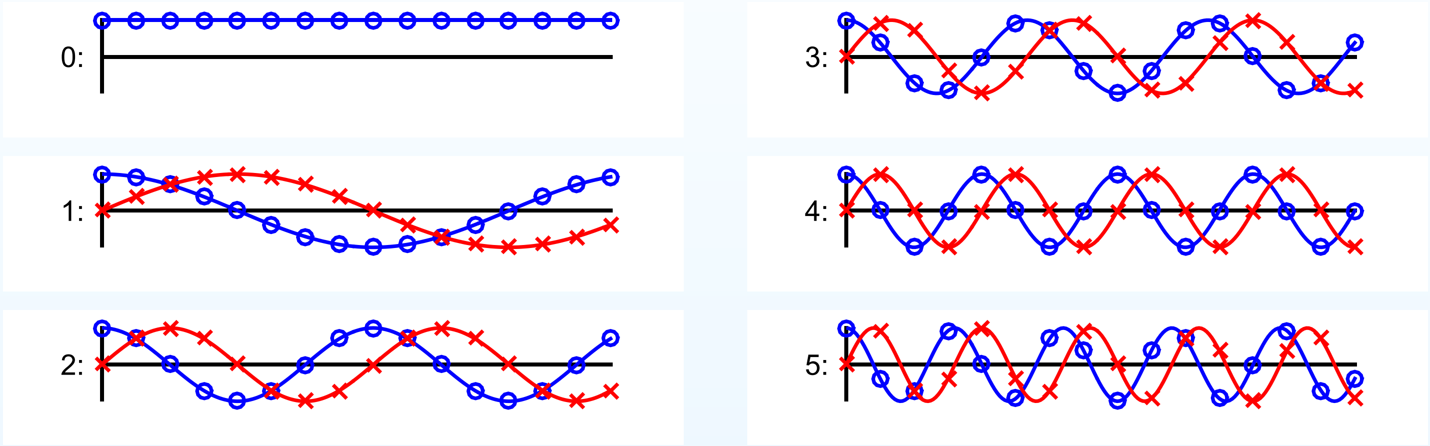


DCT basis functions: $x[n] = \frac{1}{N}X[0] + \frac{2}{N}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$

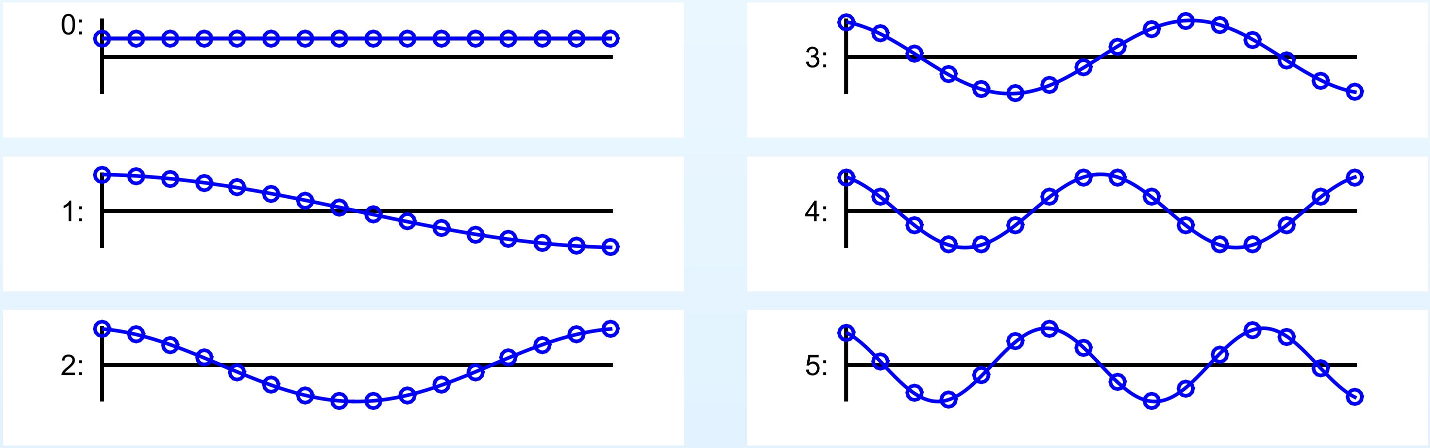


Basis Functions

DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$



DCT basis functions: $x[n] = \frac{1}{N}X[0] + \frac{2}{N}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$



DCT of sine wave

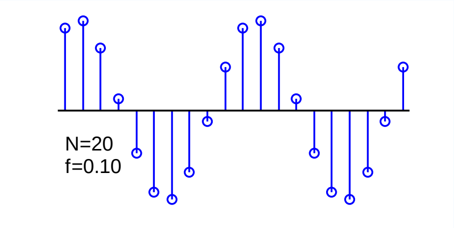
DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

DCT of sine wave

DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

$$f = \frac{m}{N}$$

$x[n]$

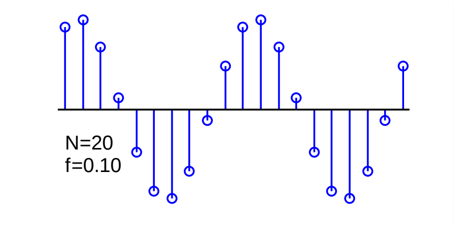


DCT of sine wave

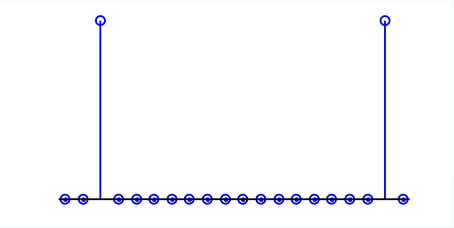
DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

$$f = \frac{m}{N}$$

$x[n]$



$|X_F[k]|$



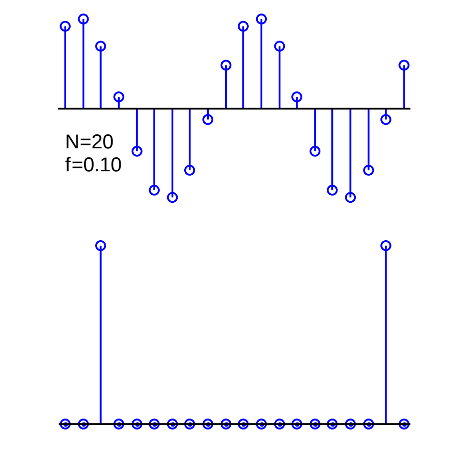
DCT of sine wave

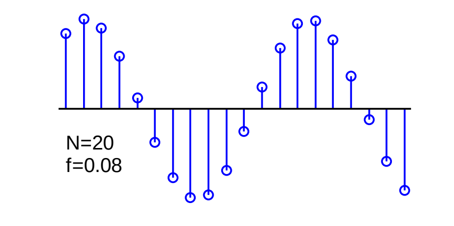
DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

$f = \frac{m}{N}$ $\qquad$ $f \neq \frac{m}{N}$

$x[n]$

$|X_F[k]|$





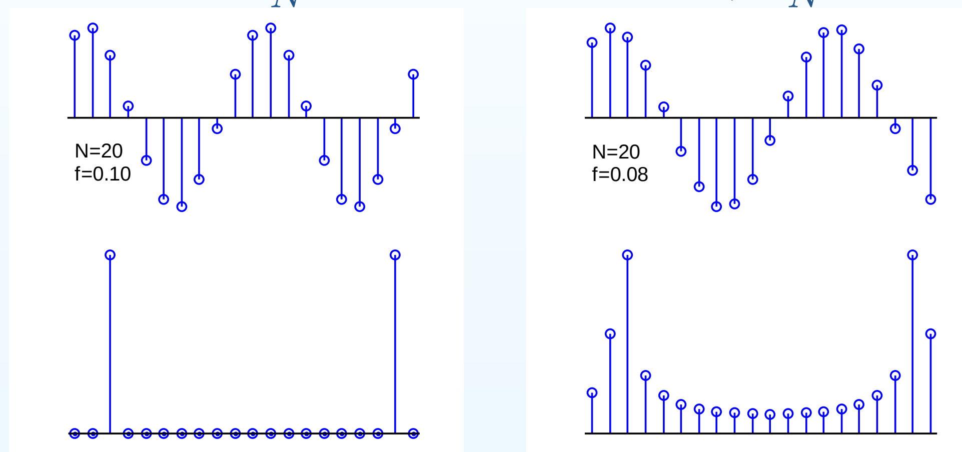
DCT of sine wave

DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

$f = \frac{m}{N}$ $\qquad$ $f \neq \frac{m}{N}$

$x[n]$

$|X_F[k]|$



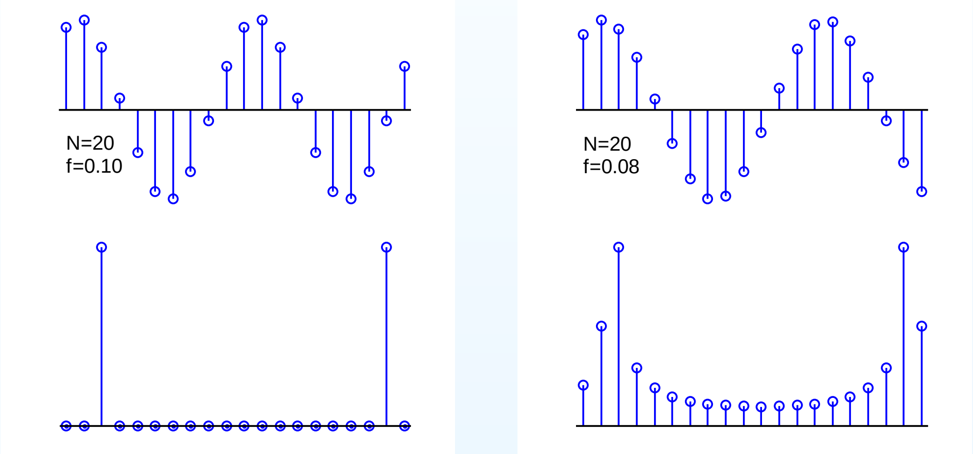
DCT of sine wave

DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

$f = \frac{m}{N}$ $\qquad$ $f \neq \frac{m}{N}$

$x[n]$

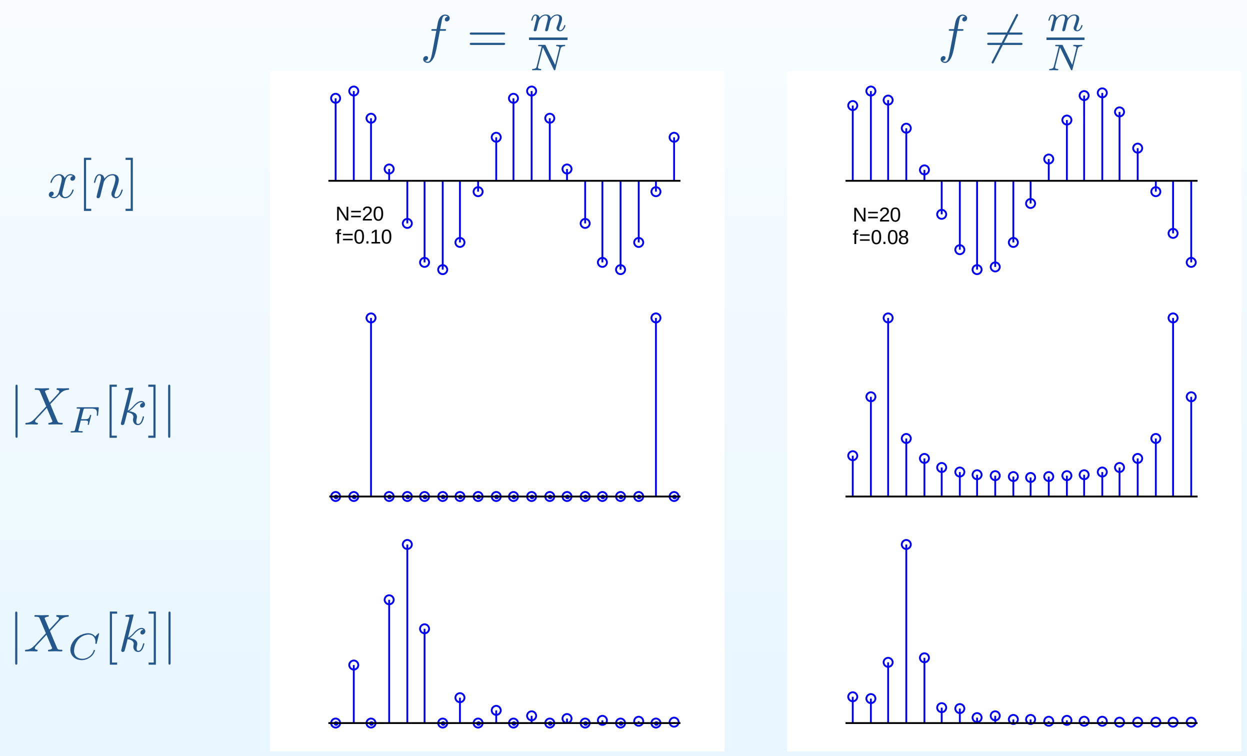
$|X_F[k]|$



DFT: Real→Complex; Freq range $[0, 1]$; Poorly localized unless $f = \frac{m}{N}$; $|X_F[k]| \propto k^{-1}$ for $Nf < k \ll \frac{N}{2}$

DCT of sine wave

DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$



DFT: Real→Complex; Freq range $[0,1]$; Poorly localized unless $f = \frac{m}{N}$; $|X_F[k]| \propto k^{-1}$ for $Nf < k \ll \frac{N}{2}$

DCT of sine wave

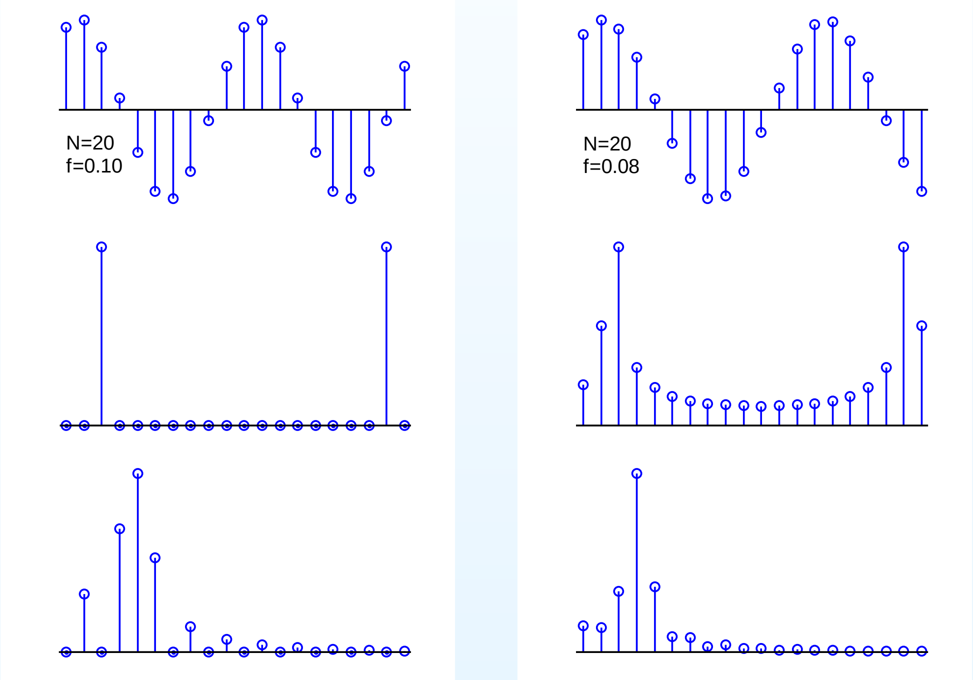
DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

$f = \frac{m}{N}$ $\quad$ $f \neq \frac{m}{N}$

$x[n]$

$|X_F[k]|$

$|X_C[k]|$



DFT: Real→Complex; Freq range $[0, 1]$; Poorly localized unless $f = \frac{m}{N}$; $|X_F[k]| \propto k^{-1}$ for $Nf < k \ll \frac{N}{2}$

DCT: Real→Real; Freq range $[0, 0.5]$; Well localized $\forall f$; $|X_C[k]| \propto k^{-2}$ for $2Nf < k < N$

DCT Properties

Definition: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

DCT Properties

Definition: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

- Linear: $\alpha x[n] + \beta y[n] \to \alpha X[k] + \beta Y[k]$

DCT Properties

Definition: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

- Linear: $\alpha x[n] + \beta y[n] \rightarrow \alpha X[k] + \beta Y[k]$
- "Convolution⟷Multiplication" property of DFT does not hold ☹

DCT Properties

Definition: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

- Linear: $\alpha x[n] + \beta y[n] \rightarrow \alpha X[k] + \beta Y[k]$
- "Convolution⟷Multiplication" property of DFT does not hold ☹
- Symmetric: $X[-k] = X[k]$ since $\cos -\alpha k = \cos +\alpha k$

DCT Properties

Definition: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

- Linear: $\alpha x[n] + \beta y[n] \rightarrow \alpha X[k] + \beta Y[k]$
- "Convolution⟷Multiplication" property of DFT does not hold ☹
- Symmetric: $X[-k] = X[k]$ since $\cos -\alpha k = \cos +\alpha k$
- Anti-periodic: $X[k+2N] = -X[k]$

DCT Properties

Definition: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

- Linear: $\alpha x[n] + \beta y[n] \to \alpha X[k] + \beta Y[k]$
- "Convolution⟷Multiplication" property of DFT does not hold ☹
- Symmetric: $X[-k] = X[k]$ since $\cos -\alpha k = \cos +\alpha k$
- Anti-periodic: $X[k+2N] = -X[k]$ because:
    - $2\pi(2n+1)(k+2N) = 2\pi(2n+1)k + 8\pi Nn + 4N\pi$
    - $\cos(\theta+\pi) = -\cos\theta$

DCT Properties

Definition: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

- Linear: $\alpha x[n] + \beta y[n] \rightarrow \alpha X[k] + \beta Y[k]$
- "Convolution⟷Multiplication" property of DFT does not hold ☹
- Symmetric: $X[-k] = X[k]$ since $\cos -\alpha k = \cos +\alpha k$
- Anti-periodic: $X[k+2N] = -X[k]$ because:
  - $2\pi(2n+1)(k+2N) = 2\pi(2n+1)k + 8\pi Nn + 4N\pi$
  - $\cos(\theta+\pi) = -\cos\theta$

$\Rightarrow X[N] = 0$ since $X[N] = X[-N] = -X[-N+2N]$

DCT Properties

Definition: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

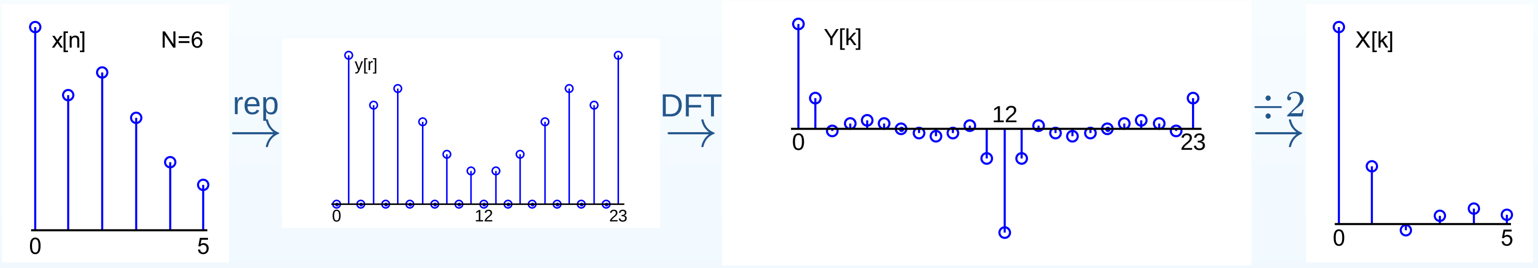
- Linear: $\alpha x[n] + \beta y[n] \rightarrow \alpha X[k] + \beta Y[k]$
- "Convolution⟷Multiplication" property of DFT does not hold ☹
- Symmetric: $X[-k] = X[k]$ since $\cos -\alpha k = \cos +\alpha k$
- Anti-periodic: $X[k+2N] = -X[k]$ because:
  - $2\pi(2n+1)(k+2N) = 2\pi(2n+1)k + 8\pi Nn + 4N\pi$
  - $\cos(\theta+\pi) = -\cos\theta$

  $\Rightarrow X[N] = 0$ since $X[N] = X[-N] = -X[-N+2N]$
- Periodic: $X[k+4N] = -X[k+2N] = X[k]$

Energy Conservation

DCT: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

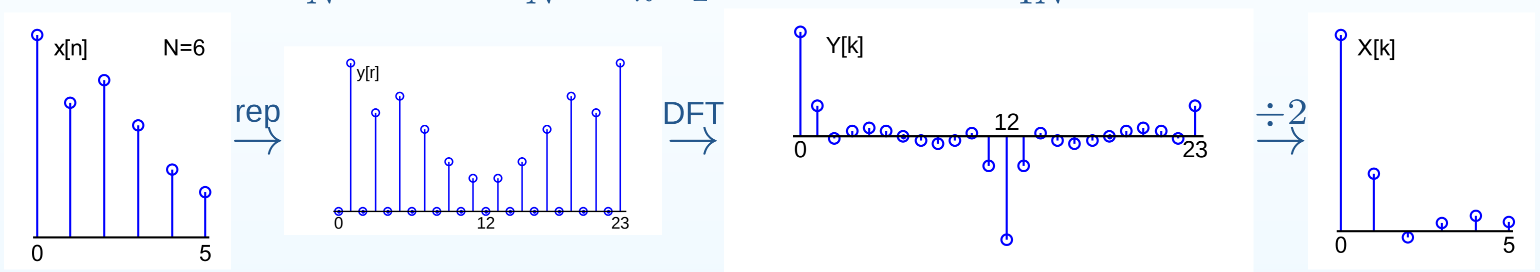
IDCT: $x[n] = \frac{1}{N} X[0] + \frac{2}{N} \sum_{k=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$



Energy Conservation

DCT: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

IDCT: $x[n] = \frac{1}{N} X[0] + \frac{2}{N} \sum_{k=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$

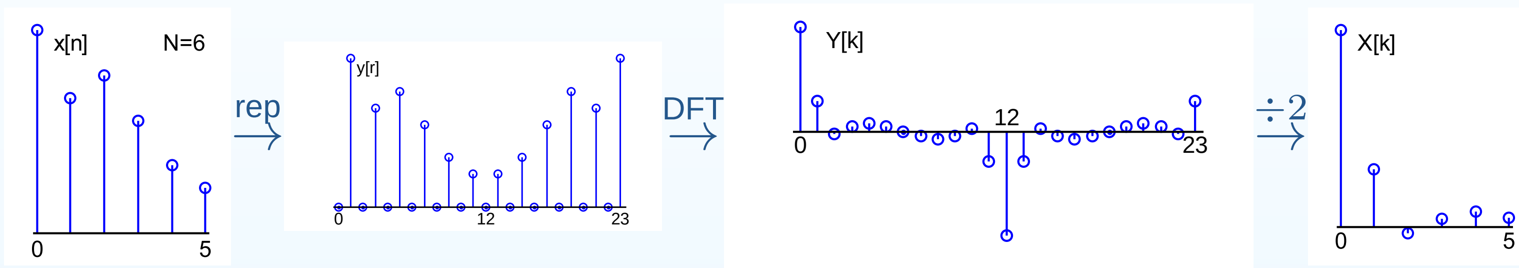


Energy: $E = \sum_{n=0}^{N-1} |x[n]|^2$

Energy Conservation

DCT: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

IDCT: $x[n] = \frac{1}{N} X[0] + \frac{2}{N} \sum_{k=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$

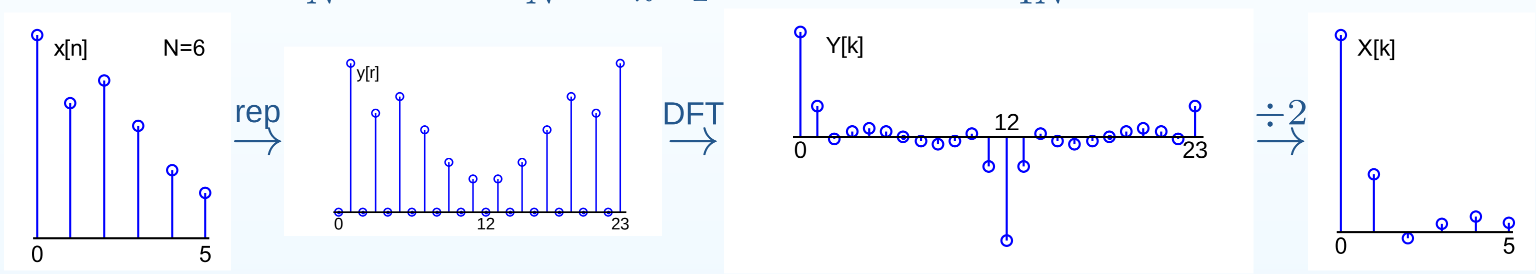


Energy: $E = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} |X[0]|^2 + \frac{2}{N} \sum_{n=1}^{N-1} |X[n]|^2$

Energy Conservation

DCT: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

IDCT: $x[n] = \frac{1}{N} X[0] + \frac{2}{N} \sum_{k=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$



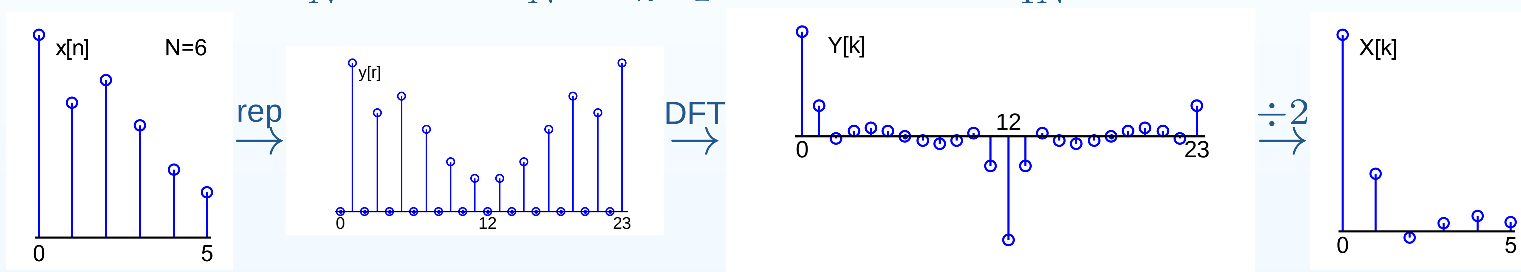
Energy: $E = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} |X[0]|^2 + \frac{2}{N} \sum_{n=1}^{N-1} |X[n]|^2$

In diagram above: $E \to 2E \to 8NE \to \approx 0.5NE$

Energy Conservation

DCT: $X[k]=\sum_{n=0}^{N-1}x[n]\cos\frac{2\pi(2n+1)k}{4N}$

IDCT: $x[n]=\frac{1}{N}X[0]+\frac{2}{N}\sum_{k=1}^{N-1}X[k]\cos\frac{2\pi(2n+1)k}{4N}$



Energy: $E=\sum_{n=0}^{N-1}|x[n]|^2=\frac{1}{N}|X[0]|^2+\frac{2}{N}\sum_{n=1}^{N-1}|X[n]|^2$

In diagram above: $E\rightarrow 2E\rightarrow 8NE\rightarrow\approx 0.5NE$

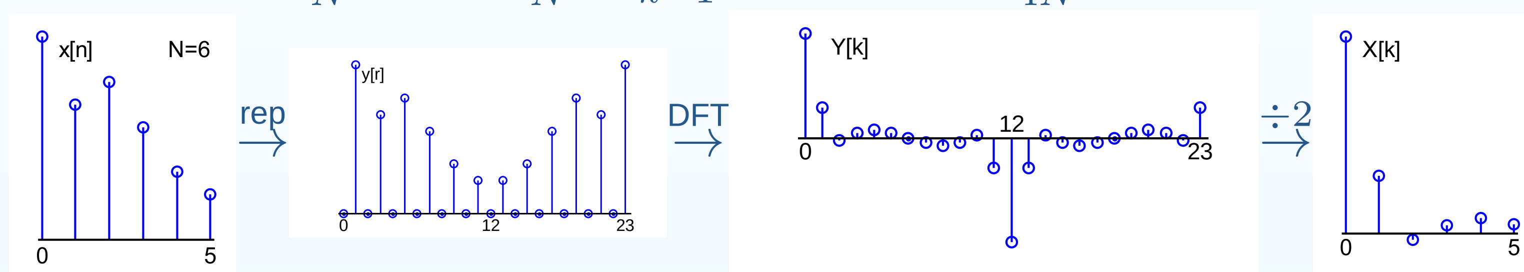
Orthogonal DCT (preserves energy: $\sum|x[n]|^2=\sum|X[n]|^2$)

Note: MATLAB dct() calculates the ODCT

Energy Conservation

DCT: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

IDCT: $x[n] = \frac{1}{N} X[0] + \frac{2}{N} \sum_{k=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$



Energy: $E = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} |X[0]|^2 + \frac{2}{N} \sum_{n=1}^{N-1} |X[n]|^2$

In diagram above: $E \to 2E \to 8NE \to \approx 0.5NE$

Orthogonal DCT (preserves energy: $\sum |x[n]|^2 = \sum |X[n]|^2$)

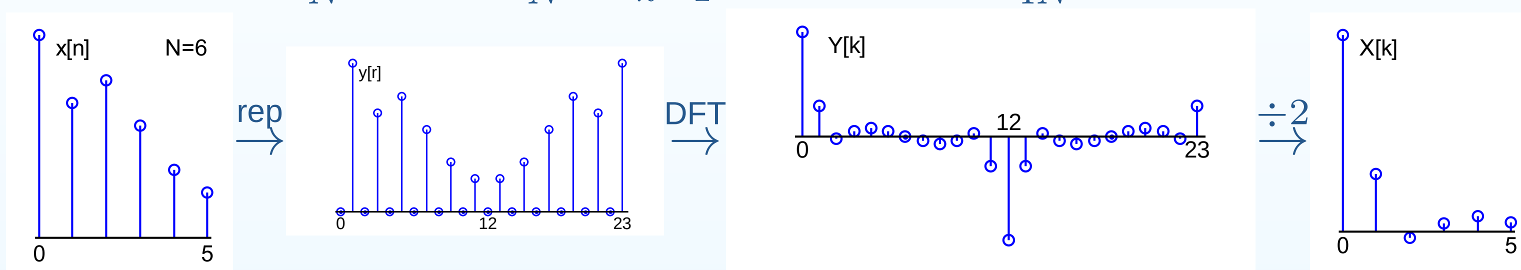
ODCT: $X[k] = \begin{cases} \sqrt{\frac{1}{N}} \sum_{n=0}^{N-1} x[n] & k = 0 \\ \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N} & k \neq 0 \end{cases}$

Note: MATLAB dct() calculates the ODCT

Energy Conservation

DCT: $X[k]=\sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N}$

IDCT: $x[n]=\frac{1}{N}X[0]+\frac{2}{N}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$



Energy: $E=\sum_{n=0}^{N-1} |x[n]|^2=\frac{1}{N}|X[0]|^2+\frac{2}{N}\sum_{n=1}^{N-1}|X[n]|^2$

In diagram above: $E\rightarrow 2E\rightarrow 8NE\rightarrow\approx 0.5NE$

Orthogonal DCT (preserves energy: $\sum|x[n]|^2=\sum|X[n]|^2$)

$$\text{ODCT: } X[k]=\begin{cases}\sqrt{\frac{1}{N}}\sum_{n=0}^{N-1} x[n] & k=0\\ \sqrt{\frac{2}{N}}\sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N} & k\neq 0\end{cases}$$

$$\text{IODCT: } x[n]=\sqrt{\frac{1}{N}}X[0]+\sqrt{\frac{2}{N}}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$$

Note: MATLAB dct() calculates the ODCT

Energy Compaction

If consecutive $x[n]$ are positively correlated, DCT concentrates energy in a few $X[k]$ and decorrelates them.

Energy Compaction

If consecutive $x[n]$ are positively correlated, DCT concentrates energy in a few $X[k]$ and decorrelates them.

Example: Markov Process: $x[n] = \rho x[n-1] + \sqrt{1-\rho^2}u[n]$
where $u[n]$ is i.i.d. unit Gaussian.
Then $\langle x^2[n] \rangle = 1$ and $\langle x[n]x[n-1] \rangle = \rho$.
Covariance of vector $\mathbf{x}$ is $\mathbf{S}_{i,j} = \langle \mathbf{x}\mathbf{x}^H \rangle_{i,j} = \rho^{|i-j|}$.

Energy Compaction

If consecutive $x[n]$ are positively correlated, DCT concentrates energy in a few $X[k]$ and decorrelates them.

Example: Markov Process: $x[n] = \rho x[n-1] + \sqrt{1-\rho^2}u[n]$

where $u[n]$ is i.i.d. unit Gaussian.

Then $\langle x^2[n] \rangle = 1$ and $\langle x[n]x[n-1] \rangle = \rho$.

Covariance of vector $\mathbf{x}$ is $\mathbf{S}_{i,j} = \langle \mathbf{x}\mathbf{x}^H \rangle_{i,j} = \rho^{|i-j|}$.

Suppose ODCT of $\mathbf{x}$ is $\mathbf{Cx}$ and DFT is $\mathbf{Fx}$.

Covariance of $\mathbf{Cx}$ is $\left\langle \mathbf{C}\mathbf{x}\mathbf{x}^H\mathbf{C}^H \right\rangle = \mathbf{CSC}^H$ (similarly $\mathbf{FSF}^H$)

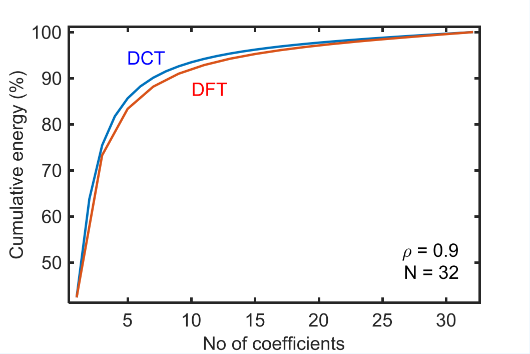
Diagonal elements give mean coefficient energy.

Energy Compaction

If consecutive $x[n]$ are positively correlated, DCT concentrates energy in a few $X[k]$ and decorrelates them.

Example: Markov Process: $x[n] = \rho x[n-1] + \sqrt{1-\rho^2}u[n]$
where $u[n]$ is i.i.d. unit Gaussian.
Then $\langle x^2[n] \rangle = 1$ and $\langle x[n]x[n-1] \rangle = \rho$.
Covariance of vector $\mathbf{x}$ is $\mathbf{S}_{i,j} = \langle \mathbf{x}\mathbf{x}^H \rangle_{i,j} = \rho^{|i-j|}$.

Suppose ODCT of $\mathbf{x}$ is $\mathbf{Cx}$ and DFT is $\mathbf{Fx}$.
Covariance of $\mathbf{Cx}$ is $\left\langle \mathbf{Cxx}^H\mathbf{C}^H \right\rangle = \mathbf{CSC}^H$ (similarly $\mathbf{FSF}^H$)
Diagonal elements give mean coefficient energy.

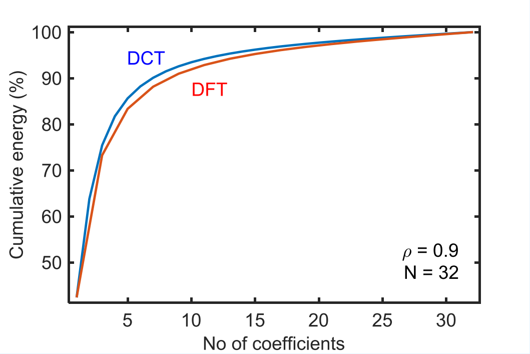


Energy Compaction

If consecutive $x[n]$ are positively correlated, DCT concentrates energy in a few $X[k]$ and decorrelates them.

Example: Markov Process: $x[n] = \rho x[n-1] + \sqrt{1-\rho^2}u[n]$
where $u[n]$ is i.i.d. unit Gaussian.
Then $\langle x^2[n]\rangle = 1$ and $\langle x[n]x[n-1]\rangle = \rho$.
Covariance of vector $\mathbf{x}$ is $\mathbf{S}_{i,j} = \langle \mathbf{x}\mathbf{x}^H \rangle_{i,j} = \rho^{|i-j|}$.

Suppose ODCT of $\mathbf{x}$ is $\mathbf{Cx}$ and DFT is $\mathbf{Fx}$.
Covariance of $\mathbf{Cx}$ is $\left\langle \mathbf{C}\mathbf{x}\mathbf{x}^H\mathbf{C}^H \right\rangle = \mathbf{CSC}^H$ (similarly $\mathbf{FSF}^H$)
Diagonal elements give mean coefficient energy.



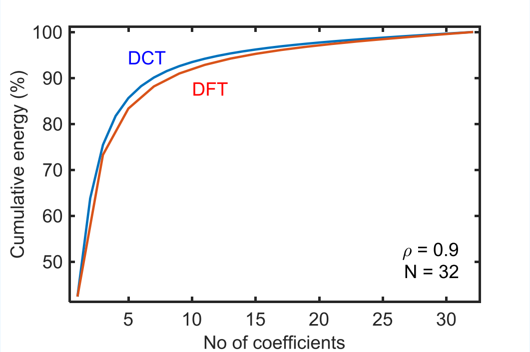
- Used in MPEG and JPEG (superseded by JPEG2000 using wavelets)
- Used in speech recognition to decorrelate spectral coeficients: DCT of log spectrum

Energy Compaction

If consecutive $x[n]$ are positively correlated, DCT concentrates energy in a few $X[k]$ and decorrelates them.

Example: Markov Process: $x[n] = \rho x[n-1] + \sqrt{1-\rho^2}u[n]$
where $u[n]$ is i.i.d. unit Gaussian.
Then $\langle x^2[n]\rangle = 1$ and $\langle x[n]x[n-1]\rangle = \rho$.
Covariance of vector $\mathbf{x}$ is $\mathbf{S}_{i,j} = \langle \mathbf{x}\mathbf{x}^H\rangle_{i,j} = \rho^{|i-j|}$.

Suppose ODCT of $\mathbf{x}$ is $\mathbf{Cx}$ and DFT is $\mathbf{Fx}$.
Covariance of $\mathbf{Cx}$ is $\left\langle \mathbf{C}\mathbf{x}\mathbf{x}^H\mathbf{C}^H \right\rangle = \mathbf{CSC}^H$ (similarly $\mathbf{FSF}^H$)
Diagonal elements give mean coefficient energy.

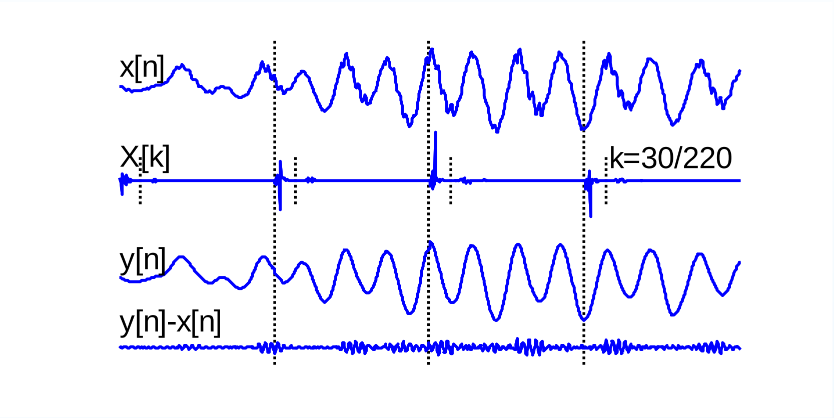


- Used in MPEG and JPEG (superseded by JPEG2000 using wavelets)
- Used in speech recognition to decorrelate spectral coeficients: DCT of log spectrum

Energy compaction good for coding (low-valued coefficients can be set to 0)
Decorrelation good for coding and for probability modelling

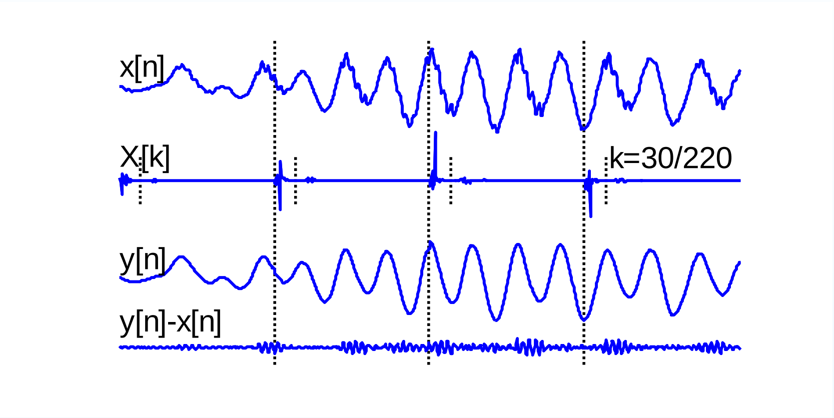
Frame-based coding

- Divide continuous signal into frames



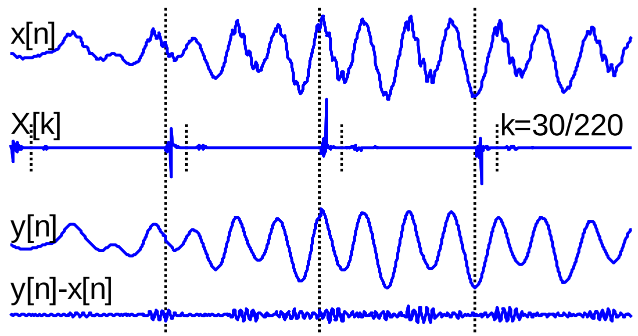
Frame-based coding

- Divide continuous signal into frames
- Apply DCT to each frame



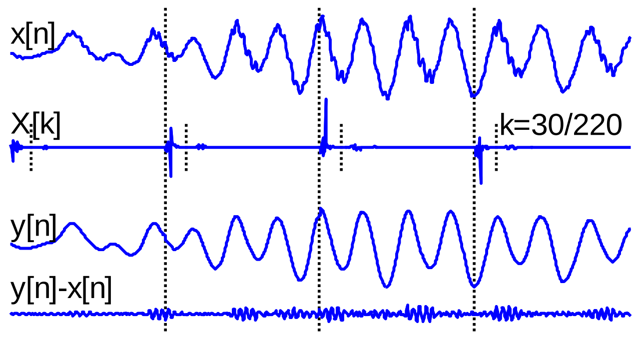
Frame-based coding

- Divide continuous signal into frames
- Apply DCT to each frame
- Encode DCT
  - e.g. keep only 30 $X[k]$



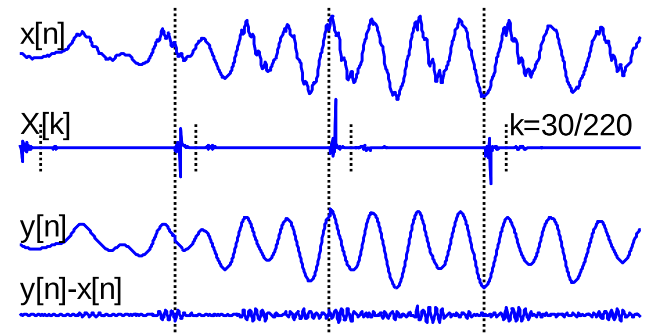
Frame-based coding

- Divide continuous signal into frames
- Apply DCT to each frame
- Encode DCT
  - e.g. keep only 30 $X[k]$
- Apply IDCT $\rightarrow y[n]$



Frame-based coding

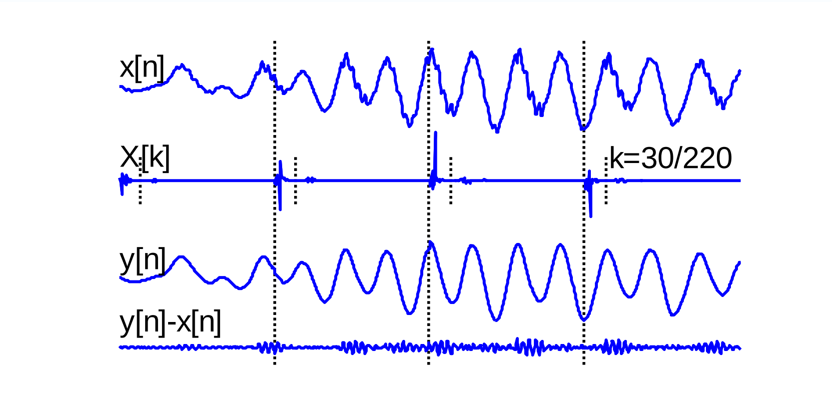
- Divide continuous signal into frames
- Apply DCT to each frame
- Encode DCT
  - e.g. keep only 30 $X[k]$
- Apply IDCT $\rightarrow y[n]$



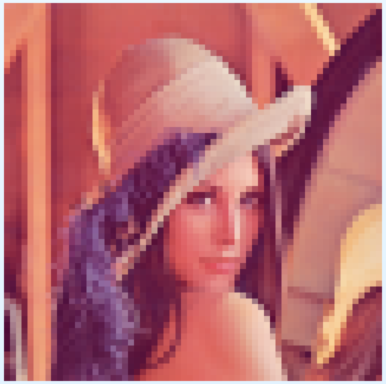
Problem: Coding may create discontinuities at frame boundaries

Frame-based coding

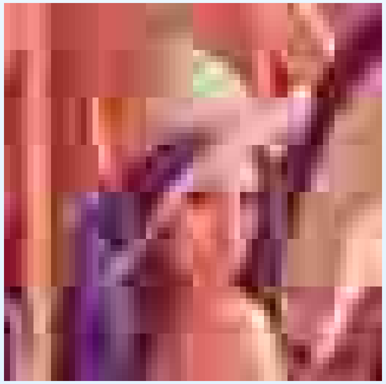
- Divide continuous signal into frames
- Apply DCT to each frame
- Encode DCT
  - e.g. keep only 30 $X[k]$
- Apply IDCT $\rightarrow y[n]$



Problem: Coding may create discontinuities at frame boundaries
e.g. JPEG, MPEG use 8×8 pixel blocks


8.3 kB (PNG)


1.6 kB (JPEG)

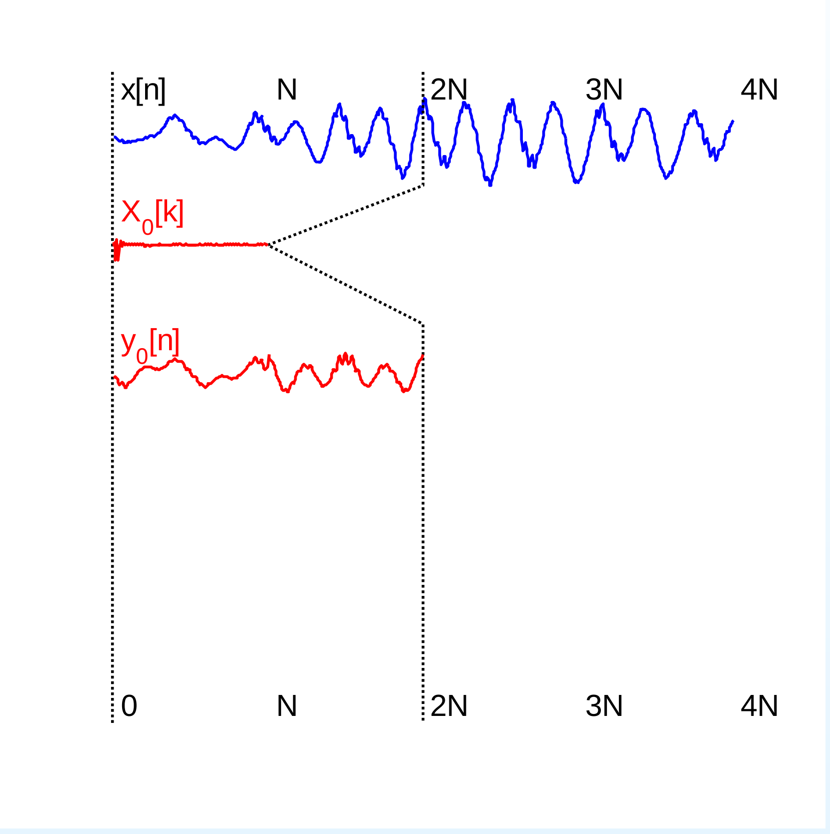

0.5 kB (JPEG)

Lapped Transform

+

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

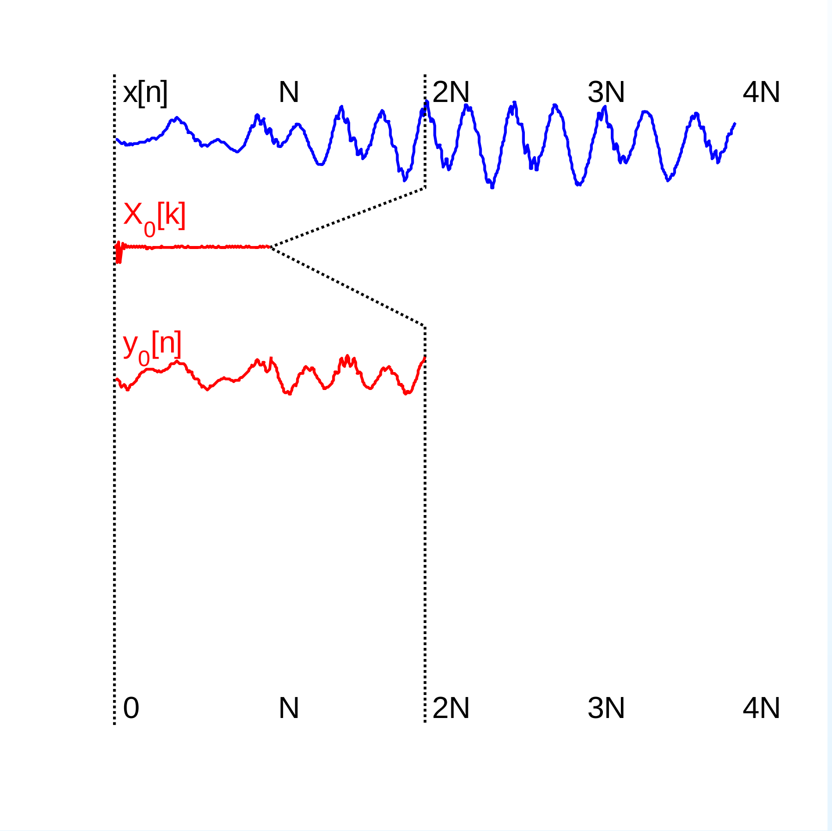
$x[0:2N-1]$



Lapped Transform

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$$x[0:2N-1] \overset{\text{MDCT}}{\to} X_0[0:N-1]$$

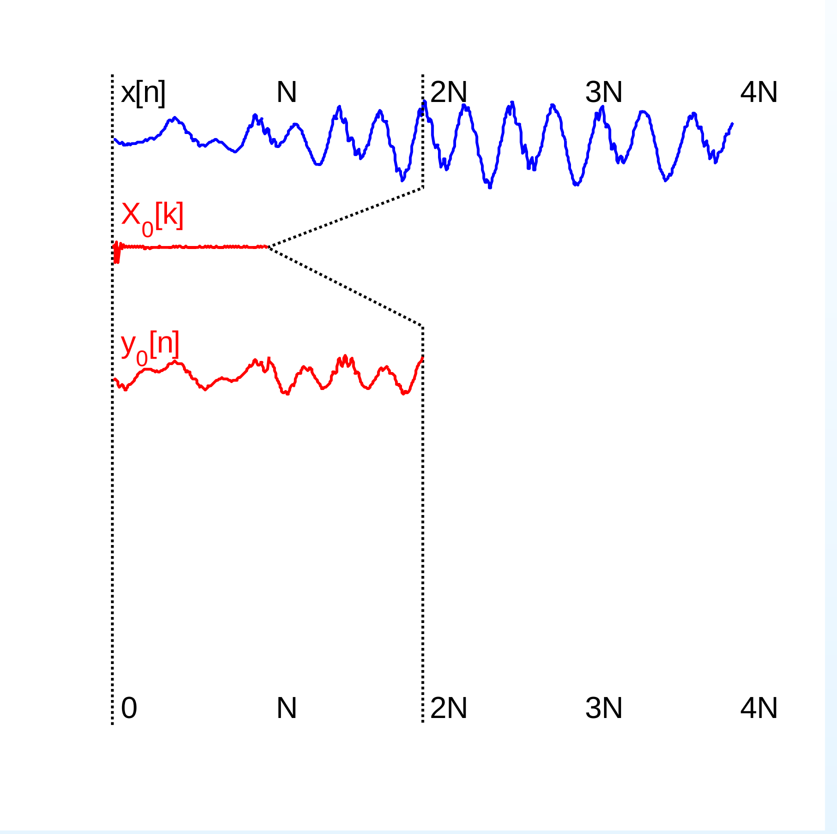


MDCT: $2N \to N$ coefficients

Lapped Transform

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$$x[0:2N-1] \overset{\text{MDCT}}{\rightarrow} X_0[0:N-1] \overset{\text{IMDCT}}{\rightarrow} y_0[0:2N-1]$$



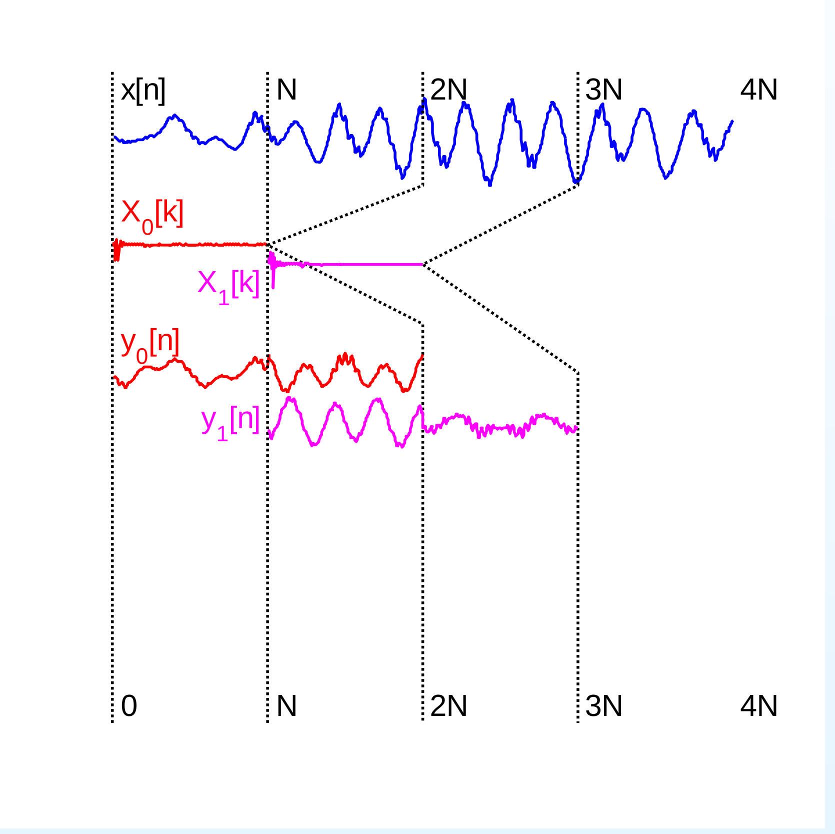
MDCT: $2N \rightarrow N$ coefficients, IMDCT: $N \rightarrow 2N$ samples

Lapped Transform

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$$x[0:2N-1] \overset{\text{MDCT}}{\rightarrow} X_0[0:N-1] \overset{\text{IMDCT}}{\rightarrow} y_0[0:2N-1]$$

$$x[N:3N-1] \overset{\text{MDCT}}{\rightarrow} X_1[N:2N-1]$$



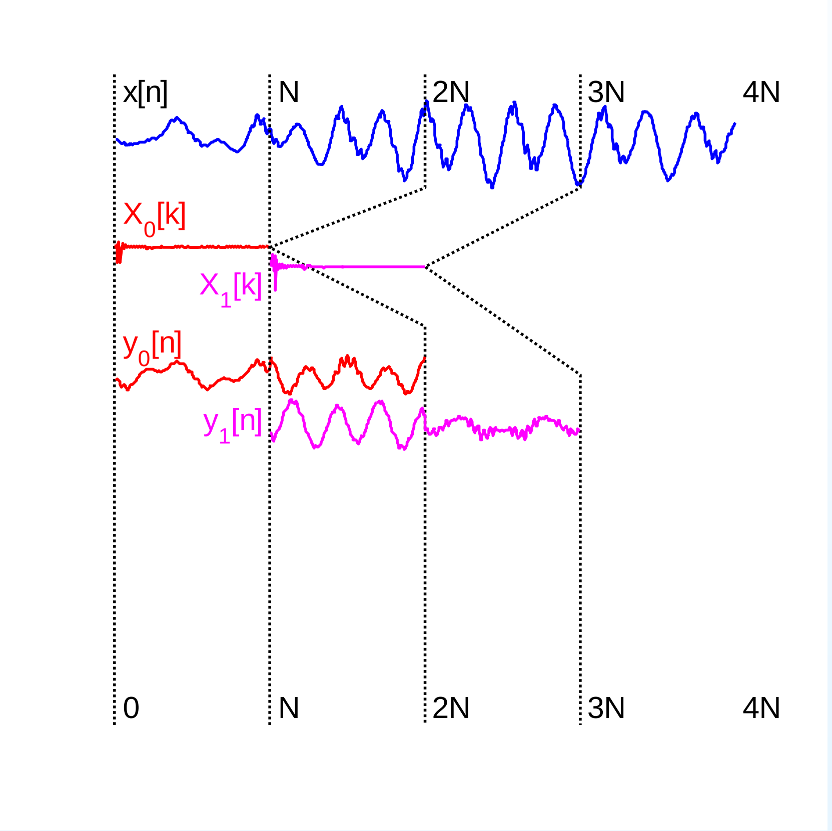
MDCT: $2N \rightarrow N$ coefficients, IMDCT: $N \rightarrow 2N$ samples

Lapped Transform

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$$x[0:2N-1] \overset{\text{MDCT}}{\rightarrow} X_0[0:N-1] \overset{\text{IMDCT}}{\rightarrow} y_0[0:2N-1]$$

$$x[N:3N-1] \overset{\text{MDCT}}{\rightarrow} X_1[N:2N-1] \overset{\text{IMDCT}}{\rightarrow} y_1[N:3N-1]$$



MDCT: $2N \rightarrow N$ coefficients, IMDCT: $N \rightarrow 2N$ samples

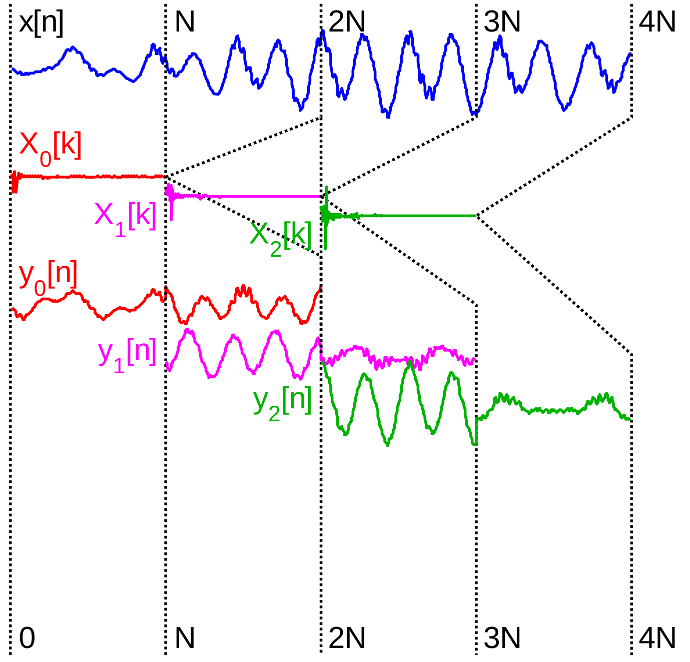
Lapped Transform

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$$x[0:2N-1] \overset{\text{MDCT}}{\rightarrow} X_0[0:N-1] \overset{\text{IMDCT}}{\rightarrow} y_0[0:2N-1]$$

$$x[N:3N-1] \overset{\text{MDCT}}{\rightarrow} X_1[N:2N-1] \overset{\text{IMDCT}}{\rightarrow} y_1[N:3N-1]$$

$$x[2N:4N-1] \overset{\text{MDCT}}{\rightarrow} X_2[2N:3N-1]$$



MDCT: $2N \rightarrow N$ coefficients, IMDCT: $N \rightarrow 2N$ samples

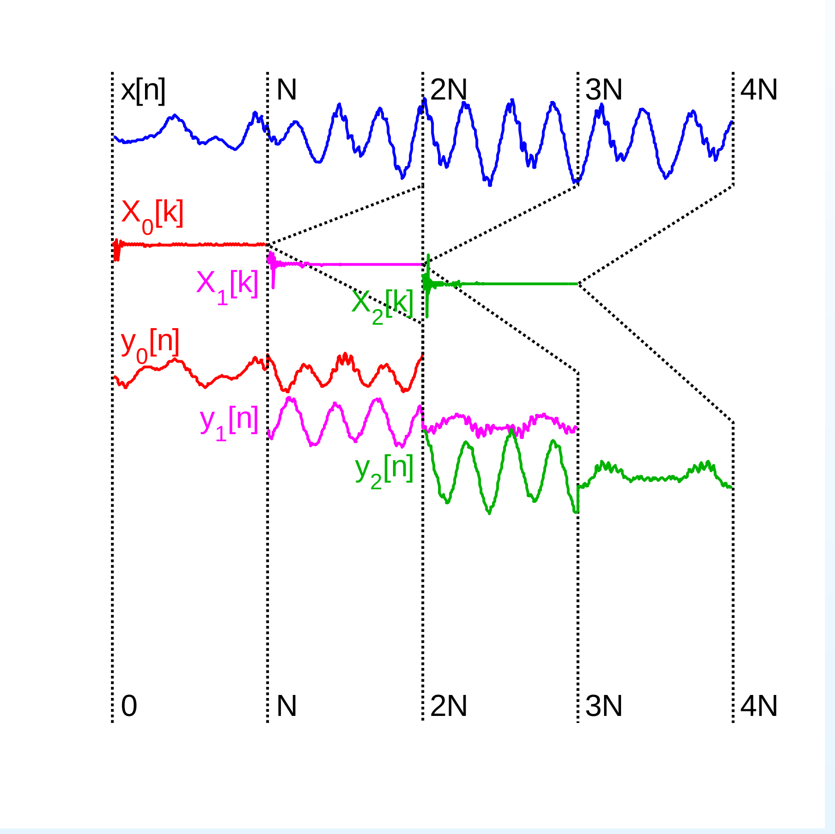
Lapped Transform

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$$x[0:2N-1] \overset{\text{MDCT}}{\rightarrow} X_0[0:N-1] \overset{\text{IMDCT}}{\rightarrow} y_0[0:2N-1]$$

$$x[N:3N-1] \overset{\text{MDCT}}{\rightarrow} X_1[N:2N-1] \overset{\text{IMDCT}}{\rightarrow} y_1[N:3N-1]$$

$$x[2N:4N-1] \overset{\text{MDCT}}{\rightarrow} X_2[2N:3N-1] \overset{\text{IMDCT}}{\rightarrow} y_2[2N:4N-1]$$



MDCT: $2N \rightarrow N$ coefficients, IMDCT: $N \rightarrow 2N$ samples

Lapped Transform

+

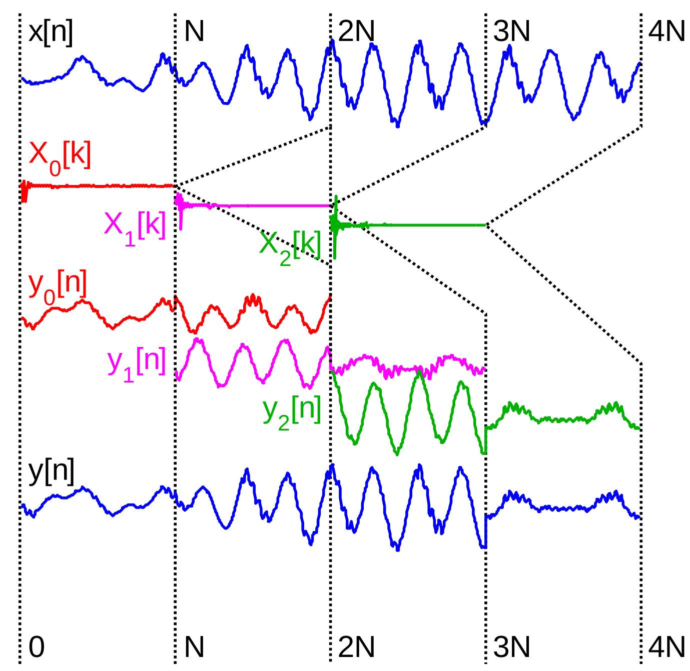
Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$$x[0:2N-1] \xrightarrow{\text{MDCT}} X_0[0:N-1] \xrightarrow{\text{IMDCT}} y_0[0:2N-1]$$

$$x[N:3N-1] \xrightarrow{\text{MDCT}} X_1[N:2N-1] \xrightarrow{\text{IMDCT}} y_1[N:3N-1]$$

$$x[2N:4N-1] \xrightarrow{\text{MDCT}} X_2[2N:3N-1] \xrightarrow{\text{IMDCT}} y_2[2N:4N-1]$$

$$y[n] = y_0[n] + y_1[n] + y_2[n]$$



MDCT: $2N \rightarrow N$ coefficients, IMDCT: $N \rightarrow 2N$ samples

Add $y_i[n]$ together to get $y[n]$. Only two non-zero terms far any n.

Lapped Transform

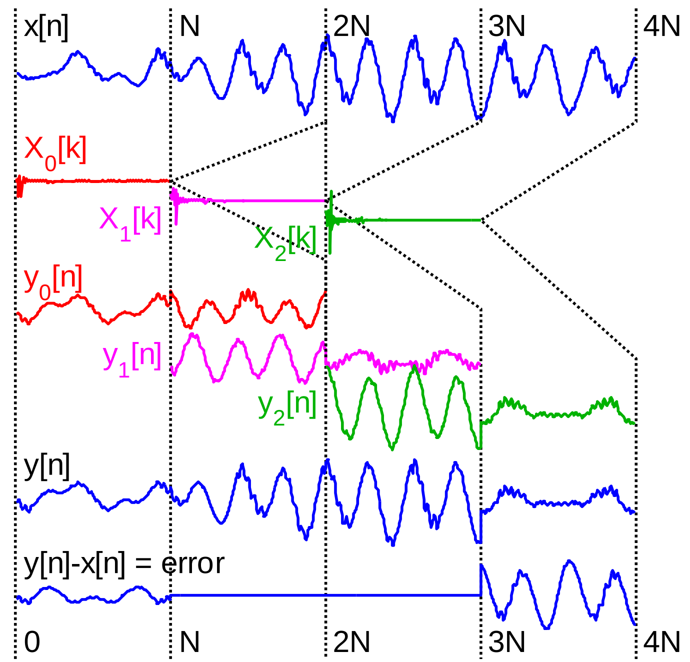
Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$$x[0:2N-1] \overset{\text{MDCT}}{\rightarrow} X_0[0:N-1] \overset{\text{IMDCT}}{\rightarrow} y_0[0:2N-1]$$

$$x[N:3N-1] \overset{\text{MDCT}}{\rightarrow} X_1[N:2N-1] \overset{\text{IMDCT}}{\rightarrow} y_1[N:3N-1]$$

$$x[2N:4N-1] \overset{\text{MDCT}}{\rightarrow} X_2[2N:3N-1] \overset{\text{IMDCT}}{\rightarrow} y_2[2N:4N-1]$$

$$y[n] = y_0[n] + y_1[n] + y_2[n]$$



MDCT: $2N \rightarrow N$ coefficients, IMDCT: $N \rightarrow 2N$ samples

Add $y_i[n]$ together to get $y[n]$. Only two non-zero terms far any n.

Errors cancel exactly: Time-domain alias cancellation (TDAC)

MDCT (Modified DCT)

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le k < N$

MDCT (Modified DCT)

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le k < N$

If $\mathbf{x}$ and $\mathbf{X}$ are column vectors, then $\mathbf{X} = \mathbf{M}\mathbf{x}$

where $\mathbf{M}$ is an $N \times 2N$ matrix with $m_{k,n} = \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$.

MDCT (Modified DCT)

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le n < 2N$

If $\mathbf{x}$ and $\mathbf{X}$ are column vectors, then $\mathbf{X} = \mathbf{M}\mathbf{x}$

where $\mathbf{M}$ is an $N \times 2N$ matrix with $m_{k,n} = \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$.

MDCT (Modified DCT)

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le n < 2N$

If $\mathbf{x}$, $\mathbf{X}$ and $\mathbf{y}$ are column vectors, then $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X}$

where $\mathbf{M}$ is an $N \times 2N$ matrix with $m_{k,n} = \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$.

MDCT (Modified DCT)

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le n < 2N$

If $\mathbf{x}$, $\mathbf{X}$ and $\mathbf{y}$ are column vectors, then $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X} = \frac{1}{N}\mathbf{M}^T\mathbf{M}\mathbf{x}$

where $\mathbf{M}$ is an $N \times 2N$ matrix with $m_{k,n} = \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$.

MDCT (Modified DCT)

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le n < 2N$

If $\mathbf{x}$, $\mathbf{X}$ and $\mathbf{y}$ are column vectors, then $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X} = \frac{1}{N}\mathbf{M}^T\mathbf{M}\mathbf{x}$

where $\mathbf{M}$ is an $N \times 2N$ matrix with $m_{k,n} = \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$.

Quasi-Orthogonality: The $2N \times 2N$ matrix, $\frac{1}{N}\mathbf{M}^T\mathbf{M}$, is almost the identity:

$$\frac{1}{N}\mathbf{M}^T\mathbf{M} = \frac{1}{2}\begin{bmatrix} \mathbf{I}-\mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}+\mathbf{J} \end{bmatrix} \text{ with } \mathbf{I} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \mathbf{J} = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & ⋰ & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$$

MDCT (Modified DCT)

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le n < 2N$

If $\mathbf{x}$, $\mathbf{X}$ and $\mathbf{y}$ are column vectors, then $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X} = \frac{1}{N}\mathbf{M}^T\mathbf{M}\mathbf{x}$

where $\mathbf{M}$ is an $N \times 2N$ matrix with $m_{k,n} = \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$.

Quasi-Orthogonality: The $2N \times 2N$matrix, $\frac{1}{N}\mathbf{M}^T\mathbf{M}$, is almost the identity:

$$\frac{1}{N}\mathbf{M}^T\mathbf{M} = \frac{1}{2}\begin{bmatrix} \mathbf{I}-\mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}+\mathbf{J} \end{bmatrix} \text{ with } \mathbf{I} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \mathbf{J} = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & ⋰ & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$$

When two consective $\mathbf{y}$ frames are overlapped by N samples, the second half of the first frame has thus been multiplied by $\frac{1}{2}(\mathbf{I}+\mathbf{J})$ and the first half of the second frame by $\frac{1}{2}(\mathbf{I}-\mathbf{J})$. When these $\mathbf{y}$ frames are added together, the corresponding $\mathbf{x}$ samples have been multiplied by $\frac{1}{2}(\mathbf{I}+\mathbf{J}) + \frac{1}{2}(\mathbf{I}-\mathbf{J}) = \mathbf{I}$ giving perfect reconstruction.

MDCT (Modified DCT)

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\quad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\quad 0 \le n < 2N$

If $\mathbf{x}$, $\mathbf{X}$ and $\mathbf{y}$ are column vectors, then $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X} = \frac{1}{N}\mathbf{M}^T\mathbf{M}\mathbf{x}$

where $\mathbf{M}$ is an $N \times 2N$ matrix with $m_{k,n} = \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$.

Quasi-Orthogonality: The $2N \times 2N$matrix, $\frac{1}{N}\mathbf{M}^T\mathbf{M}$, is almost the identity:

$$\frac{1}{N}\mathbf{M}^T\mathbf{M} = \frac{1}{2}\begin{bmatrix} \mathbf{I}-\mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}+\mathbf{J} \end{bmatrix} \text{ with } \mathbf{I} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \mathbf{J} = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & ⋰ & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$$

When two consective $\mathbf{y}$ frames are overlapped by N samples, the second half of the first frame has thus been multiplied by $\frac{1}{2}(\mathbf{I}+\mathbf{J})$ and the first half of the second frame by $\frac{1}{2}(\mathbf{I}-\mathbf{J})$. When these $\mathbf{y}$ frames are added together, the corresponding $\mathbf{x}$ samples have been multiplied by $\frac{1}{2}(\mathbf{I}+\mathbf{J}) + \frac{1}{2}(\mathbf{I}-\mathbf{J}) = \mathbf{I}$ giving perfect reconstruction.

Normally the $2N$-long $\mathbf{x}$ and $\mathbf{y}$ frames are windowed before the MDCT and again after the IMDCT to avoid any discontinuities; if the window is symmetric and satisfies $w^2[i] + w^2[i+N] = 2$ the perfect reconstruction property is still true.

MDCT Basis Elements

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N} \qquad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N} \qquad 0 \le n < 2N$

MDCT Basis Elements

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le n < 2N$

In vector notation: $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X} = \frac{1}{N}\mathbf{M}^T\mathbf{M}\mathbf{x}$

MDCT Basis Elements

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\quad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\quad 0 \le n < 2N$

In vector notation: $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X} = \frac{1}{N}\mathbf{M}^T\mathbf{M}\mathbf{x}$

The rows of $\mathbf{M}$ form the MDCT basis elements.

MDCT Basis Elements

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\quad 0 \leq k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\quad 0 \leq n < 2N$

In vector notation: $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X} = \frac{1}{N}\mathbf{M}^T\mathbf{M}\mathbf{x}$

The rows of $\mathbf{M}$ form the MDCT basis elements.

Example ($N = 4$):

$$\mathbf{M} = \begin{bmatrix} 0.56 & 0.20 & -0.20 & -0.56 & -0.83 & -0.98 & -0.98 & -0.83 \\ -0.98 & -0.56 & 0.56 & 0.98 & 0.20 & -0.83 & -0.83 & 0.20 \\ 0.20 & 0.83 & -0.83 & -0.20 & 0.98 & -0.56 & -0.56 & 0.98 \\ 0.83 & -0.98 & 0.98 & -0.83 & 0.56 & -0.20 & -0.20 & 0.56 \end{bmatrix}$$

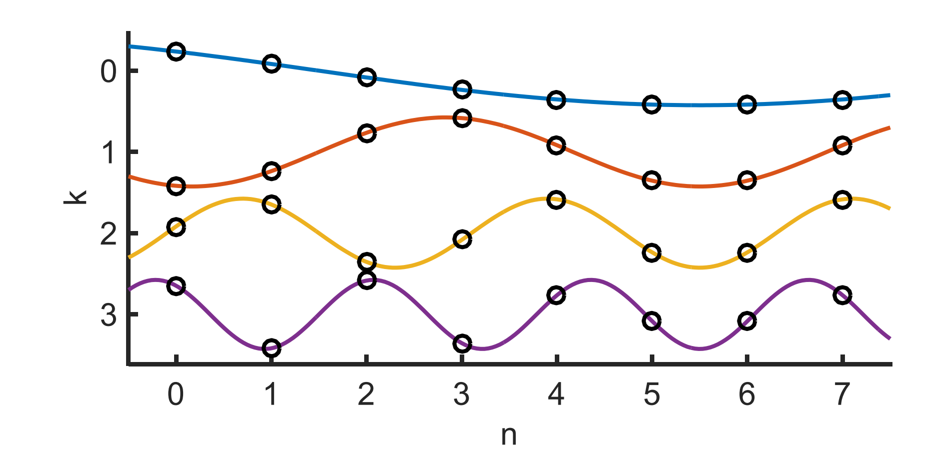
MDCT Basis Elements

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N} \qquad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N} \qquad 0 \le n < 2N$

In vector notation: $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X} = \frac{1}{N}\mathbf{M}^T\mathbf{M}\mathbf{x}$

The rows of $\mathbf{M}$ form the MDCT basis elements.



Example ($N = 4$):

$$\mathbf{M} = \begin{bmatrix} 0.56 & 0.20 & -0.20 & -0.56 & -0.83 & -0.98 & -0.98 & -0.83 \\ -0.98 & -0.56 & 0.56 & 0.98 & 0.20 & -0.83 & -0.83 & 0.20 \\ 0.20 & 0.83 & -0.83 & -0.20 & 0.98 & -0.56 & -0.56 & 0.98 \\ 0.83 & -0.98 & 0.98 & -0.83 & 0.56 & -0.20 & -0.20 & 0.56 \end{bmatrix}$$

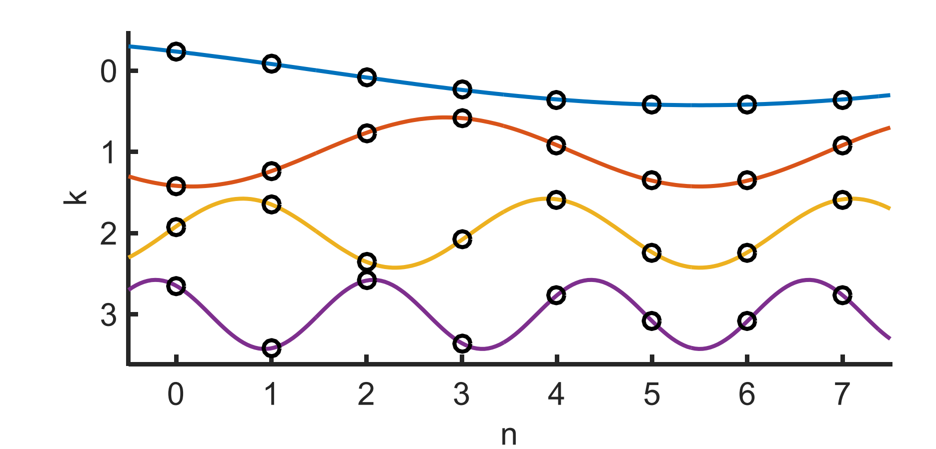
MDCT Basis Elements

MDCT: $X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le k < N$

IMDCT: $y[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$ $\qquad 0 \le n < 2N$

In vector notation: $\mathbf{X} = \mathbf{M}\mathbf{x}$ and $\mathbf{y} = \frac{1}{N}\mathbf{M}^T\mathbf{X} = \frac{1}{N}\mathbf{M}^T\mathbf{M}\mathbf{x}$

The rows of $\mathbf{M}$ form the MDCT basis elements.



Example ($N = 4$):

$$\mathbf{M} = \begin{bmatrix} 0.56 & 0.20 & -0.20 & -0.56 & -0.83 & -0.98 & -0.98 & -0.83 \\ -0.98 & -0.56 & 0.56 & 0.98 & 0.20 & -0.83 & -0.83 & 0.20 \\ 0.20 & 0.83 & -0.83 & -0.20 & 0.98 & -0.56 & -0.56 & 0.98 \\ 0.83 & -0.98 & 0.98 & -0.83 & 0.56 & -0.20 & -0.20 & 0.56 \end{bmatrix}$$

The basis frequencies are $\{0.5,\ 1.5,\ 2.5,\ 3.5\}$ times the fundamental.

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\left[\begin{array}{cc}\mathbf{x} & \overleftarrow{\mathbf{x}}\end{array}\right]$

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\left[\begin{array}{cc}\mathbf{x} & \overleftarrow{\mathbf{x}}\end{array}\right]$
- Often scaled to make an orthogonal transform (ODCT)

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\begin{bmatrix} \mathbf{x} & \overleftarrow{\mathbf{x}} \end{bmatrix}$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\left[\mathbf{x} \quad \overleftarrow{\mathbf{x}} \right]$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
  - Energy Compaction: Most energy is in only a few coefficients

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\left[\mathbf{x} \quad \overleftarrow{\mathbf{x}} \right]$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
  - Energy Compaction: Most energy is in only a few coefficients
  - Decorrelation: The coefficients are uncorrelated with each other

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\begin{bmatrix} \mathbf{x} & \overleftarrow{\mathbf{x}} \end{bmatrix}$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
  - Energy Compaction: Most energy is in only a few coefficients
  - Decorrelation: The coefficients are uncorrelated with each other
- Nice convolution property of DFT is lost ☹

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\begin{bmatrix} \mathbf{x} & \overleftarrow{\mathbf{x}} \end{bmatrix}$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
  - Energy Compaction: Most energy is in only a few coefficients
  - Decorrelation: The coefficients are uncorrelated with each other
- Nice convolution property of DFT is lost ☹

MDCT: Modified Discrete Cosine Transform

- Lapped transform: $2N \to N \to 2N$

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\left[\mathbf{x} \quad \overleftarrow{\mathbf{x}}\right]$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
  - Energy Compaction: Most energy is in only a few coefficients
  - Decorrelation: The coefficients are uncorrelated with each other
- Nice convolution property of DFT is lost ☹

MDCT: Modified Discrete Cosine Transform

- Lapped transform: $2N \rightarrow N \rightarrow 2N$
- Aliasing errors cancel out when overlapping output frames are added

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\begin{bmatrix} \mathbf{x} & \overleftarrow{\mathbf{x}} \end{bmatrix}$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
    - Energy Compaction: Most energy is in only a few coefficients
    - Decorrelation: The coefficients are uncorrelated with each other
- Nice convolution property of DFT is lost ☹

MDCT: Modified Discrete Cosine Transform

- Lapped transform: $2N \to N \to 2N$
- Aliasing errors cancel out when overlapping output frames are added
- Similar to DCT for energy compaction and decorrelation ☺

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\left[\mathbf{x} \quad \overleftarrow{\mathbf{x}} \right]$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
    - Energy Compaction: Most energy is in only a few coefficients
    - Decorrelation: The coefficients are uncorrelated with each other
- Nice convolution property of DFT is lost ☹

MDCT: Modified Discrete Cosine Transform

- Lapped transform: $2N \rightarrow N \rightarrow 2N$
- Aliasing errors cancel out when overlapping output frames are added
- Similar to DCT for energy compaction and decorrelation ☺
- Overlapping windowed frames can avoid edge discontinuities ☺

Summary

DCT: Discrete Cosine Transform

- Equivalent to a DFT of time-shifted double-length $\left[\begin{array}{cc} \mathbf{x} & \overleftarrow{\mathbf{x}} \end{array}\right]$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
  - Energy Compaction: Most energy is in only a few coefficients
  - Decorrelation: The coefficients are uncorrelated with each other
- Nice convolution property of DFT is lost ☹

MDCT: Modified Discrete Cosine Transform

- Lapped transform: $2N \rightarrow N \rightarrow 2N$
- Aliasing errors cancel out when overlapping output frames are added
- Similar to DCT for energy compaction and decorrelation ☺
- Overlapping windowed frames can avoid edge discontinuities ☺
- Used in audio coding: MP3, WMA, AC-3, AAC, Vorbis, ATRAC

Summary

DCT: Discrete Cosine Transform
- Equivalent to a DFT of time-shifted double-length $\left[\begin{array}{cc}\mathbf{x} & \overleftarrow{\mathbf{x}}\end{array}\right]$
- Often scaled to make an orthogonal transform (ODCT)
- Better than DFT for energy compaction and decorrelation ☺
  - Energy Compaction: Most energy is in only a few coefficients
  - Decorrelation: The coefficients are uncorrelated with each other
- Nice convolution property of DFT is lost ☹

MDCT: Modified Discrete Cosine Transform
- Lapped transform: $2N \rightarrow N \rightarrow 2N$
- Aliasing errors cancel out when overlapping output frames are added
- Similar to DCT for energy compaction and decorrelation ☺
- Overlapping windowed frames can avoid edge discontinuities ☺
- Used in audio coding: MP3, WMA, AC-3, AAC, Vorbis, ATRAC

For further details see Mitra: 5.

MATLAB routines

dct, idct	ODCT with optional zero-padding