9: Optimal IIR Design

Error choices

We want to find a filter $H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})}$ that approximates a target response $D(\omega)$. Assume A is order N and B is order M.

Two possible error measures:

Solution Error: $E_S(\omega) = W_S(\omega)\left(\frac{B(e^{j\omega})}{A(e^{j\omega})} - D(\omega)\right)$

Equation Error: $E_E(\omega) = W_E(\omega)\left(B(e^{j\omega}) - D(\omega)A(e^{j\omega})\right)$

We may know $D(\omega)$ completely or else only $|D(\omega)|$

We minimize $\int_{-\pi}^{\pi} |E_*(\omega)|^p \, d\omega$
where $p = 2$ (least squares) or ∞ (minimax).

Weight functions $W_*(\omega)$ are chosen to control relative errors at different frequencies. $W_S(\omega) = |D(\omega)|^{-1}$ gives constant dB error.

We actually want to minimize E_S but E_E is easier because it gives rise to linear equations.

However if $W_E(\omega) = \frac{W_S(\omega)}{|A(e^{j\omega})|}$, then $|E_E(\omega)| = |E_S(\omega)|$

Overdetermined set of equations $\mathbf{Ax} = \mathbf{b}$ (#equations > #unknowns)

We want to minimize $||\mathbf{e}||^2$ where $\mathbf{e} = \mathbf{Ax} - \mathbf{b}$

$$||\mathbf{e}||^2 = \mathbf{e}^T\mathbf{e} = \left(\mathbf{x}^T\mathbf{A}^T - \mathbf{b}^T\right)(\mathbf{Ax} - \mathbf{b})$$

Differentiate with respect to $\mathbf{x}$:

$$d\left(\mathbf{e}^T\mathbf{e}\right) = d\mathbf{x}^T\mathbf{A}^T(\mathbf{Ax} - \mathbf{b}) + \left(\mathbf{x}^T\mathbf{A}^T - \mathbf{b}^T\right)\mathbf{A}d\mathbf{x}$$

[since $d(\mathbf{uv}) = d\mathbf{u}\,\mathbf{v} + \mathbf{u}\,d\mathbf{v}$]

$$= 2d\mathbf{x}^T\mathbf{A}^T(\mathbf{Ax} - \mathbf{b})$$

[since $\mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$]

$$= 2d\mathbf{x}^T\left(\mathbf{A}^T\mathbf{Ax} - \mathbf{A}^T\mathbf{b}\right)$$

This is zero for any $d\mathbf{x}$ iff $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$

Thus $||\mathbf{e}||^2$ is minimized if $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$

These are the Normal Equations ("Normal" because $\mathbf{A}^T\mathbf{e} = 0$)

The pseudoinverse $\mathbf{x} = \mathbf{A}^+\mathbf{b}$ works even if $\mathbf{A}^T\mathbf{A}$ is singular and finds the $\mathbf{x}$ with minimum $||\mathbf{x}||^2$ that minimizes $||\mathbf{e}||^2$.

This is a <u>very</u> widely used technique.

Frequency Sampling

For every ω we want: $0 = W(\omega)\left(B(e^{j\omega}) - D(\omega)A(e^{j\omega})\right)$

$$= W(\omega)\left(\sum_{m=0}^{M} b[m]e^{-jm\omega} - D(\omega)\left(1 + \sum_{n=1}^{N} a[n]e^{-jn\omega}\right)\right)$$

$$\Rightarrow \begin{pmatrix} \mathbf{u}(\omega)^T & \mathbf{v}(\omega)^T \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = W(\omega)D(\omega)$$

where $\mathbf{u}(\omega)^T = -W(\omega)D(\omega)\begin{bmatrix} e^{-j\omega} & e^{-j2\omega} & \cdots & e^{-jN\omega} \end{bmatrix}$

$$\mathbf{v}(\omega)^T = W(\omega)\begin{bmatrix} 1 & e^{-j\omega} & e^{-j2\omega} & \cdots & e^{-jM\omega} \end{bmatrix}$$

Choose K values of ω, $\{\ \omega_1 \ \cdots \ \omega_K\ \}$ [with $K \geq \frac{M+N+1}{2}$]

$$\begin{pmatrix} \mathbf{U}^T & \mathbf{V}^T \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \mathbf{d}$$

[K equations, $M+N+1$ unkowns]

where $\mathbf{U} = \begin{bmatrix} \mathbf{u}(\omega_1) & \cdots & \mathbf{u}(\omega_K) \end{bmatrix}$,

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}(\omega_1) & \cdots & \mathbf{v}(\omega_K) \end{bmatrix},$$

$$\mathbf{d} = \begin{bmatrix} W(\omega_1)D(\omega_1) & \cdots & W(\omega_K)D(\omega_K) \end{bmatrix}^T$$

We want to force $\mathbf{a}$ and $\mathbf{b}$ to be real; find least squares solution to

$$\begin{pmatrix} \Re\left(\mathbf{U}^T\right) & \Re\left(\mathbf{V}^T\right) \\ \Im\left(\mathbf{U}^T\right) & \Im\left(\mathbf{V}^T\right) \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \Re\left(\mathbf{d}\right) \\ \Im\left(\mathbf{d}\right) \end{pmatrix}$$

Iterative Solution

Least squares solution minimizes the E_E rather than E_S.

However $E_E = E_S$ if $W_E(\omega) = \frac{W_S(\omega)}{|A(e^{j\omega})|}$.

We can use an iterative solution technique:

1. Select K frequencies $\{\omega_k\}$ (e.g. uniformly spaced)
2. Initialize $W_E(\omega_k) = W_S(\omega_k)$
3. Find least squares solution to
   $W_E(\omega_k)\left(B(e^{j\omega_k}) - D(\omega_k)A(e^{j\omega_k})\right) = 0 \forall k$
4. Force $A(z)$ to be stable
   Replace pole p_i by $(p_i^*)^{-1}$ whenever $|p_i| \geq 1$
5. Update weights: $W_E(\omega_k) = \frac{W_S(\omega_k)}{\left|A(e^{j\omega_k})\right|}$
6. Return to step 3 until convergence

But for faster convergence use Newton-Raphson . . .

Newton-Raphson

Newton: To solve $f(x) = 0$ given an initial guess x_0, we write

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0) \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Converges very rapidly once x_0 is close to the solution

So for each ω_k, we can write (omitting the ω and $e^{j\omega}$ arguments)

$$\begin{aligned} E_S &\approx W_S \left(\frac{B_0}{A_0} - D \right) + \frac{W_S}{A_0} (B - B_0) - \frac{W_S B_0}{A_0^2} (A - A_0) \\ &= \frac{W_S}{A_0} \left(B_0 - A_0 D + B - B_0 - \frac{B_0}{A_0} (A - 1) - \frac{B_0}{A_0} + B_0 \right) \end{aligned}$$

From which we get a linear equation for each ω_k :

$$\begin{pmatrix} \frac{B_0}{D A_0} \mathbf{u}^T & \mathbf{v}^T \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = W \left(A_0 D + \frac{B_0}{A_0} - B_0 \right)$$

where $W = \frac{W_S}{A_0}$ and, as before, $u_n(\omega) = -W(\omega) D(\omega) e^{-jn\omega}$ for $n \in 1 : N$ and $v_m(\omega) = W(\omega) e^{-jm\omega}$ for $m \in 0 : M$.

At each iteration, calculate $A_0(e^{j\omega_k})$ and $B_0(e^{j\omega_k})$ based on $\mathbf{a}$ and $\mathbf{b}$ from the previous iteration.

Then use linear least squares to minimize the linearized E_S using the above equation replicated for each of the ω_k.

Magnitude-only Specification

If the filter specification only dictates the target magnitude: $|D(\omega)|$, we need to select the target phase.

Solution:

Make an initial guess of the phase and then at each iteration update $\angle D(\omega) = \angle \frac{B(e^{j\omega})}{A(e^{j\omega})}$.

Initial Guess:

If $H(e^{j\omega})$ is causal and minimum phase then the magnitude and phase are not independent:

$$\angle H(e^{j\omega}) = -\ln\left|H(e^{j\omega})\right| \circledast \cot\tfrac{\omega}{2}$$
$$\ln\left|H(e^{j\omega})\right| = \ln|H(\infty)| + \angle H(e^{j\omega}) \circledast \cot\tfrac{\omega}{2}$$

where $\circledast$ is circular convolution and $\cot x$ is taken to be zero for $-\epsilon < x < \epsilon$ for some small value of ϵ and we take the limit as $\epsilon \to 0$.

This result is a consequence of the Hilbert Relations.

Hilbert Relations

We define $t[n] = u[n-1] - u[-1-n]$

$$T(z) = \frac{z^{-1}}{1-z^{-1}} - \frac{z}{1-z} = \frac{1+z^{-1}}{1-z^{-1}}$$

$$T(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-e^{-j\omega}} = \frac{e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}$$

$$= \frac{2\cos\frac{\omega}{2}}{2j\sin\frac{\omega}{2}} = -j\cot\frac{\omega}{2}$$

$h[n] \rightarrow$ even/odd parts: $h_e[n] = \frac{1}{2}\left(h[n] + h[-n]\right)$

$$h_o[n] = \frac{1}{2}\left(h[n] - h[-n]\right)$$

so $\Re\left(H(e^{j\omega})\right) = H_e(e^{j\omega})$

$$\Im\left(H(e^{j\omega})\right) = -jH_o(e^{j\omega})$$

If $h[n]$ is causal: $h_o[n] = h_e[n]t[n]$

$$h_e[n] = h[0]\delta[n] + h_o[n]t[n]$$

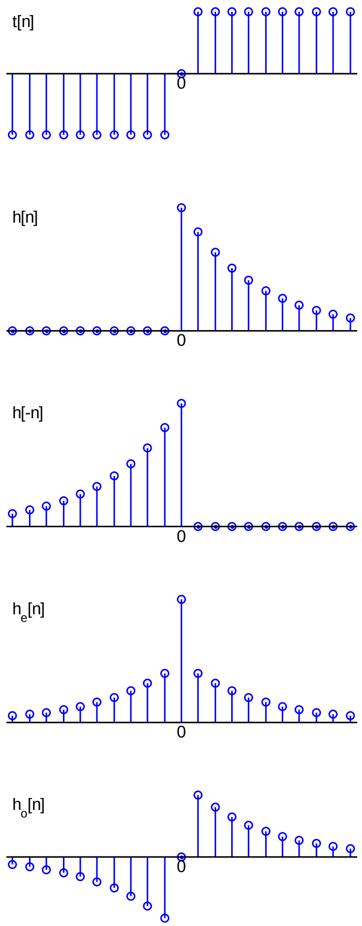
Hence, for causal $h[n]$:

$$\Im\left(H(e^{j\omega})\right) = -j\left(\Re\left(H(e^{j\omega})\right) \circledast -j\cot\frac{\omega}{2}\right)$$

$$= -\Re\left(H(e^{j\omega})\right) \circledast \cot\frac{\omega}{2}$$

$$\Re\left(H(e^{j\omega})\right) = H(\infty) + j\Im\left(H(e^{j\omega})\right) \circledast -j\cot\frac{\omega}{2}$$

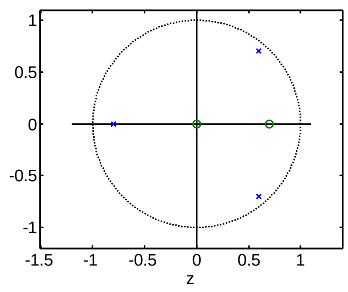
$$= H(\infty) + \Im\left(H(e^{j\omega})\right) \circledast \cot\frac{\omega}{2}$$



Magnitude ↔ Phase Relation

Given $H(z) = g\frac{\prod(1-q_m z^{-1})}{\prod(1-p_n z^{-1})}$

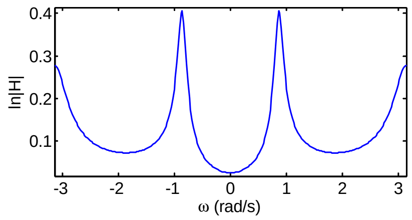
$$\begin{aligned}\ln H(z) &= \ln(g) + \sum \ln\left(1 - q_m z^{-1}\right)\\ &\quad - \sum \ln\left(1 - p_n z^{-1}\right)\\ &= \ln|H(z)| + j\angle H(z)\end{aligned}$$



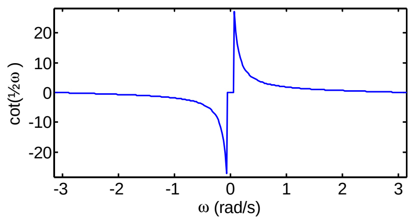
Taylor Series:

$$\ln\left(1 - az^{-1}\right) = -az^{-1} - \frac{a^2}{2}z^{-2} - \frac{a^3}{3}z^{-3} - \ldots$$

causal and stable provided $|a| < 1$



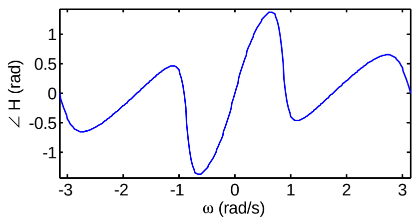
So, if $H(z)$ is minimum phase (all p_n and q_m inside unit circle) then $\ln H(z)$ is the z-transform of a stable causal sequence and:



$$\angle H(e^{j\omega}) = -\ln\left|H(e^{j\omega})\right| \circledast \cot\frac{\omega}{2}$$

$$\ln\left|H(e^{j\omega})\right| = \ln|g| + \angle H(e^{j\omega}) \circledast \cot\frac{\omega}{2}$$

Example: $H(z) = \frac{10-7z^{-1}}{100-40z^{-1}-11z^{-2}+68z^{-3}}$



Note symmetric dead band in $\cot\frac{\omega}{2}$ for $|\omega| < \epsilon$

Summary

- Want to minimize solution error, E_S, but E_E gives linear equations:
  - $E_S(\omega) = W_S(\omega)\left(\frac{B(e^{j\omega})}{A(e^{j\omega})} - D(\omega)\right)$
  - $E_E(\omega) = W_E(\omega)\left(B(e^{j\omega}) - D(\omega)A(e^{j\omega})\right)$
  - use $W_*(\omega)$ to weight errors at different ω.
- Linear least squares: solution to overdetermined $\mathbf{Ax} = \mathbf{b}$
  - Least squares error: $\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$
- Closed form solution: least squares E_E at $\{\omega_k\}$
  - use $W_E(\omega) = \frac{W_S(\omega)}{|A(e^{j\omega})|}$ to approximate E_S
  - use Taylor series to approximate E_S better (Newton-Raphson)
- Hilbert relations
  - relate $\Re\left(H\left(e^{j\omega}\right)\right)$ and $\Im\left(H\left(e^{j\omega}\right)\right)$ for causal stable sequences
  - $\Rightarrow$ relate $\ln\left|H\left(e^{j\omega}\right)\right|$ and $\angle H\left(e^{j\omega}\right)$ for causal stable minimum phase sequences

For further details see Mitra: 9.

MATLAB routines

invfreqz	IIR design for complex response