7: Optimal FIR filters

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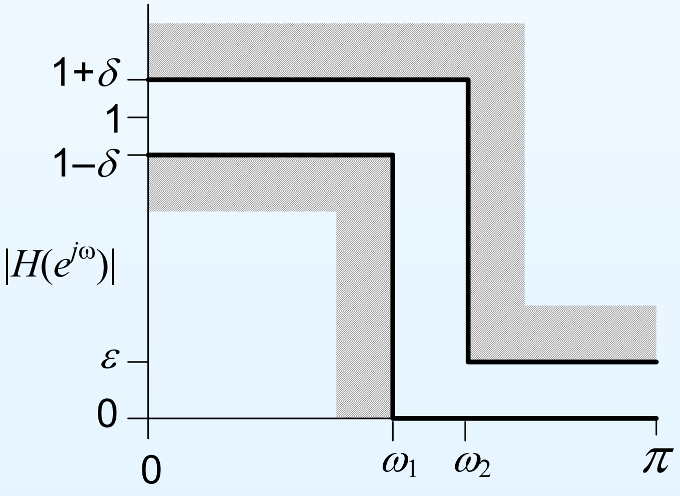

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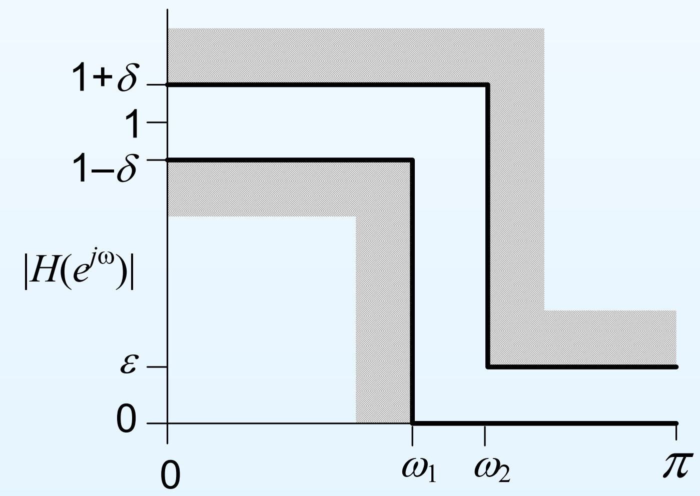

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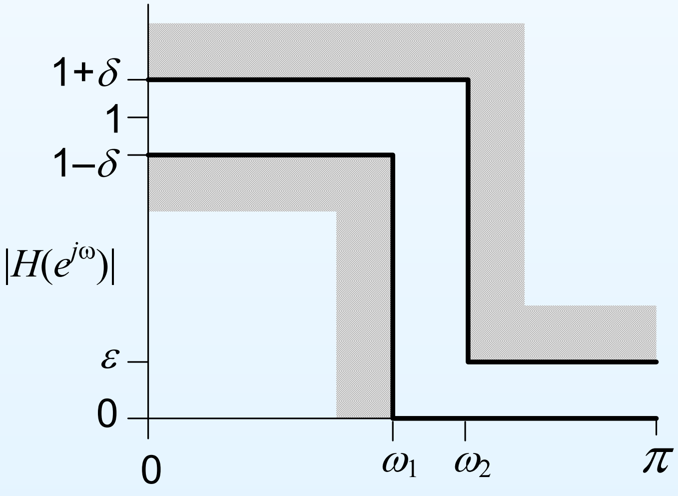

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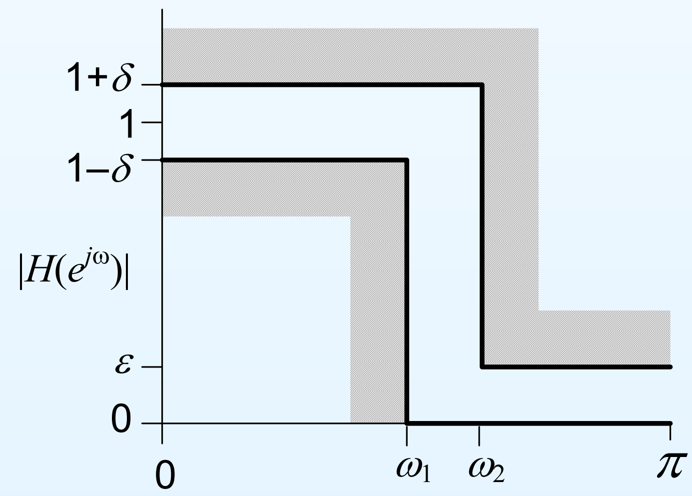

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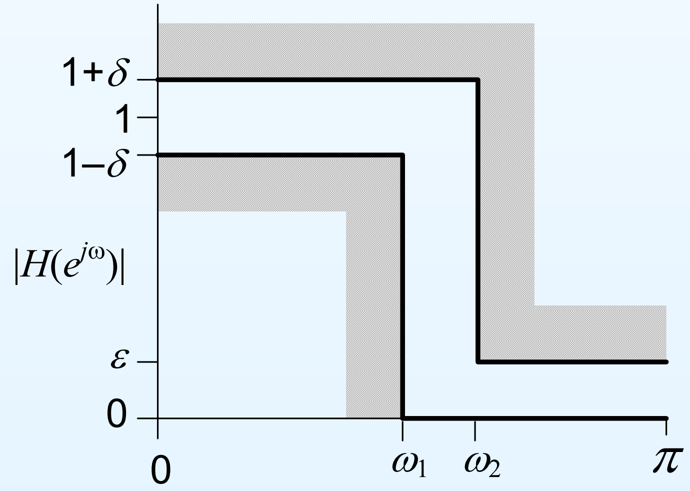

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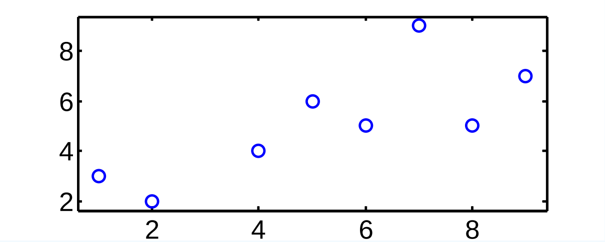
Minimax criterion: $h[n] = \arg\min_{h[n]} \max_{\omega} |e(\omega)|$: minimize max error

Alternation Theorem

Want to find the best fit line: with the smallest maximal error.



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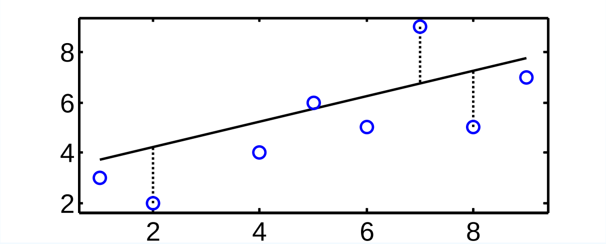
Best fit line always attains the maximal error three times with alternate signs



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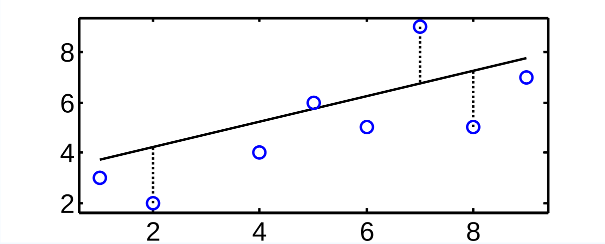

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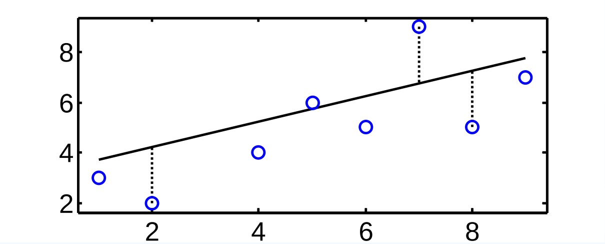

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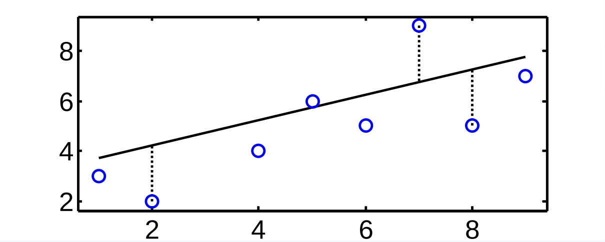

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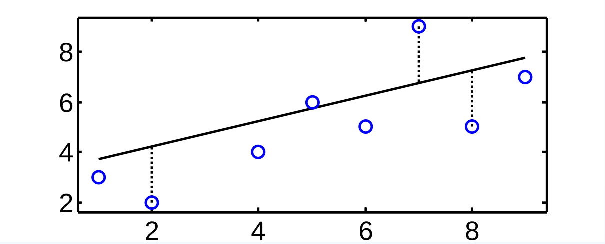

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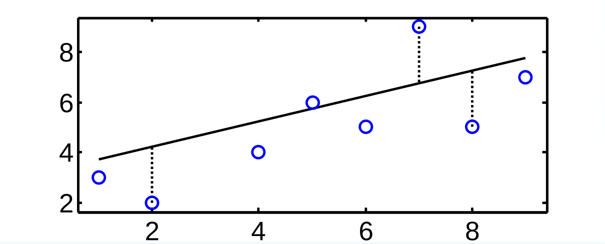

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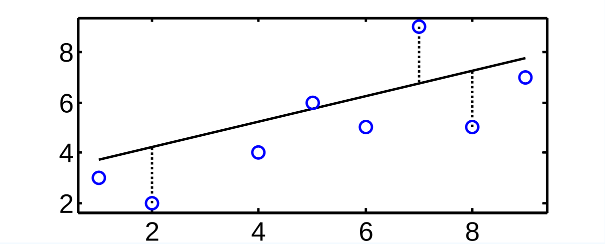

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Fitting to a continuous function is the same as to an infinite number of points.

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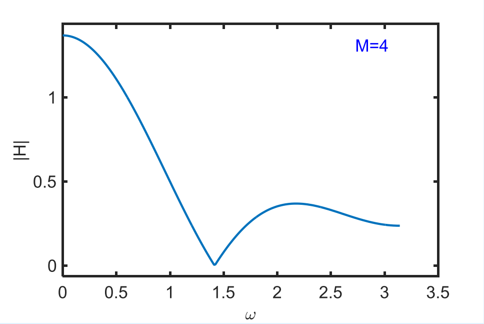

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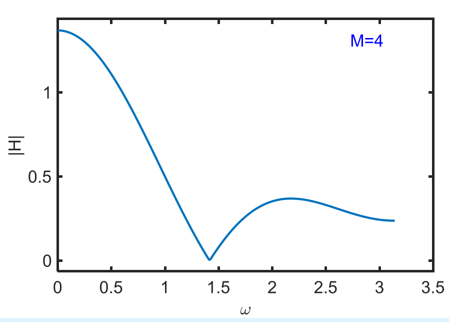

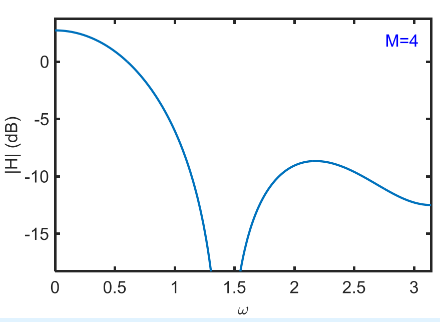

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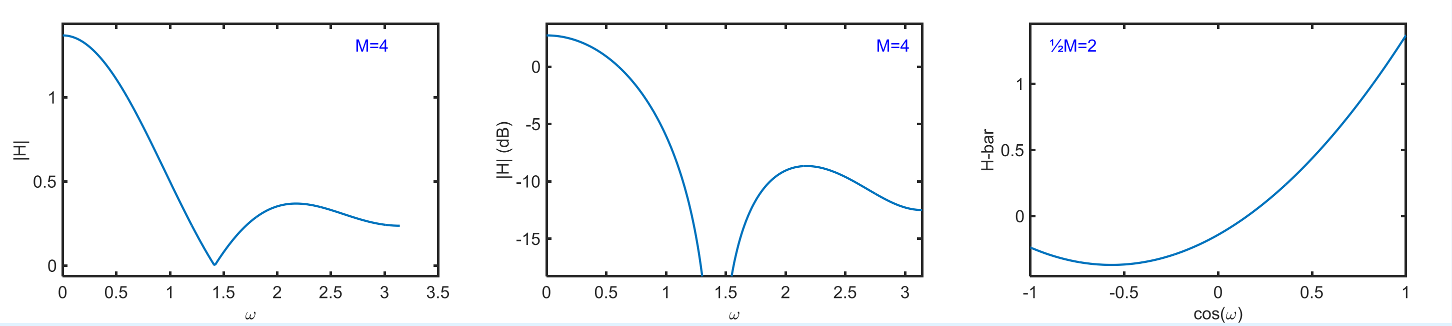

Maximal Error Locations

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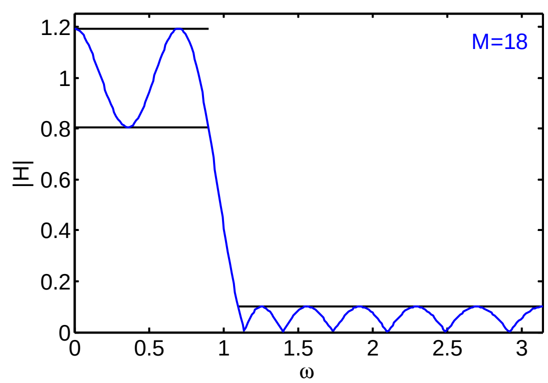

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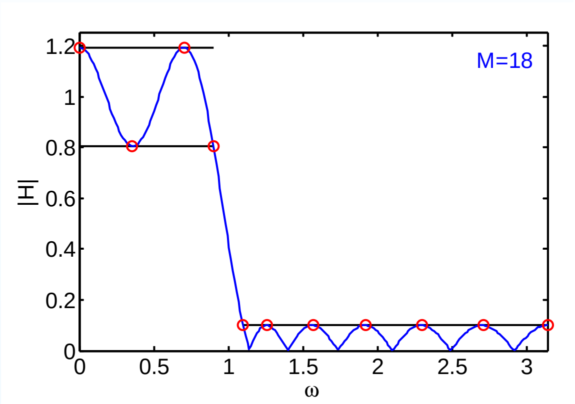

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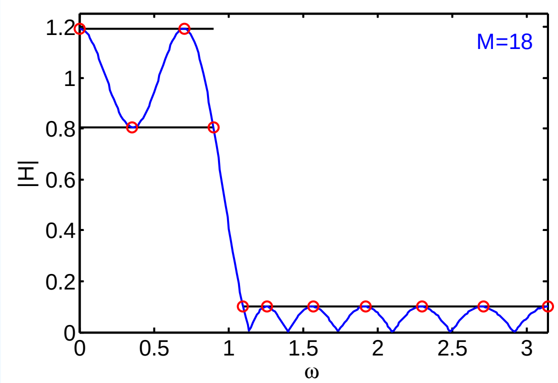

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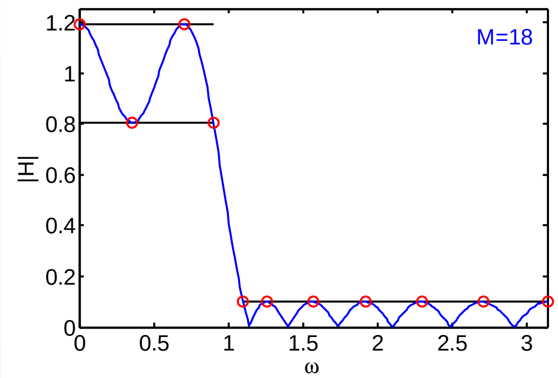

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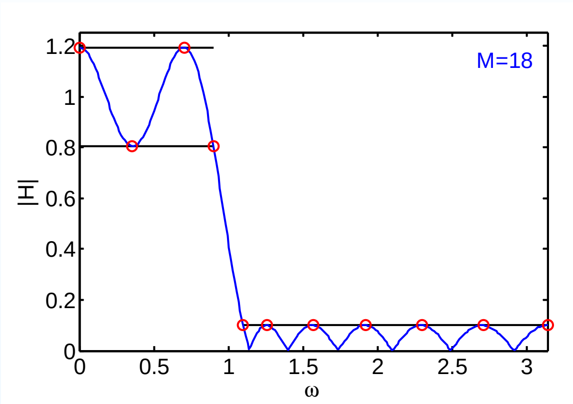

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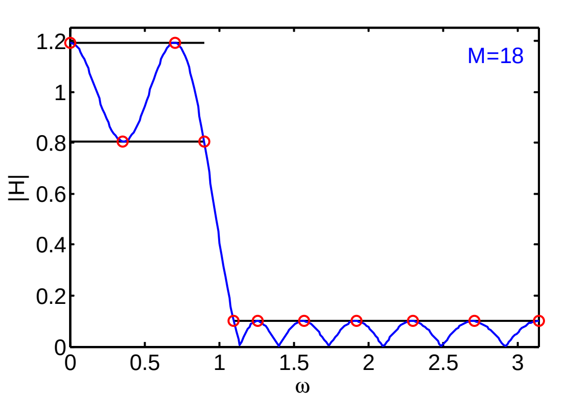

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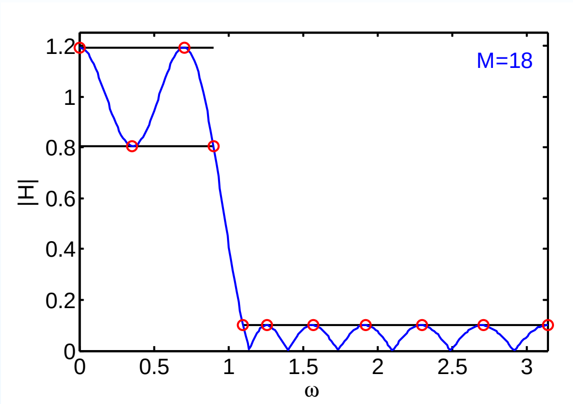

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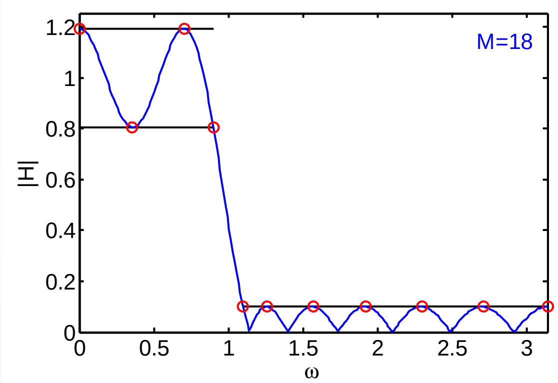

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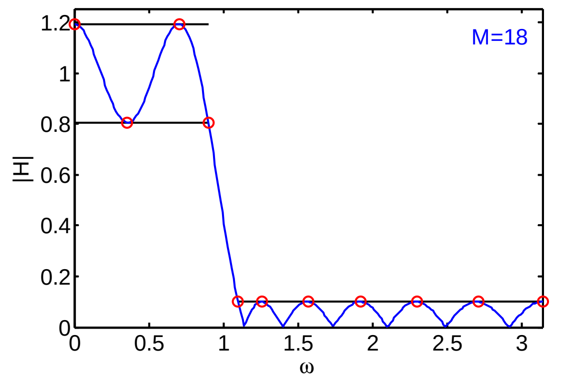

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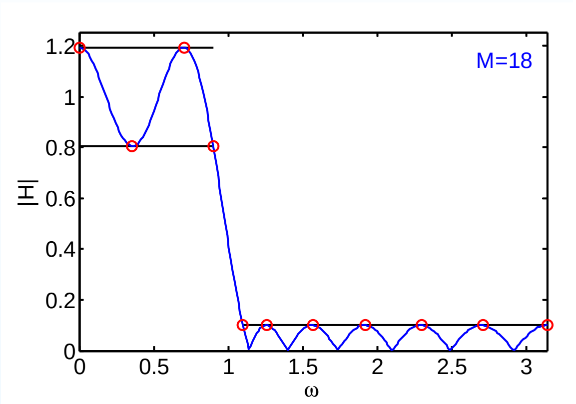

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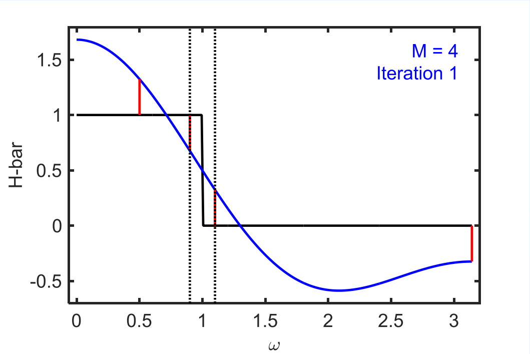

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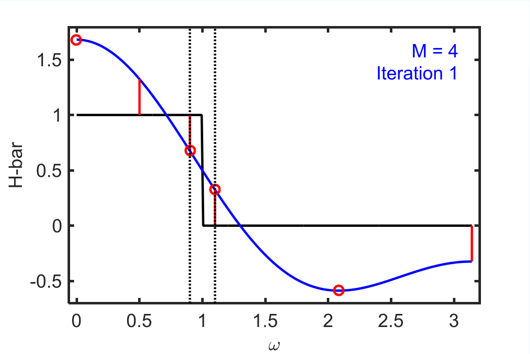

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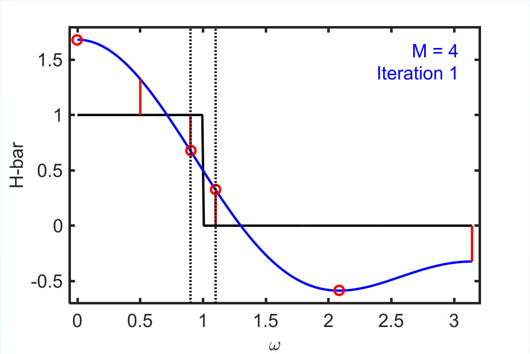

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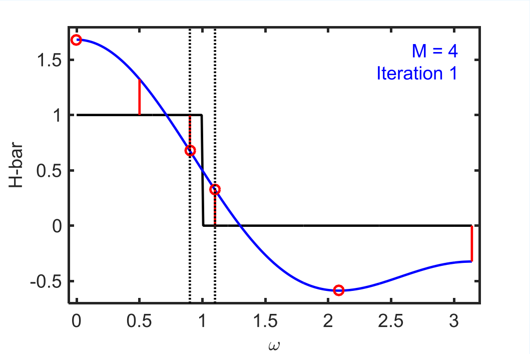

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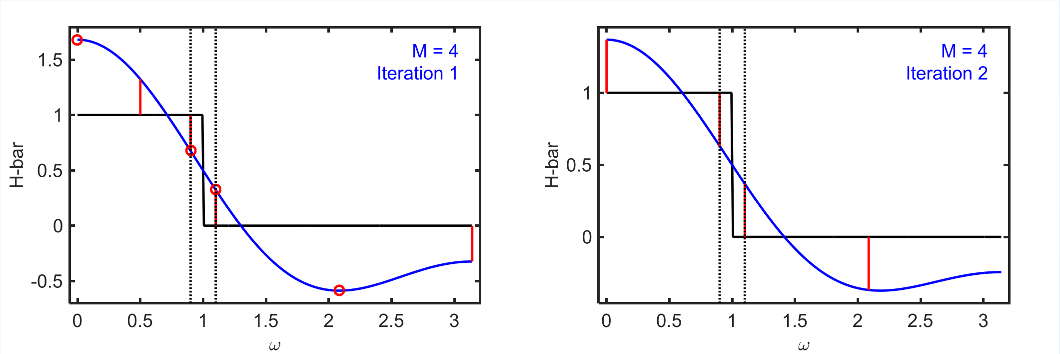

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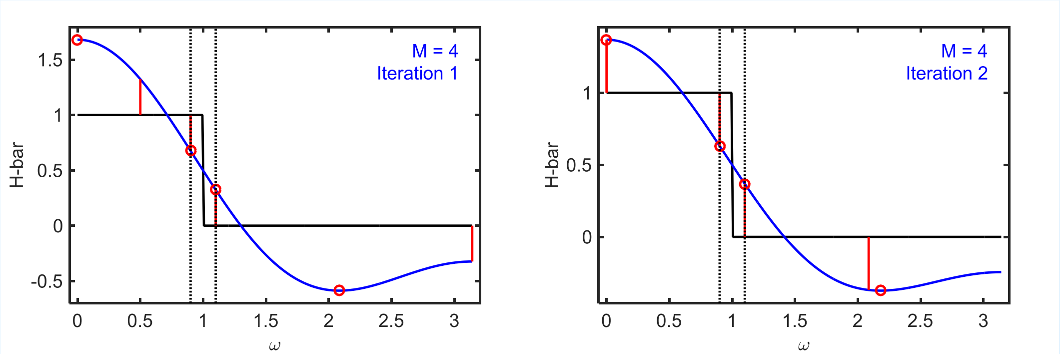

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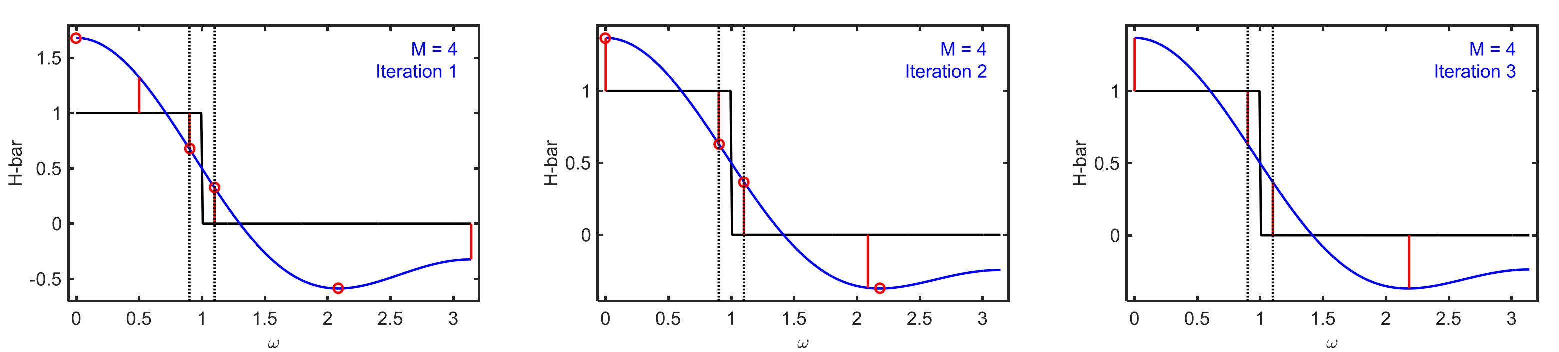

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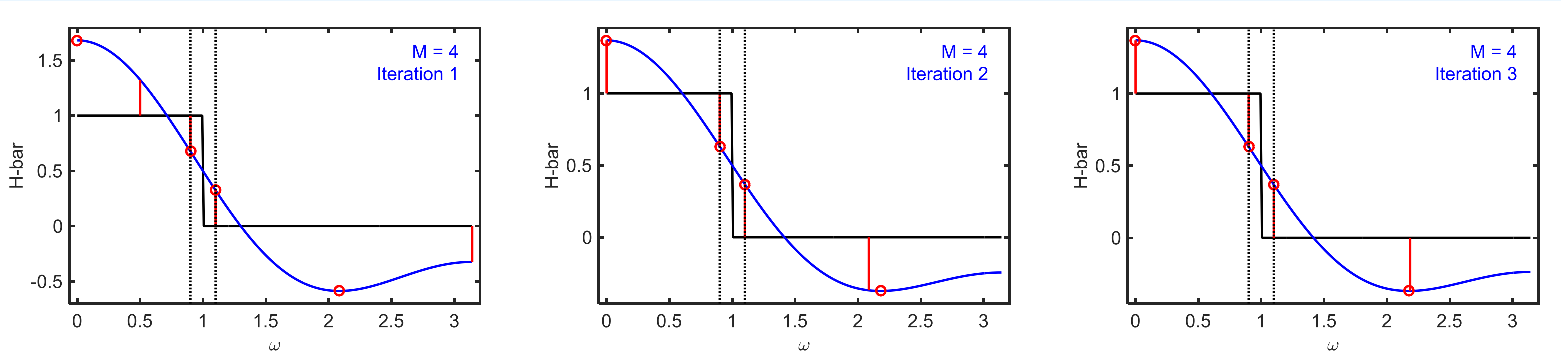

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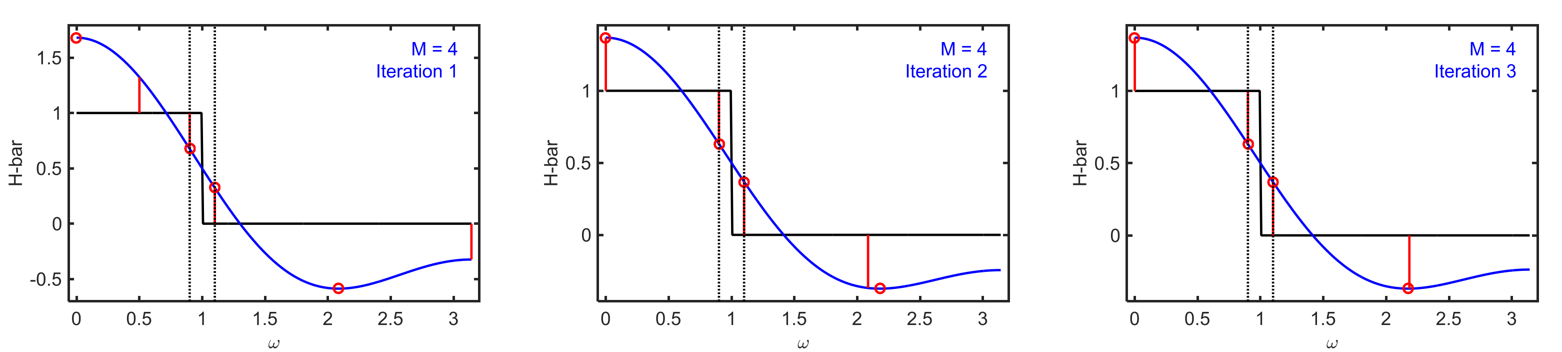
5. **Evaluate** $\overline{H}(\omega)$ on $M+1$ evenly spaced ω and do an **IDFT** to get $h[n]$.



Remex Step 2: Determine Polynomial

For each extremal frequency, ω_i for $1 \le i \le \frac{M}{2} + 2$

$$d(\omega_i) = \overline{H}(\omega_i) + \frac{(-1)^i \epsilon}{s(\omega_i)}$$

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$\left(\frac{M}{2} + 1\right)$-polynomial going through all the $\overline{H}(\omega_i)$ [actually order $\frac{M}{2}$]

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$$\sum_{i=1}^{\frac{M}{2}+2} \frac{(-1)^i c_i}{s(\omega_i)} \epsilon = \sum_{i=1}^{\frac{M}{2}+2} c_i d(\omega_i)$$

Solve for ϵ then calculate the $\overline{H}(\omega_i)$ then use Lagrange interpolation:

$$\overline{H}(\omega) = P(\cos\omega) = \sum_{i=1}^{\frac{M}{2}+2} \overline{H}(\omega_i) \prod_{j \ne i} \frac{\cos\omega - \cos\omega_j}{\cos\omega_i - \cos\omega_j}$$

$\left(\frac{M}{2} + 1\right)$-polynomial going through all the $\overline{H}(\omega_i)$ [actually order $\frac{M}{2}$]

Example Design

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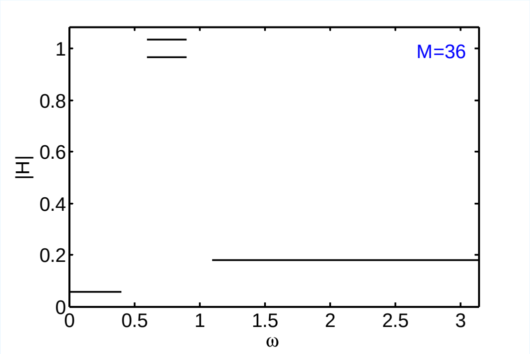
Bandpass $\omega = [0.5,\ 1]$,



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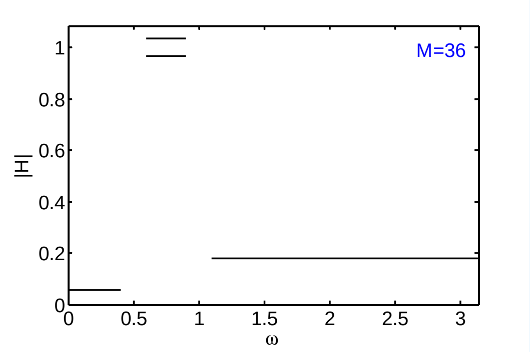

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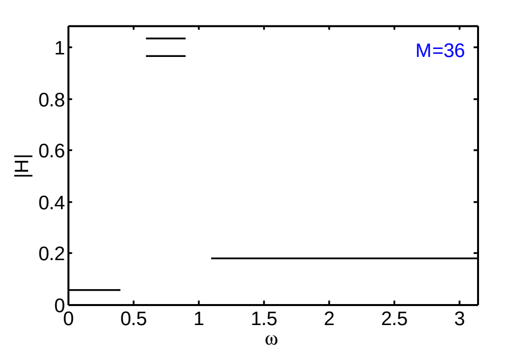

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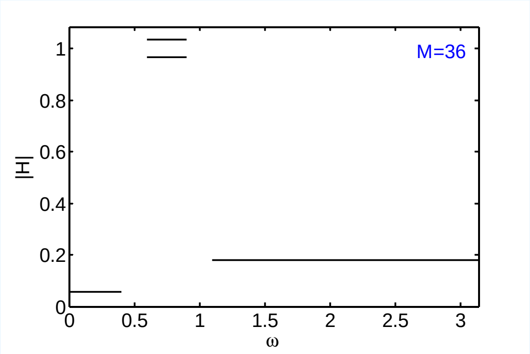

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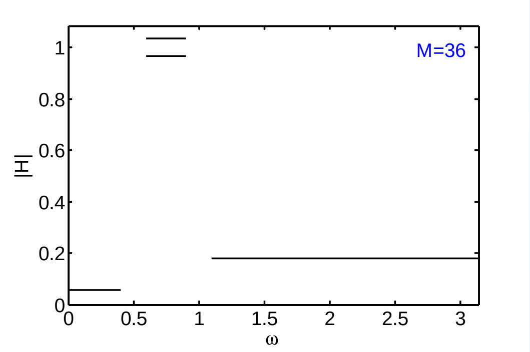

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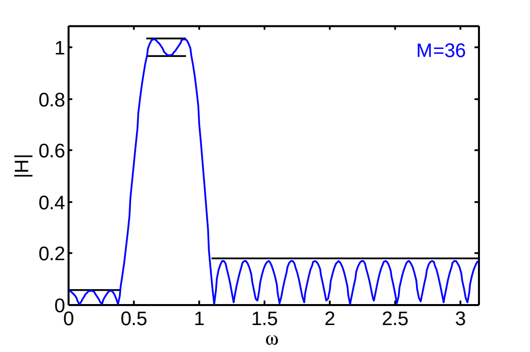

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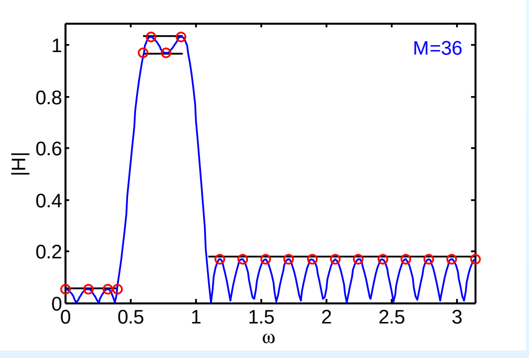

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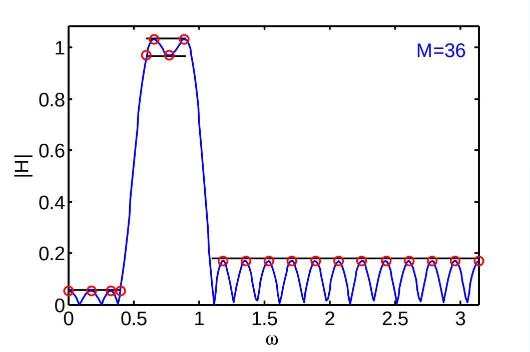

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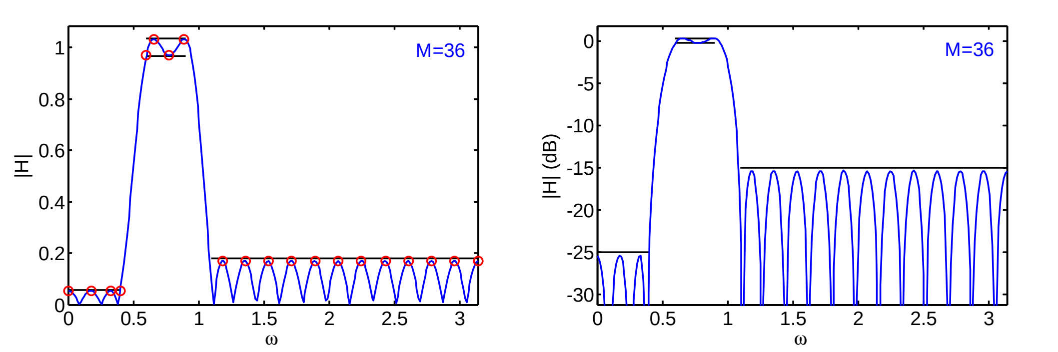

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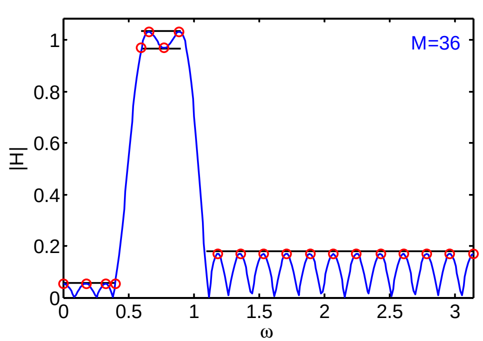

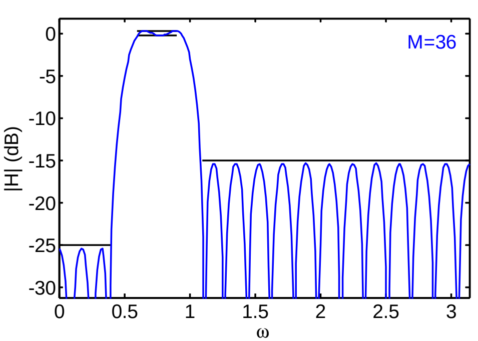

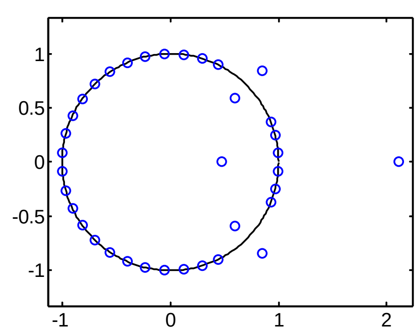

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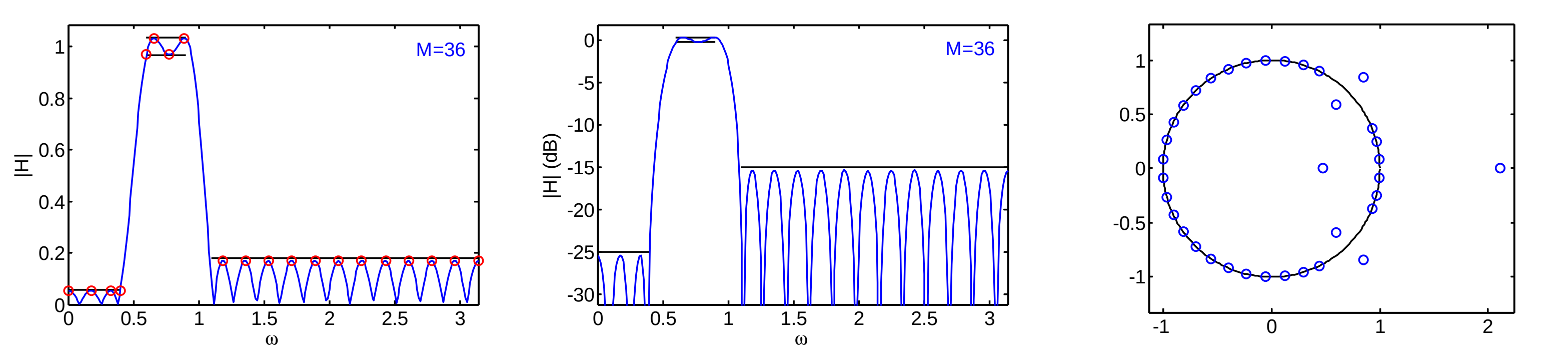

FIR Pros and Cons

- Can have linear phase
  - no envelope distortion, all frequencies have the same delay ☺
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For further details see Mitra: 10.

MATLAB routines

firpm	optimal FIR filter design
firpmord	estimate require order for firpm
cfirpm	arbitrary-response filter design
remez	[obsolete] optimal FIR filter design