IMPERIAL COLLEGE LONDON

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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

MSc and EEE/ISE PART IV: MEng and ACGI

Corrected Copy

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Tuesday, 19 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.T. Stathaki
Second Marker(s) : M.K. Gurcan

1.

(a) Let $G(z)$ be the transfer function of a causal system. Show that $g[0] = \lim_{z\to+\infty} G(z)$ where $g[n]$ denotes the impulse response of that system.

[5]

(b) Let $A(z)$ be a real coefficient stable allpass transfer function with order greater than zero. Show that $|A(z)| < 1$ for $|z| > 1$.

[Hint: Since the poles occur in complex conjugate pairs you can prove the required result for the transfer function $\frac{1-d^*z}{z-d}$ and then generalize it].

[5]

(c) Let $G(z)$ be a causal stable nonminimum phase transfer function.

(i) Show that $G(z) = H(z)A(z)$, where $A(z)$ is a causal stable allpass transfer function and $H(z)$ is another causal stable transfer function that is minimum phase with $|G(e^{j\omega})| = |H(e^{j\omega})|$.

[5]

(ii) If $g[n]$ and $h[n]$ denote the impulse responses of the transfer functions $G(z)$ and $H(z)$ respectively, show using the results of parts (a) and (b) above that $|g(0)| \leq |h(0)|$.

[5]

2.

(a) Show that the analog transfer function $H(s) = \frac{bs}{s^2 + bs + \Omega_o^2}, b > 0$ has a bandpass magnitude response with $|H(j0)| = |H(j\infty)| = 0$ and $|H(j\Omega_o)| = 1$. Determine the frequencies Ω_1 and Ω_2, where $\Omega_2 > \Omega_1$, at which the gain is 3dB below the maximum value of 0dB at Ω_o. Show that $\Omega_1\Omega_2 = \Omega_o^2$. The difference $\Omega_2 - \Omega_1$ is called the 3dB bandwidth of the bandpass transfer function. Show that $b = \Omega_2 - \Omega_1$

[6]

(b) The bandpass transfer function of part (a) above can be expressed in the form $H(s) = \frac{1}{2}[A_1(s) - A_2(s)]$ where $A_1(s)$ and $A_2(s)$ are stable analog allpass transfer functions of the form $A_i(s) = \frac{D_i(-s)}{D_i(s)}, i = 1,2$ with $D_i(s)$ a polynomial of s. Determine $A_1(s)$ and $A_2(s)$.

[6]

(c) The magnitude squared response of an analog Butterworth filter $H(s)$ of N^{th} order is given by

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega / \Omega_c)^{2N}}, \Omega_c \text{ constant}$$

Show that the poles of the Butterworth filter are of the form $p_l = \Omega_c e^{j[\pi(N+2l-1)/2N]}, l = 1,2,\ldots,N$, using the relationship $|H(j\Omega)|^2 = H(s)H(-s)$, where $s = j\Omega$.

[8]

3.

(a) This problem illustrates how aliasing can be suitably exploited in order to realize interesting frequency response characteristics. An ideal causal analog lowpass filter with an impulse response $h_a(t)$ has a frequency response given by

$$H_a(j\Omega) = \begin{cases} 1, & |\Omega| \le |\Omega_c| \\ 0, & \text{otherwise.} \end{cases}$$

Let $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ be the frequency responses of digital filters obtained by sampling $h_a(t)$ at $t = nT$, where $T = 3\pi / 2\Omega_c$ and $T = \pi / \Omega_c$, respectively. Assume that the transfer functions are later normalized so that $H_1(e^{j0}) = H_2(e^{j0}) = 1$.

(i) Sketch the frequency responses $H_1(e^{j\omega})$, $H_2(e^{j\omega})$.

[5]

(ii) What type of filter is $G(z) = H_2(z) - H_1(z)$ (lowpass, highpass, etc.)?

[5]

(b) Consider the Finite Impulse Response filter transfer functions $G_1(z) = \frac{1}{N}\sum_{i=0}^{N-1} z^{-i}$ and $G_2(z) = \frac{1}{N}\sum_{i=0}^{N-1} (-1)^i z^{-i}$, $N > 1$. Show that $G_1(z)$ is a lowpass filter and $G_2(z)$ is a highpass filter, by using the following two approaches:

(i) Intuitive approach. In this approach you can describe the effect of filters $G_1(z)$ and $G_2(z)$ on an input signal without necessarily using mathematical relationships.

[5]

(ii) Mathematical approach. In this approach you must find the frequency response of the two filters.

[5]

For part (b) (ii) you may wish to use the relationship $\sum_{i=0}^{N-1} z^{-i} = \frac{1 - z^{-N}}{1 - z^{-1}}, z \ne 1$.

4.

(a) Verify the cascade equivalences of Figure 1 below.

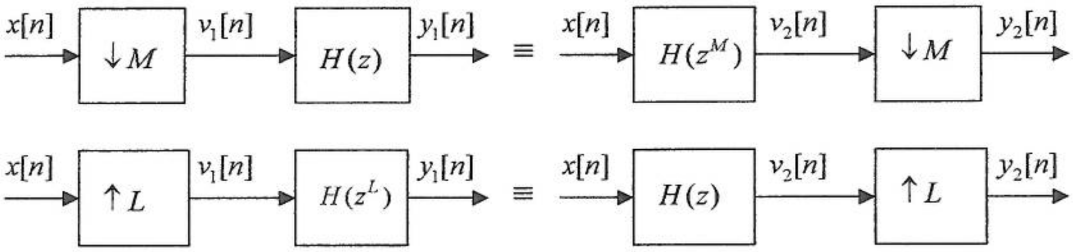


Figure 1

[10]

(b) Consider the multirate structure of Figure 2 below, where $H_0(z), H_1(z)$ and $H_2(z)$ are ideal zero-phase real-coefficient lowpass, bandpass and highpass filters respectively, with frequency responses as indicated in Figure 3. If the input is a real sequence with a discrete-time Fourier transform $X(e^{j\omega})$ as shown below, sketch the discrete-time Fourier transform of the outputs $y_0[n], y_1[n]$ and $y_2[n]$.

[10]

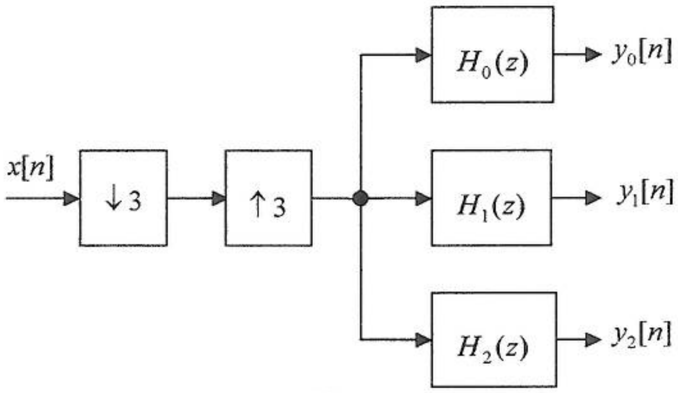


Figure 2

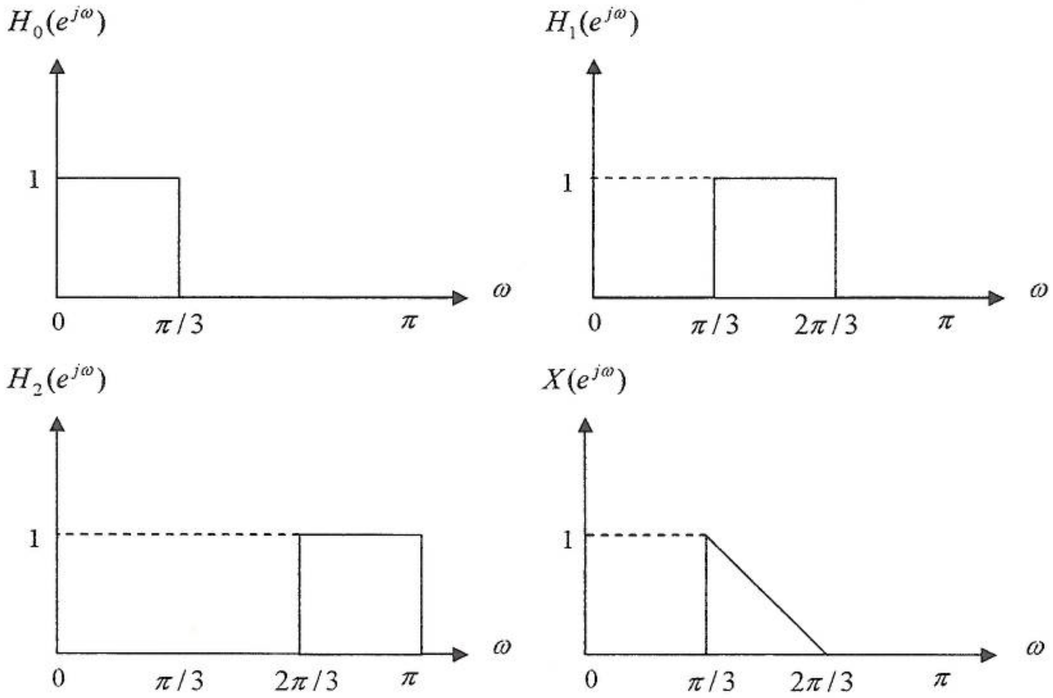


Figure 3

1.

(a) Let $G(z)$ be a causal transfer function. Show that $g[0]=\lim_{z\to+\infty} G(z)$ where $g[n]$ denotes the impulse response of the transfer function $G(z)$.

[5]

Answer

$$G(z)=\sum_{n=0}^{+\infty} g[n]z^{-n} \Rightarrow \lim_{z\to+\infty}\sum_{n=0}^{+\infty} g[n]z^{-n}=g[0]$$

(b) Let $A(z)$ be a real coefficient allpass transfer function. Show that $|A(z)|<1$ for $|z|>1$.
[Hint: Since the poles occur in complex conjugate pairs you can prove the required result for the transfer function $\frac{1-d^*z}{z-d}$ and then generalize it].

[5]

Answer

$$A_1(z)=\frac{1-d^*z}{z-d}$$

$$|A_1(z)|^2=A_1(z)A_1^*(z)=\frac{1-d^*z}{z-d}\frac{1-dz^*}{z^*-d^*}$$

$$1-|A_1(z)|^2=1-A_1(z)A_1^*(z)=1-\frac{1-d^*z}{z-d}\frac{1-dz^*}{z^*-d^*}=\frac{(z-d)(z^*-d^*)-(1-d^*z)(1-dz^*)}{(z-d)(z^*-d^*)}$$

$$=\frac{|z|^2+|d|^2-zd^*-dz^*-1-|z|^2|d|^2+zd^*+dz^*}{(z-d)(z^*-d^*)}=\frac{(|z|^2-1)(1-|d|^2)}{(z-d)(z^*-d^*)}$$

Hence, $1-|A_1(z)|^2>0$ if $|z|>1$. Therefore, $|A_1(z)|^2<1$ if $|z|>1$.

(c) Let $G(z)$ be a causal stable nonminimum phase transfer function, and let $H(z)$ denote another causal stable transfer function that is minimum phase with $|G(e^{j\omega})|=|H(e^{j\omega})|$.

(i) Show that $G(z)=H(z)A(z)$, where $A(z)$ is a causal stable allpass transfer function.

[5]

Answer

Since $G(z)$ is non-minimum phase but causal, it will have some zeros outside the unit circle. Let $z=a$ be one such zero. We can then write:

$$G(z)=P(z)(1-az^{-1})=P(z)(1-az^{-1})\frac{(1-az)}{(1-az)}=P(z)(1-az)\frac{(1-az^{-1})}{(1-az)}$$

Note that $\frac{(1-az^{-1})}{(1-az)}$ is a stable first order allpass function. If we carry out this operation for all zeros of $G(z)$ outside the unit circle we can write $G(z)=H(z)A(z)$ where $H(z)$ will have all zeros inside the unit circle and will thus be a minimum phase function and $A(z)$ will be a product of first order allpass functions and hence and allpass function.

If $g[n]$ and $h[n]$ denote their respective impulse responses, show using results of questions (a) and (b) above that:

(ii) $|g(0)| \le |h(0)|$

[5]

Answer

$$|g[0]| = \left|\lim_{z \to +\infty} G(z)\right| = \left|\lim_{z \to +\infty} H(z)A(z)\right| = \left|\lim_{z \to +\infty} H(z)\right|\left|\lim_{z \to +\infty} A(z)\right| \le \left|\lim_{z \to +\infty} H(z)\right| = |h[0]|$$

because $\left|\lim_{z \to +\infty} A(z)\right| \le 1$

2.

(a) Show that the analog transfer function $H(s) = \frac{bs}{s^2 + bs + \Omega_0^2}, b > 0$ has a bandpass magnitude response with $|H(j0)| = |H(j\infty)| = 0$ and $|H(j\Omega_0)| = 1$. Determine the frequencies Ω_1 and Ω_2 at which the gain is 3dB below the maximum value of 0dB at Ω_0. Show that $\Omega_1\Omega_2 = \Omega_0^2$. The difference $\Omega_2 - \Omega_1$ is called the 3dB bandwidth of the bandpass transfer function. Show that $b = \Omega_2 - \Omega_1$

[6]

Answer

$H(s) = \frac{bs}{s^2 + bs + \Omega_0^2}, b > 0$. Thus, $H(j\Omega) = \frac{jb\Omega}{-\Omega^2 + bj\Omega + \Omega_0^2}, b > 0$, hence

$|H(j\Omega)|^2 = \frac{b^2\Omega^2}{b^2\Omega^2 + (\Omega_0^2 - \Omega^2)^2}$. Now at $\Omega = 0$, $|H(j0)| = 0$, at $\Omega = +\infty$, $|H(j\infty)| = 0$ and at $\Omega = \Omega_0$, $|H(j\Omega_0)| = 1$. Hence, $H(s)$ has a band pass response. The 3-dB frequencies are given by $\frac{b^2\Omega_c^2}{b^2\Omega_c^2 + (\Omega_0^2 - \Omega_c^2)^2} = \frac{1}{2}$. Thus, $(\Omega_0^2 - \Omega_c^2)^2 = b^2\Omega_c^2$ or $\Omega_c^4 - (b^2 + 2\Omega_0^2)\Omega_c^2 + \Omega_0^4 = 0$. Hence, if Ω_1 and Ω_2 are the roots of this equation, then so are $-\Omega_1$, $-\Omega_2$, and the product of the roots is Ω_0^4. This implies $\Omega_1\Omega_2 = \Omega_0^2$. Also $\Omega_1^2 + \Omega_2^2 = b^2 + 2\Omega_0^2$. Hence, $(\Omega_2 - \Omega_1)^2 = b^2$ which gives the desired result $\Omega_2 - \Omega_1 = b$.

(b) The bandpass transfer function of question (a) above can be expressed in the form $H(s) = \frac{1}{2}[A_1(s) - A_2(s)]$ where $A_1(s)$ and $A_2(s)$ are stable analog allpass transfer functions of the form $A_i(s) = \frac{D_i(-s)}{D_i(s)}, i = 1,2$ with $D(s)$ polynomial of s. Determine $A_1(s)$ and $A_2(s)$.

[6]

Answer

$$H(s) = \frac{1}{2}\frac{2bs}{s^2 + bs + \Omega_0^2} = \frac{1}{2}\left[\frac{s^2 + bs + \Omega_0^2}{s^2 + bs + \Omega_0^2} - \frac{s^2 - bs + \Omega_0^2}{s^2 + bs + \Omega_0^2}\right] = \frac{1}{2}\left[1 - \frac{s^2 - bs + \Omega_0^2}{s^2 + bs + \Omega_0^2}\right]$$

$A_1(s) = 1$

$$A_2(s) = \frac{s^2 - bs + \Omega_0^2}{s^2 + bs + \Omega_0^2}$$

(c) The magnitude squared response of an analog Butterworth filter $H(s)$ of N^{th} order is given by

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}, \Omega_c \text{ constant}$$

Show that the poles of the Butterworth filter are of the form $p_l = \Omega_c e^{j[\pi(N+2l-1)/2N]}, l = 1,2,\ldots,N$, using the relationship $|H(j\Omega)|^2 = H(s)H(-s)$

[8]

Answer

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+(-s^2/\Omega_c^2)^N}$$

The poles of this expression occur on a circle of radius Ω_c at equally spaced points. The transfer function itself will be specified by just the poles in the negative real half-plane of s. The l^{th} pole is specified by:

$$\frac{-p_l^2}{\Omega_c^2} = (-1)^{\frac{1}{N}} = e^{\frac{j(2l-1)\pi}{N}} \text{ and hence, } \frac{(jp_l)^2}{\Omega_c^2} = e^{\frac{j(2l-1)\pi}{N}} \Rightarrow \frac{jp_l}{\Omega_c} = e^{\frac{j(2l-1)\pi}{2N}} \Rightarrow$$

$$\frac{e^{-j\frac{\pi}{2}} p_l}{\Omega_c} = e^{\frac{j(2l-1)\pi}{2N}} \Rightarrow \frac{e^{-j\frac{\pi N}{2N}} p_l}{\Omega_c} = e^{\frac{j(2l-1)\pi}{2N}} \Rightarrow p_l = \Omega_c e^{j\frac{\pi N}{2N}} e^{\frac{j(2l-1)\pi}{2N}} \Rightarrow$$

$$p_l = \Omega_c e^{j\frac{\pi N}{2N}} e^{\frac{j(2l-1)\pi}{2N}} \Rightarrow$$

$$p_l = \Omega_c e^{j[\pi(N+2l-1)/2N]}, l = 1,2,\ldots,N$$

3.

(a) This problem illustrates how aliasing can be suitably exploited in order to realize interesting frequency response characteristics. An ideal causal analog lowpass filter with an impulse response $h_a(t)$ has a frequency response given by

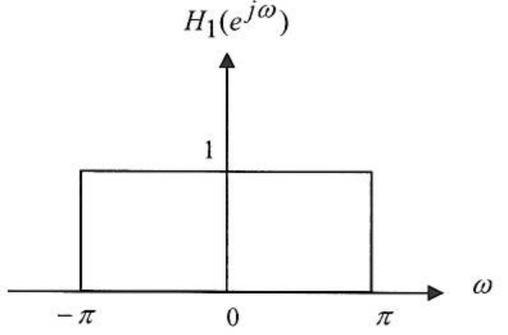
$$H_a(j\Omega) = \begin{cases} 1, & |\Omega| \le |\Omega_c| \\ 0, & \text{otherwise.} \end{cases}$$

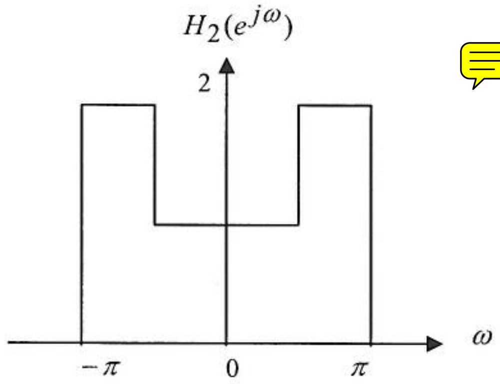
Let $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ be the frequency responses of digital filters obtained by sampling $h_a(t)$ at $t = nT$, where $T = 3\pi / 2\Omega_c$ and $T = \pi / \Omega_c$, respectively. Assume that the transfer functions are later normalized so that $H_1(e^{j0}) = H_2(e^{j0}) = 1$.

(i) Sketch the frequency responses $H_1(e^{j\omega})$, $H_1(e^{j\omega})$.

[5]

Answer



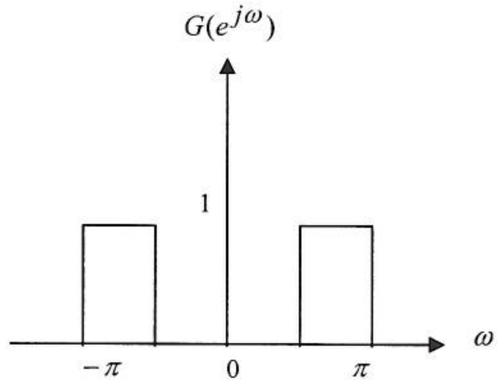


(ii) What type of filter is $G(z) = H_2(z) - H_1(z)$ (lowpass, highpass, etc.)?

[5]

Answer

$G(z)$ has the form shown below and therefore, it is a highpass filter



(b) Consider the Finite Impulse Response filter transfer functions $G_1(z) = \frac{1}{N}\sum_{i=0}^{N-1} z^{-i}$ and $G_2(z) = \frac{1}{N}\sum_{i=0}^{N-1} (-1)^i z^{-i}$. Show that $G_1(z)$ is a lowpass filter and $G_2(z)$ is a highpass filter, by using the following two approaches:

(i) Intuitive approach. In this approach you can describe the effect of filters $G_1(z)$ and $G_2(z)$ on an input signal without necessarily using mathematical relationships.

[5]

Answer

$G_1(z)$ is a lowpass filter because when we mix signal samples with positive weights we destroy the abrupt changes of the signals, which are related to the high frequencies of the signal.

$G_2(z)$ is a highpass filter because when we mix signal samples with some negative weights we might enhance the abrupt changes of the signals, which are related to the high frequencies of the signal.

(ii) Mathematical approach. In this approach you must find the frequency response of the two filters.

[5]

For part (b) (ii) you may wish to use the relationship $\sum_{i=0}^{N-1} z^{-i} = \frac{1-z^{-N}}{1-z^{-1}}$.

Answer

$$G_1(z) = \frac{1}{N}\sum_{i=0}^{N-1} z^{-i} = \frac{1}{N}\frac{1-z^{-N}}{1-z^{-1}} = \frac{1}{N}\frac{z^{-N/2}}{z^{-1/2}}\frac{z^{N/2}-z^{-N/2}}{z^{1/2}-z^{-1/2}}$$

$$G_1(e^{j\omega}) = \frac{1}{N}\frac{z^{-N/2}}{z^{-1/2}}\frac{z^{N/2}-z^{-N/2}}{z^{1/2}-z^{-1/2}}\bigg|_{z=e^{j\omega}} = \frac{1}{N}e^{-j\omega(N-1)/2}\frac{2j\sin\omega N/2}{2j\sin\omega/2} \Rightarrow$$

$$\left|G_1(e^{j\omega})\right| = \frac{1}{N}\left|\frac{\sin\omega N/2}{\sin\omega/2}\right|$$

The above function is the well know sinc function which decays as the frequency tends to π.

$$G_2(z) = \frac{1}{N}\sum_{i=0}^{N-1} (-z^{-1})^i = \frac{1}{N}\frac{1-(-z^{-1})^N}{1-(-z^{-1})}$$

N even:

$$G_2(z) = \frac{1}{N}\frac{1-(z^{-1})^N}{1+z^{-1}} = \frac{1}{N}\frac{z^{-N/2}}{z^{-1/2}}\frac{z^{N/2}-z^{-N/2}}{z^{1/2}+z^{-1/2}} \Rightarrow$$

$$G_2(e^{j\omega}) = \frac{1}{N}e^{-j\omega(N-1)/2}\frac{2j\sin\omega N/2}{2\cos\omega/2} \Rightarrow$$

$$\left|G_1(e^{j\omega})\right| = \frac{1}{N}\left|\frac{\sin\omega N/2}{\cos\omega/2}\right|$$

N odd:

$$G_2(z) = \frac{1}{N}\frac{1+(z^{-1})^N}{1+z^{-1}} = \frac{1}{N}\frac{z^{-N/2}}{z^{-1/2}}\frac{z^{N/2}+z^{-N/2}}{z^{1/2}+z^{-1/2}} \Rightarrow$$

$$G_2(e^{j\omega}) = \frac{1}{N}e^{-j\omega(N-1)/2}\frac{2\cos\omega N/2}{2\cos\omega/2} \Rightarrow$$

$$\left|G_1(e^{j\omega})\right| = \frac{1}{N}\left|\frac{\cos\omega N/2}{\cos\omega/2}\right|$$

In both cases the denominator decreases as frequency approaches π, therefore, the entire response increases and therefore, we have highpass filters.

4.

(a) Verify the cascade equivalence of Figure 1 below.

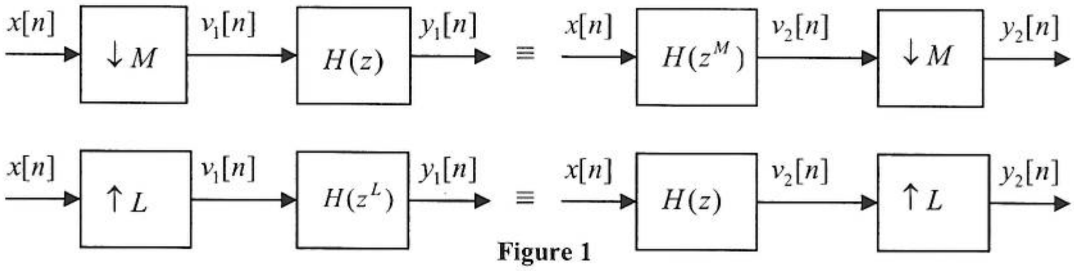
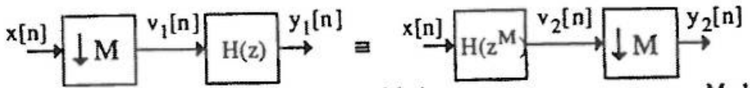


Figure 1

[10]

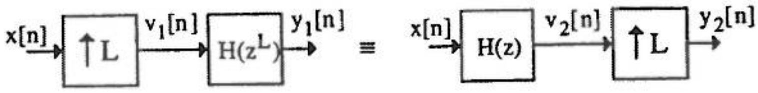
Answer



For the left-hand side figure, we have $V_1(z)=\frac{1}{M}\sum_{k=0}^{M-1}X(z^{1/M}W_M^k)$, $Y_1(z)=\frac{1}{M}\sum_{k=0}^{M-1}H(z)X(z^{1/M}W_M^k)$,

For the right-hand side figure,we have $V_2(z)=H(z^M)X(z)$, $Y_2(z)=\frac{1}{M}\sum_{k=0}^{M-1}H(zW_M^{kM})X(z^{1/M}W_M^k)$

$=\frac{1}{M}\sum_{k=0}^{M-1}H(z)X(z^{1/M}W_M^k)$. Hence, $Y_1(z)=Y_2(z)$.



For the left-hand side figure, we have $V_1(z)=X(z^L)$, $Y_1(z)=H(z^L)X(z^L)$. For the right-hand side figure,we have $V_2(z)=H(z)X(z)$, $Y_2(z)=H(z^L)X(z^L)$. Hence, $Y_1(z)=Y_2(z)$.

(b) Consider the multirate structure of Figure 2 below, where $H_0(z), H_1(z)$ and $H_2(z)$ are ideal zero-phase real-coefficient lowpass, bandpass and highpass filters respectively, with frequency responses as indicated in Figure 3. If the input is a real sequence with a discrete-time Fourier transform $X(e^{j\omega})$ as shown below, sketch the discrete-time Fourier transform of the outputs $y_0(t), y_1(t)$ and $y_2(t)$

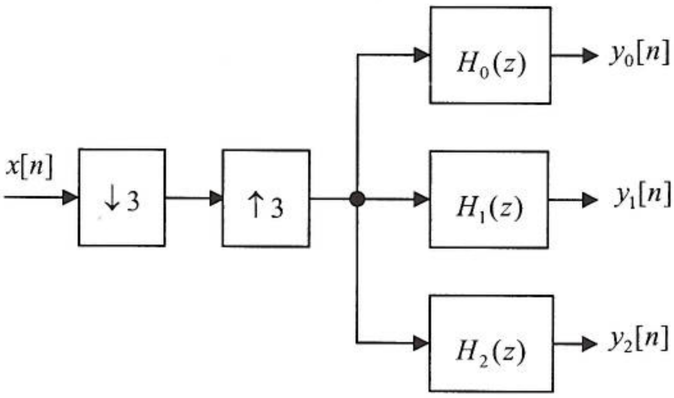
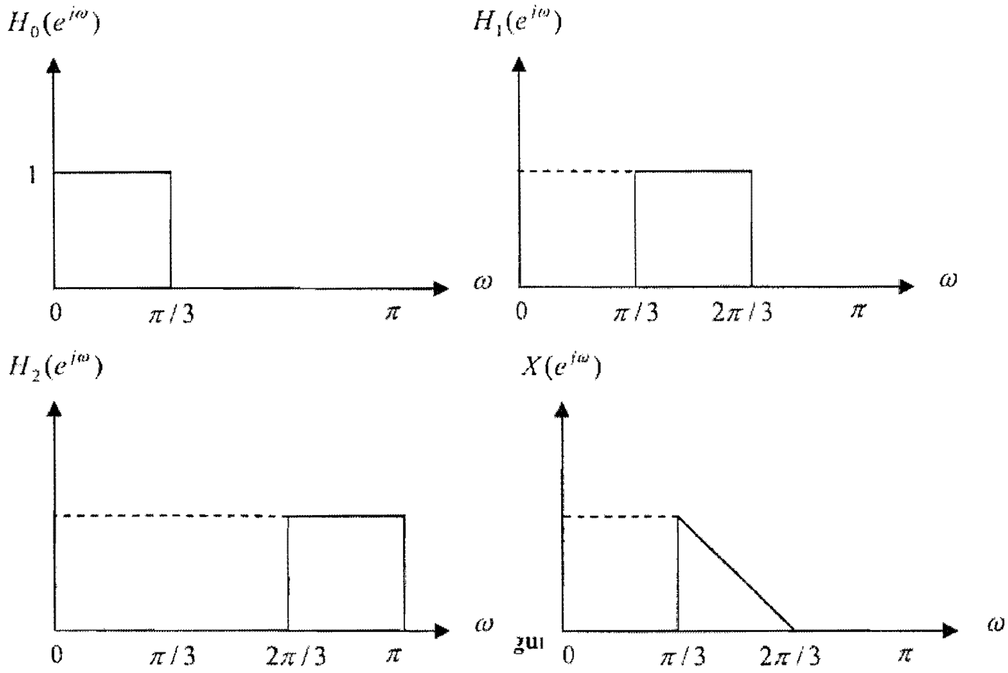


Figure 2



Answer

After decimation by 3 we obtain the signal $W(e^{j\omega})$. After interpolation by 3 of the signal $W(e^{j\omega})$ we obtain the signal $U(e^{j\omega})$. The three outputs are shown below.