

6: Window Filter

▷ Design

Inverse DTFT

Rectangular window

Dirichlet Kernel +

Window relationships

Common Windows

Order Estimation

Example Design

Frequency sampling

Summary

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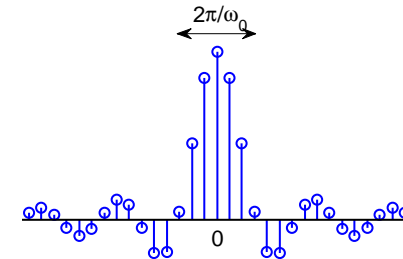
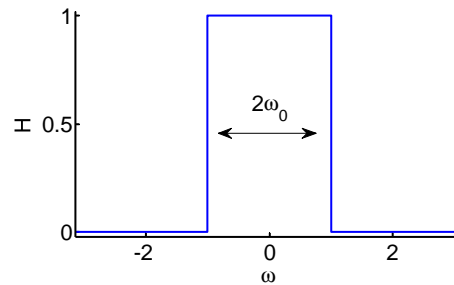
For any BIBO stable filter, $H(e^{j\omega})$ is the DTFT of $h[n]$

$$H(e^{j\omega}) = \sum_{-\infty}^{\infty} h[n]e^{-j\omega n} \Leftrightarrow h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n} d\omega$$

If we know $H(e^{j\omega})$ exactly, the IDTFT gives the ideal $h[n]$

Example: Ideal Lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases} \Leftrightarrow h[n] = \frac{\sin \omega_0 n}{\pi n}$$



Note: Width in ω is $2\omega_0$, width in n is $\frac{2\pi}{\omega_0}$: **product is 4π always**

Sadly $h[n]$ is **infinite** and **non-causal**. **Solution:** multiply $h[n]$ by a window

Rectangular window

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Truncate to $\pm \frac{M}{2}$ to make finite; $h_1[n]$ is now of length $M + 1$

MSE Optimality:

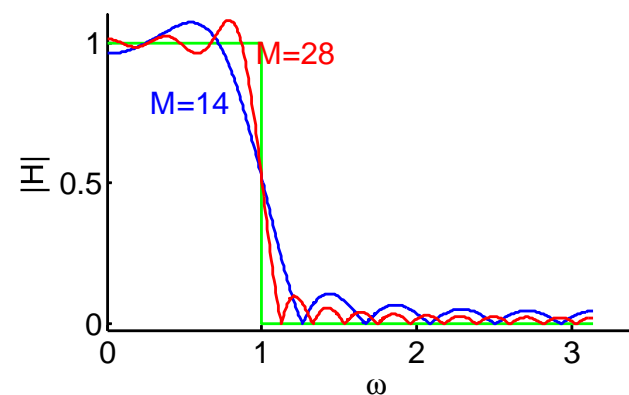
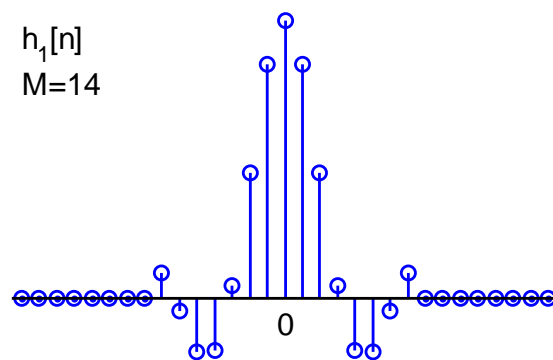
Define mean square error (MSE) in frequency domain

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega}) - H_1(e^{j\omega})|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H(e^{j\omega}) - \sum_{-\frac{M}{2}}^{\frac{M}{2}} h_1[n] e^{-j\omega n} \right|^2 d\omega \end{aligned}$$

Minimum E is when $h_1[n] = h[n]$.

Proof: From Parseval: $E = \sum_{-\frac{M}{2}}^{\frac{M}{2}} |h[n] - h_1[n]|^2 + \sum_{|n| > \frac{M}{2}} |h[n]|^2$

However: 9% overshoot at a discontinuity even for large n .



Normal to delay by $\frac{M}{2}$ to make causal. Multiplies $H(e^{j\omega})$ by $e^{-j\frac{M}{2}\omega}$.

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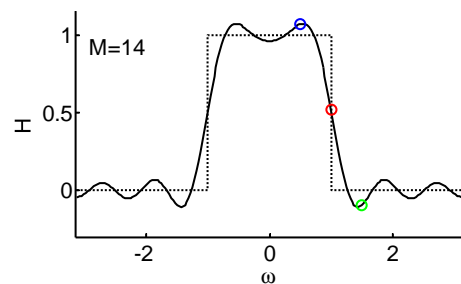
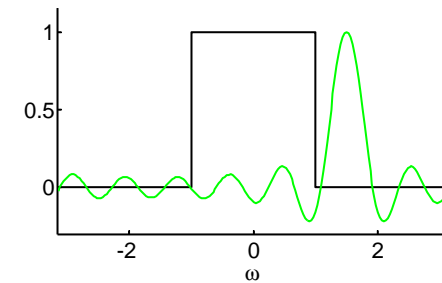
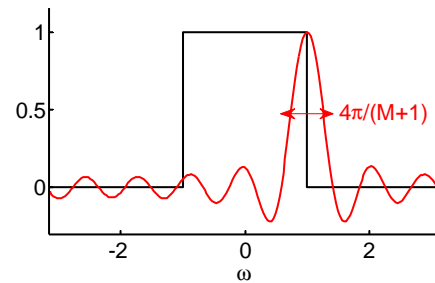
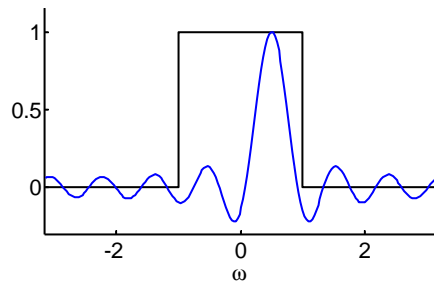
Truncation \Leftrightarrow Multiply $h[n]$ by a rectangular window, $w[n] = \delta_{-\frac{M}{2} \leq n \leq \frac{M}{2}}$

\Leftrightarrow **Circular Convolution** $H_{M+1}(e^{j\omega}) = \frac{1}{2\pi} H(e^{j\omega}) \circledast W(e^{j\omega})$

$$W(e^{j\omega}) = \sum_{-\frac{M}{2}}^{\frac{M}{2}} e^{-j\omega n} \stackrel{(i)}{=} 1 + 2 \sum_1^{0.5M} \cos(n\omega) \stackrel{(ii)}{=} \frac{\sin 0.5(M+1)\omega}{\sin 0.5\omega}$$

Proof: (i) $e^{-j\omega(-n)} + e^{-j\omega(+n)} = 2 \cos(n\omega)$ (ii) Sum geom. progression

Effect: convolve ideal freq response with **Dirichlet kernel (aliased sinc)**



Provided that $\frac{4\pi}{M+1} \ll 2\omega_0 \Leftrightarrow M+1 \gg \frac{2\pi}{\omega_0}$:

Passband ripple: $\Delta\omega \approx \frac{4\pi}{M+1}$, stopband $\frac{2\pi}{M+1}$

Transition pk-to-pk: $\Delta\omega \approx \frac{4\pi}{M+1}$

Transition Gradient: $\left. \frac{d|H|}{d\omega} \right|_{\omega=\omega_0} \approx \frac{M+1}{2\pi}$

[Dirichlet Kernel]

Other properties of $W(e^{j\omega})$:

The DTFT of a symmetric rectangular window of length $M + 1$ is $W(e^{j\omega}) = \sum_{-\frac{M}{2}}^{\frac{M}{2}} e^{-j\omega n} = e^{j\omega \frac{M}{2}} \sum_0^M e^{-j\omega n} = e^{j\omega \frac{M}{2}} \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} = \frac{e^{j0.5\omega(M+1)} - e^{-j0.5\omega(M+1)}}{e^{j0.5\omega} - e^{-j0.5\omega}} = \frac{\sin 0.5(M+1)\omega}{\sin 0.5\omega}$.

For small x we can approximate $\sin x \approx x$; the error is $< 1\%$ for $x < 0.25$. So, for $\omega < 0.5$, we have $W(e^{j\omega}) \approx 2\omega^{-1} \sin 0.5(M+1)\omega$.

The peak value is at $\omega = 0$ and equals $M + 1$; this means that the peak gradient of $H_{M+1}(e^{j\omega})$ will be $\frac{M+1}{2\pi}$.

The minimum value of $W(e^{j\omega})$ is approximately equal to the minimum of $2\omega^{-1} \sin 0.5(M+1)\omega$ which is when $\sin 0.5(M+1)\omega = -1$ i.e. when $\omega = \frac{1.5\pi}{0.5(M+1)} = \frac{3\pi}{M+1}$.

Hence $\min W(e^{j\omega}) \approx \min 2\omega^{-1} \sin 0.5(M+1)\omega = -\frac{M+1}{1.5\pi}$.

Passband and Stopband ripple:

The ripple in $W(e^{j\omega}) = \frac{\sin 0.5(+1)\omega}{\sin 0.5\omega}$ has a period of $\Delta\omega = \frac{2\pi}{0.5(+1)} = \frac{4\pi}{M+1}$ and this gives rise to ripple with this period in both the passband and stopband of $H_{M+1}(e^{j\omega})$.

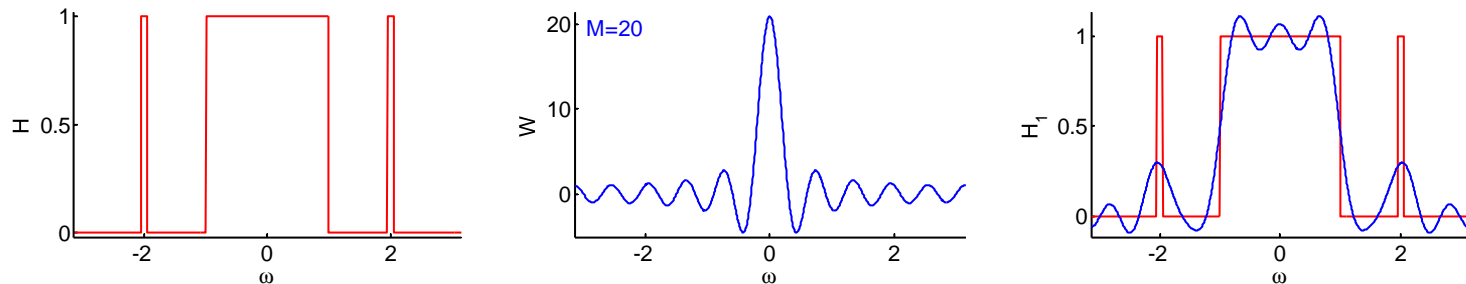
However the stopband ripple takes the value of $H_{M+1}(e^{j\omega})$ alternately positive and negative. If you plot the magnitude response, $|H_{M+1}(e^{j\omega})|$ then this ripple will be full-wave rectified and will double in frequency so its period will now be $\frac{2\pi}{M+1}$.

Window relationships

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When you multiply an impulse response by a window $M + 1$ long

$$H_{M+1}(e^{j\omega}) = \frac{1}{2\pi} H(e^{j\omega}) \circledast W(e^{j\omega})$$



(a) passband gain $\approx w[0]$; peak $\approx \frac{w[0]}{2} + \frac{0.5}{2\pi} \int_{\text{mainlobe}} W(e^{j\omega}) d\omega$
 rectangular window: passband gain = 1; peak gain = 1.09

(b) transition bandwidth, $\Delta\omega =$ width of the main lobe
 transition amplitude, $\Delta H =$ integral of main lobe $\div 2\pi$
 rectangular window: $\Delta\omega = \frac{4\pi}{M+1}$, $\Delta H \approx 1.18$

(c) stopband gain is an integral over oscillating sidelobes of $W(e^{j\omega})$
 rect window: $|\min H(e^{j\omega})| = 0.09 \ll |\min W(e^{j\omega})| = \frac{M+1}{1.5\pi}$

(d) features narrower than the main lobe will be broadened and attenuated

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Rectangular: $w[n] \equiv 1$
 don't use

Hanning: $0.5 + 0.5c_1$
 $c_k = \cos \frac{2\pi kn}{M+1}$
 rapid sidelobe decay

Hamming: $0.54 + 0.46c_1$
 best peak sidelobe

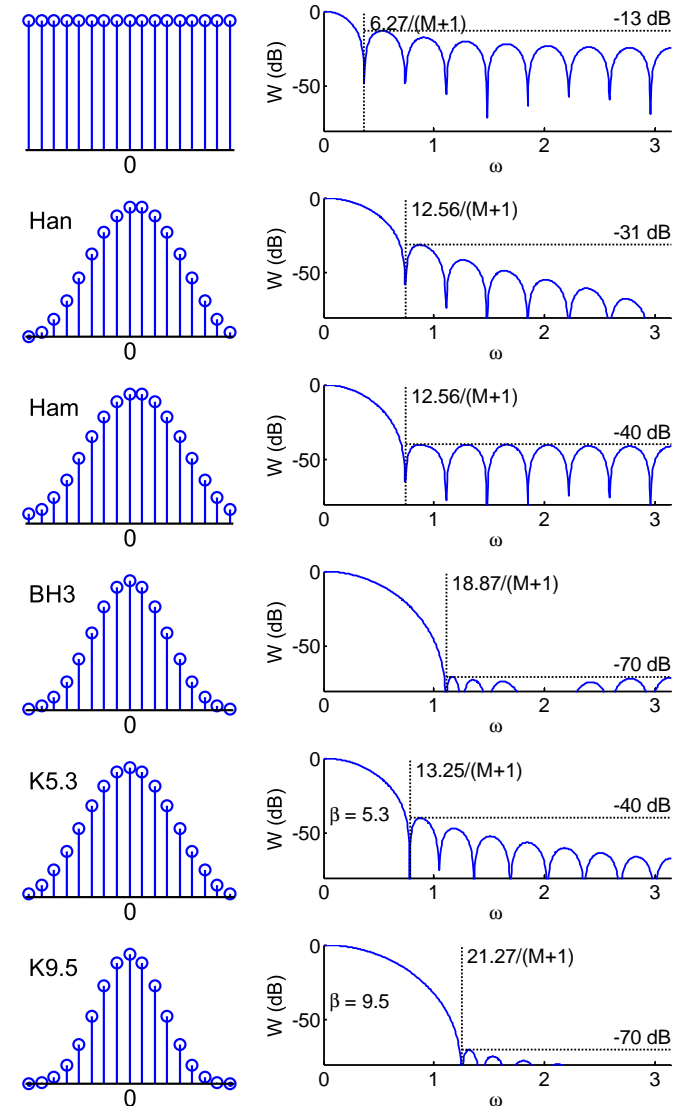
Blackman-Harris 3-term:
 $0.42 + 0.5c_1 + 0.08c_2$
 best peak sidelobe

Kaiser: $\frac{I_0\left(\beta\sqrt{1-\left(\frac{2n}{M}\right)^2}\right)}{I_0(\beta)}$

β controls width v sidelobes

Good compromise:

Width v sidelobe v decay



Order Estimation

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Several formulae estimate the required order of a filter, M .

E.g. for lowpass filter

Estimated order is

$$M \approx \frac{-5.6 - 4.3 \log_{10}(\delta\epsilon)}{\omega_2 - \omega_1} \approx \frac{-8 - 20 \log_{10} \epsilon}{2.2\Delta\omega}$$

Required M increases as either the transition width, $\omega_2 - \omega_1$, or the gain tolerances δ and ϵ get smaller.

Only approximate.

Example:

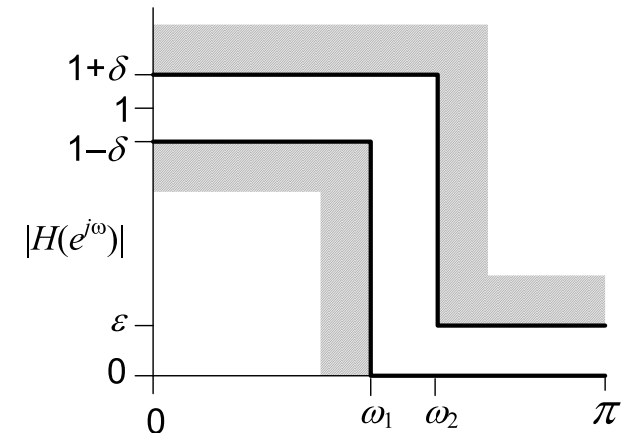
Transition band: $f_1 = 1.8$ kHz, $f_2 = 2.0$ kHz, $f_s = 12$ kHz,

$$\omega_1 = \frac{2\pi f_1}{f_s} = 0.943, \quad \omega_2 = \frac{2\pi f_2}{f_s} = 1.047$$

Ripple: $20 \log_{10}(1 + \delta) = 0.1$ dB, $20 \log_{10} \epsilon = -35$ dB

$$\delta = 10^{\frac{0.1}{20}} - 1 = 0.0116, \quad \epsilon = 10^{\frac{-35}{20}} = 0.0178$$

$$M \approx \frac{-5.6 - 4.3 \log_{10}(2 \times 10^{-4})}{1.047 - 0.943} = \frac{10.25}{0.105} = 98 \quad \text{or} \quad \frac{35 - 8}{2.2\Delta\omega} = 117$$



Example Design

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Specifications:

Bandpass: $\omega_1 = 0.5$, $\omega_2 = 1$

Transition bandwidth: $\Delta\omega = 0.1$

Ripple: $\delta = \epsilon = 0.02$

$$20 \log_{10} \epsilon = -34 \text{ dB}$$

$$20 \log_{10} (1 + \delta) = 0.17 \text{ dB}$$

Order:

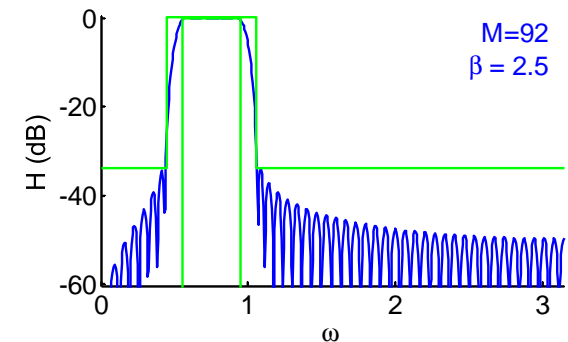
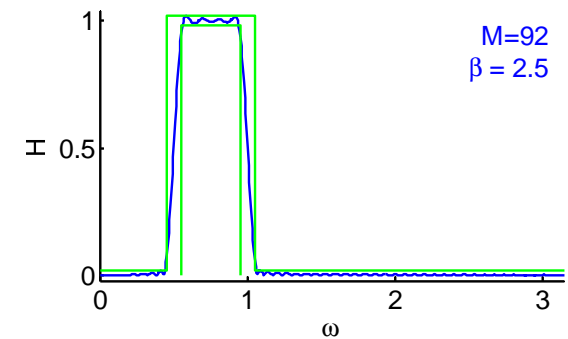
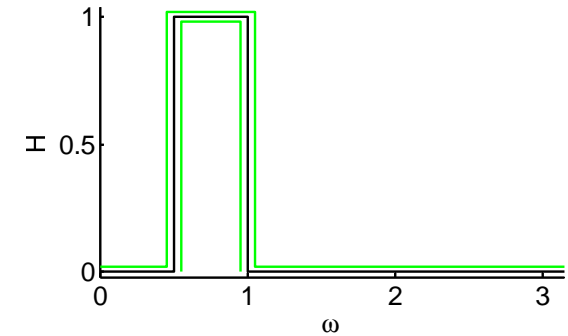
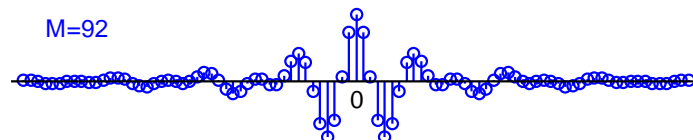
$$M \approx \frac{-5.6 - 4.3 \log_{10}(\delta\epsilon)}{\omega_2 - \omega_1} = 92$$

Ideal Impulse Response:

Difference of two lowpass filters

$$h[n] = \frac{\sin \omega_2 n}{\pi n} - \frac{\sin \omega_1 n}{\pi n}$$

Kaiser Window: $\beta = 2.5$



Frequency sampling

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Take $M + 1$ uniform samples of $H(e^{j\omega})$; take IDFT to obtain $h[n]$

Advantage:

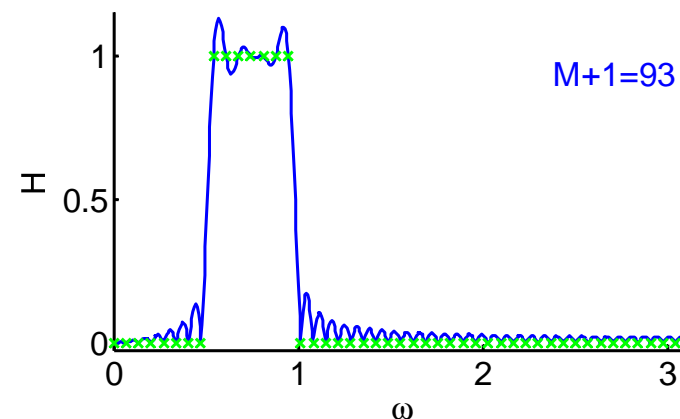
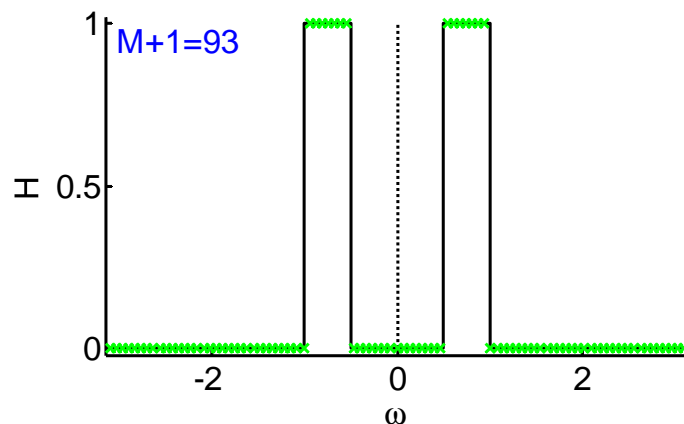
exact match at sample points

Disadvantage:

poor intermediate approximation if spectrum is varying rapidly

Solutions:

- (1) make the filter transitions smooth over $\Delta\omega$ width
- (2) oversample and do least squares fit (can't use IDFT)
- (3) use non-uniform points with more near transition (can't use IDFT)



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- Make an FIR filter by windowing the IDTFT of the ideal response
 - Ideal lowpass has $h[n] = \frac{\sin \omega_0 n}{\pi n}$
 - Add/subtract lowpass filters to make any piecewise constant response
- Ideal filter response is \circledast with the DTFT of the window
 - Rectangular window ($W(z) = \text{Dirichlet kernel}$) has -13 dB sidelobes and is always a bad idea
 - Hamming, Blackman-Harris are good
 - Kaiser good with β trading off main lobe width v. sidelobes
- **Uncertainty principle:** cannot be concentrated in both time and frequency
- **Frequency sampling:** IDFT of uniform frequency samples: not so great

For further details see Mitra: 7, 10.

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| | |
|------------------------------|--|
| diric(x,n) | Dirichlet kernel: $\frac{\sin 0.5nx}{\sin 0.5x}$ |
| hanning hamming kaiser | Window functions (Note 'periodic' option) |
| kaiserord | Estimate required filter order and β |