EE4-12
EE9-AO2
EE9-SC1

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

MSc and EEE/EIE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Tuesday, 6 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	D.M. Brookes
	Second Marker(s) :	P.T. Stathaki

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Information for Candidates:

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
SNR	Signal-to-Noise Ratio

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ or, more generally, $\sum_{n=q}^{r} \alpha^n z^{-n} = \frac{\alpha^q z^{-q} - \alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$	$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N}\sum_{n=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N}\sum_{0}^{N-1} X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r]y[n-r] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

$v[n] = x[n]y[n] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right) = \frac{1}{2\pi}X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r]y[(n-r) \mod N] \quad\Leftrightarrow\quad V[k] = X[k]Y[k]$

$v[n] = x[n]y[n] \quad\Leftrightarrow\quad V[k] = \frac{1}{N}X[k] \circledast_N Y[k] \triangleq \frac{1}{N}\sum_{r=0}^{N-1} X[r]Y[(k-r) \mod N]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in normalized rad/s.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

1. a) i) Explain what is meant by saying that a linear time-invariant system is "BIBO-stable". [1]

ii) The impulse response, $h[n]$, of a linear time-invariant system satisfies $\sum_{n=-\infty}^{\infty} |h[n]| = S$ where $S < \infty$. Prove that the system is BIBO-stable and also that $H(z)$ converges for $|z| = 1$. [2]

b) A filter, with input $x[n]$ and output $y[n]$, is defined by the difference equation

$$y[n] = \alpha y[n-1] + (1-\alpha)x[n]$$

where $0 < \alpha < 1$ is a real constant.

i) Determine the system function of the filter, $H(z)$, and the impulse response, $h[n]$, for $n = -1, 0, 1, 2$. [2]

ii) State the values of z at which $H(z)$ has a pole or zero. [2]1

iii) Determine the frequency at which the filter has a gain of $-3\,\text{dB}$.[3]

iv) If the sample frequency is f_s, show that, for $n \geq 0$, the impulse response, $h[n]$, is equal to a sampled version of $g(t) = Ae^{-\frac{t}{\tau}}$ and determine the values of the constants A and τ. [2]

c) Figure 1.1 shows the block diagram of a filter implementation comprising two delays, five multipliers with real-valued coefficients $c_1, \cdots, c_5$ and four adder elements.

i) Show that transfer function $\frac{Y(z)}{X(z)} = \frac{c_3+c_4z^{-1}+c_5z^{-2}}{1-c_1z^{-1}-c_2z^{-2}}$. [3]

ii) Suppose that each multiplier introduces independent additive white noise at its output with power spectral density $S(\omega) = S_0$ and that the noise signals are uncorrelated with $x[n]$. Show that the combined effect of the five noise sources is equivalent to two additive white noise signals at $x[n]$ and $y[n]$ respectively. Hence determine the overall power spectral density, $N(\omega)$, of the noise at $y[n]$. [3]

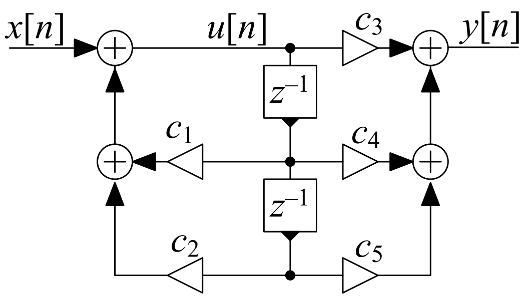

Figure 1.1

d) The impulse response of an antisymmetric FIR filter, $H(z)$, of order M satisfies the relation $h[n] = -h[M-n]$.

i) Show that the magnitude response $\left|H(e^{j\omega})\right|$ can be expressed as the absolute value of the sum of N sine waves where $N = \frac{M}{2}$ if M is even and $N = \frac{M+1}{2}$ if M is odd. [3]

ii) Show that $H(e^{j\omega})$ is necessarily zero at $\omega = 0$ but may be non-zero at $\omega = \pi$ if M is odd. Give an example of a filter for which this is the case. [2]

iii) Derive an expression for the phase response, $\angle H(e^{j\omega})$, and determine the group delay, $\tau_H\left(e^{j\omega}\right) = -\frac{d\angle H(e^{j\omega})}{d\omega}$. [2]

e) Figure 1.2 shows the analysis and synthesis sections of a subband processing system. The input and output signals are $x[n]$ and $y[n]$ respectively and the intermediate signals are $v_m[n]$, $u_m[r]$ and $w_m[n]$ where $m = 0$ or 1 according to the subband. The corresponding z-transforms are $X(z)$, $Y(z)$ etc.

i) Show that it is possible to express the overall transfer function in the form $Y(z) = \begin{bmatrix} T(z) & A(z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$ and determine expressions for $T(z)$ and $A(z)$.

You may assume without proof that for $m = 0$ or 1, [3]

$$\begin{aligned} U_m(z) &= \frac{1}{2}\left\{V_m\left(z^{\frac{1}{2}}\right) + V_m\left(-z^{\frac{1}{2}}\right)\right\} \\ W_m(z) &= U_m\left(z^2\right). \end{aligned}$$

ii) Explain why it is normally desirable to have $A(z) \equiv 0$. [2]

iii) Suppose that $H_0(z) = H_1(-z) = G_0(z) = -G_1(-z)$. Show that in this case $A(z) = 0$ and explain how the frequency responses $H_1(e^{j\omega})$, $G_0(e^{j\omega})$ and $G_1(e^{j\omega})$ are related to $H_0(e^{j\omega})$ assuming that $H_0(z)$ is an FIR or IIR filter with real coefficients. [2]

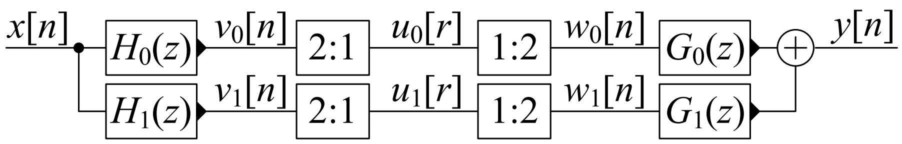

Figure 1.2

f) Figure 1.3 shows an upsampler with real-valued input $x[n]$ and output

$$y[r] = \begin{cases} x\left[\frac{r}{K}\right] & \text{if } K \mid r \\ 0 & \text{otherwise} \end{cases}$$

where $K \mid r$ means K is a factor of r.

i) Show that $Y(z) = X\left(z^K\right)$. [1]

ii) The energy and average power of $x[n]$ are defined respectively as

$$\begin{aligned} E_x &= \sum_{n=-\infty}^{\infty} |x[n]|^2 \\ P_x &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 . \end{aligned}$$

Give expressions for the energy and average power of $y[r]$ in terms of E_x and P_x. [2]

iii) Figure 1.4 shows the power spectral density of $x[n]$ which comprises white noise of unit magnitude together with a bandpass signal component occupying the range $0.5 < |\omega| < 1$. Sketch the power spectral density of $y[r]$ when $K = 3$ and give the magnitude of its white noise component and the magnitude and frequency range of all bandpass components. [3]

iv) The diagram of Fig. 1.3 is followed by a lowpass filter to remove spectral images. If $K = 3$ and $x[n]$ is as specified in part iii) above, determine the transition bandwidth and transition band centre frequency of a suitable lowpass filter and explain the reasons for your choices.

[2]

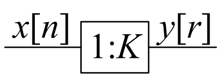

Figure 1.3

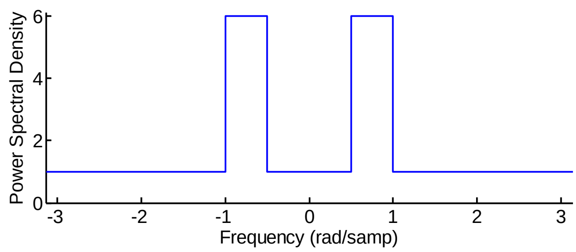

Figure 1.4

2. a) Suppose that $G_1(z) = 1 - pz^{-1}$ and $G_2(z) = 1 - qz^{-1}$ where the constants p and q may be complex. If $q = \frac{1}{p^*}$ show that $\left|G_1(e^{j\omega})\right| = \alpha \left|G_2(e^{j\omega})\right|$ for all ω and determine an expression for the constant α. [4]

b) Suppose that $H_1(z) = 4 + 14z^{-1} - 8z^{-2}$. Determine the coefficients of $H_2(z)$ such that $\left|H_1(e^{j\omega})\right| = \left|H_2(e^{j\omega})\right|$ for all ω and that all the zeros of $H_2(z)$ lie inside the unit circle. [4]

c) When designing an IIR filter $H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})}$ to approximate a complex target response $D(\omega)$ two error measures that may be used are the weighted solution error, $E_S(\omega)$, and the weighted equation error, $E_E(\omega)$, defined respectively by

$$
\begin{aligned}
E_S(\omega) &= W_S(\omega)\left(\frac{B(e^{j\omega})}{A(e^{j\omega})} - D(\omega)\right) \\
E_E(\omega) &= W_E(\omega)\left(B(e^{j\omega}) - D(\omega)A(e^{j\omega})\right).
\end{aligned}
$$

Explain the relative advantages of the two error measures and explain the purpose of the real-valued non-negative weighting functions $W_S(\omega)$ and $W_E(\omega)$. [2]

d) Suppose that $0 \le \omega_1 < \omega_2 < \ldots < \omega_K \le \pi$ is a set of K frequencies and that $A(z) = 1 + [z^{-1}\ z^{-2}\ \cdots\ z^{-N}]\mathbf{a}$ and $B(z) = [1\ z^{-1}\ z^{-2}\ \cdots\ z^{-M}]\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are real-valued coefficient column vectors.

i) Show that it is possible to express the equations $E_E(\omega_k) = 0$ for $1 \le k \le K$ as a set of K simultaneous linear equations in the form $(\mathbf{P}\ \mathbf{Q})\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix} = \mathbf{d}$.

State the dimensions of the matrices $\mathbf{P}$ and $\mathbf{Q}$ and of the vector $\mathbf{d}$ and derive expressions for the elements of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{d}$. [4]

ii) Explain how, by separating the real and imaginary parts of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{d}$, it is possible to obtain a set of simultaneous linear equations for $\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$ in which all coefficients are real-valued. Explain the circumstances under which some of the resultant equations will necessarily have all-zero coefficients. [4]

iii) Explain why it may be desirable to apply the transformation of part b) after obtaining the solution to the equations of part d) ii). [2]

iv) Assuming that $\omega_1 = 0$ and $\omega_K = \pi$, determine the minimum value of K to ensure that the equations of part d) ii) are not underdetermined. [4]

e) Suppose now that $H(z) = \frac{b}{1+az^{-1}}$, that $K = 3$, that $\omega_k = \left\{0, \frac{\pi}{2}, \pi\right\}$, that

$$
\begin{aligned}
D(\omega) &= \begin{cases} 2 & \text{for } \omega \le 0.25\pi \\ 1 & \text{for } \omega > 0.25\pi \end{cases}. \\
W_E(\omega) &\equiv 1
\end{aligned}
$$

Determine the numerical values of the elements of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{d}$ and hence determine the numerical values of a and b that minimize $\sum_k \left|E_E(\omega_k)\right|^2$. [6]

You may assume without proof that the least squares solution to an overdetermined set of real-valued linear equations, $\mathbf{Rx} = \mathbf{q}$, is given by $\mathbf{x} = \left(\mathbf{R}^T\mathbf{R}\right)^{-1}\mathbf{R}^T\mathbf{q}$ assuming that $\mathbf{R}$ has full column rank.

3. a) Figure 3.1 shows the block diagram of a system that multiplies the input sample rate by $\frac{P}{Q}$ where P and Q are coprime with $P < Q$.

i) Explain why the cutoff frequency of the lowpass filter $H(z)$ should be placed at the Nyquist rate of the output signal, $y[m]$ and give the normalized cutoff frequency, ω_0, in rad/sample in terms of P and/or Q.

Using the approximation formula $M \approx \frac{a}{3.5\Delta\omega}$, determine the required filter order M in terms of P and/or Q if the stopband attenuation in dB is $a = 60$ and the normalized transition bandwidth is $\Delta\omega = 0.1\omega_0$. [4]

ii) Using the value of M from part a) i), estimate the average number of multiplications per input sample, $x[n]$, needed to implement the system in the form of Figure 3.1. [2]

iii) The filter $H(z)$ has a symmetrical impulse response $h[r] = g[r]w[r]$ for $0 \le r \le M$ where $g[r]$ is the impulse response of an ideal lowpass filter with cutoff frequency ω_0 and $w[r]$ is a symmetrical window function.

Derive an expression for the ideal response, $g[r]$, in terms of ω_0, M and r. [4]

b) The filter $H(z)$ is now implemented as a polyphase filter as shown in Fig. 3.2. The filter implementation uses a single set of delays and multipliers with commutated coefficients.

i) State the length of the filter impulse response $h_0[n]$ in terms of M, P and/or Q and give an expression for the coefficients $h_0[n]$ in terms of $h[r]$. [2]

ii) If $x[n] = 0$ for $n < 0$, give expressions for $v[0]$, $v[1]$, $v[2P+1]$ in terms of the input $x[n]$ and the coefficients $h_p[n]$. [2]

iii) Explain how it is possible to eliminate the output decimator by changing both the sequence and rate at which the coefficient sets, $h_p[n]$ are accessed.

Determine the new coefficient set order for the case $P = 5$ and $Q = 7$. [3]

iv) Determine the number of multiplications per input sample for the system of part b) iii) and the number of distinct coefficients that must be stored. You may assume that $M + 1$ is a multiple of P. [2]

c) Suppose now that the sample rate of the input, $x[n]$, is 18 kHz and that the system is implemented as in part b) iii) with the values of a and $\Delta\omega$ as given in part a) i).

Determine the values of P, Q and M when the sample rate of the output, $y[m]$, is (i) 10 kHz and (ii) 10.1 kHz [note that 101 is a prime number].

For each of these cases estimate the number of multiplications per input sample and the number of distinct coefficients that must be stored. [5]

d) In a Farrow filter, the coefficients, $h_p[n]$, are approximated by a low-order polynomial $f_n(t)$ where $t = \frac{p}{P}$ for $0 \le p \le P-1$.

i) Assuming that a rectangular window, $w[r] \equiv 1$, is used in the design of $H(z)$ and that $\omega_0 = \frac{\pi}{P}$, give an expression for the target value of $f_0(t)$ in terms of t, M , P and Q. [3]

ii) If the polynomials, $f_n(v)$, are of order $K = 5$, determine the number of coefficients that must be stored for each of the cases defined in part c).

[3]

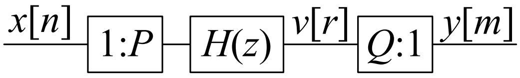

Figure 3.1

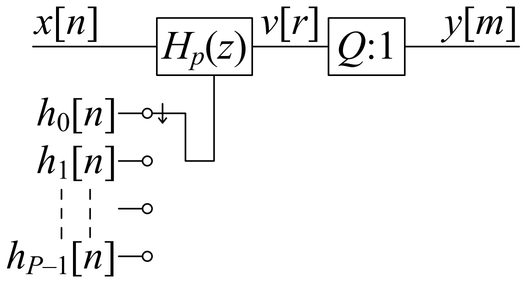

Figure 3.2

4. A complex-valued frequency-modulated signal, $x(t) = a(t)e^{j\phi(t)}$, has a $0\,Hz$ carrier frequency and a peak frequency deviation of $d = 75\,\text{kHz}$. The amplitude, $a(t)$, is approximately constant with $a(t) \approx 1$ and the phase is $\phi(t) = k\int_0^t m(\tau)d\tau$ where k is a constant and $m(t)$ is a baseband audio signal with bandwidth $b = 15\,\text{kHz}$. The signal $x(t)$ is sampled at $400\,\text{kHz}$ to obtain the discrete-time signal $x[n]$.

a) Carson's rule for the bandwidth of a double-sideband FM signal is $B = 2(d+b)$. Use this to determine the single-sided bandwidth, ω_0, of $x[n]$ in radians/sample. [2]

b) Show that $m(t) = k^{-1}a^{-2}(t)\Im\left(x^*(t)\frac{dx(t)}{dt}\right)$ where $\Im\,()$ denotes the imaginary part. [4]

c) Figure 4.1 shows a block diagram that implements the equation of part b) in discrete time. Complex-valued signals are shown as bold lines and are represented using their real and imaginary parts. The block labelled "Conj" takes the complex conjugate of its input. The differentiation block, $D(z)$, is designed as an FIR filter using the window method with a target response

$$\overline{D}(e^{j\omega}) = \begin{cases} jc\omega & \text{for } |\omega| \le \omega_1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a scaling constant.

i) Determine the impulse response $\bar{d}[n]$ of $\overline{D}(z)$ in simplified form. [4]

ii) Assuming that $\omega_1 = \frac{\omega_0+\pi}{2}$, draw dimensioned sketches showing the magnitude and phase responses of $\overline{D}(e^{j\omega})$ over the range $-\pi \le \omega \le \pi$. [3]

iii) Assume that the DTFT of the window function used when designing $D(z)$ has a main lobe width of $\omega = \pm\frac{18}{M+1}$ for a window of length $M+1$. If ω_1 is chosen as $\omega_1 = \frac{\omega_0+\pi}{2}$, determine the smallest value of M that will ensure that the transition in the response of $D(e^{j\omega})$ near $\omega = \omega_1$ lies completely within the range (ω_0, π). [3]

iv) Stating any assumptions, determine the maximum value of c that will ensure $|s[n]| \le 1$ where $s[n]$ is the output of the differentiation block, $D(z)$, as shown in Figure 4.1. [4]

d) An alternative choice for the target response is

$$\widetilde{D}(e^{j\omega}) = \begin{cases} \frac{-jc\omega_1(\pi+\omega)}{\pi-\omega_1} & \text{for } -\pi < \omega \le -\omega_1 \\ jc\omega & \text{for } |\omega| \le \omega_1 \\ \frac{jc\omega_1(\pi-\omega)}{\pi-\omega_1} & \text{for } \omega_1 < \omega \le \pi \end{cases}$$

i) Assuming that $\omega_1 = \frac{\omega_0+\pi}{2}$, draw dimensioned sketches showing the magnitude and phase responses of $\widetilde{D}(e^{j\omega})$ over the range $-\pi \le \omega \le \pi$. [4]

ii) Outline the relative advantages and disadvantages of using $\widetilde{D}(e^{j\omega})$ rather than $\overline{D}(e^{j\omega})$ as the target response when designing $D(e^{j\omega})$. [2]

e) An alternative structure that avoids any divisions is shown in Fig. 4.2 where the polynomial $f(v)$ is the truncated Taylor series for v^{-1} expanded around $v = 1$. Determine $f(v)$ for the cases when it is (i) a linear expression and (ii) a quadratic expression. In each case determine the gain error (expressed in dB) resulting from the approximation when $a(t) = 1.1$. [4]

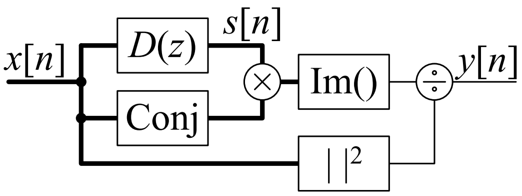

Figure 4.1

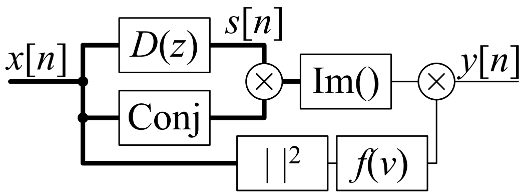

Figure 4.2

Digital Signal Processing and Digital Filters

********* **Solutions** *********

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- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \dots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
SNR	Signal-to-Noise Ratio

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ or, more generally, $\sum_{n=q}^{r} \alpha^n z^{-n} = \frac{\alpha^q z^{-q}-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$

Forward and Inverse Transforms

z:	$X(z)=\sum_{-\infty}^{\infty}x[n]z^{-n}$	$x[n]=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz$
CTFT:	$X(j\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt$	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\Omega)e^{j\Omega t}d\Omega$
DTFT:	$X(e^{j\omega})=\sum_{-\infty}^{\infty}x[n]e^{-j\omega n}$	$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$
DFT:	$X[k]=\sum_{0}^{N-1}x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n]=\frac{1}{N}\sum_{0}^{N-1}X[k]e^{j2\pi\frac{kn}{N}}$
DCT:	$X[k]=\sum_{n=0}^{N-1}x[n]\cos\frac{2\pi(2n+1)k}{4N}$	$x[n]=\frac{X[0]}{N}+\frac{2}{N}\sum_{n=1}^{N-1}X[k]\cos\frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k]=\sum_{n=0}^{2N-1}x[n]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n]=\frac{1}{N}\sum_{0}^{N-1}X[k]\cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n]=x[n]*y[n]\triangleq\sum_{r=-\infty}^{\infty}x[r]y[n-r] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$

$v[n]=x[n]y[n] \quad\Leftrightarrow\quad V\left(e^{j\omega}\right)=\frac{1}{2\pi}X\left(e^{j\omega}\right)\circledast Y\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$

DFT: $v[n]=x[n]\circledast_N y[n]\triangleq\sum_{r=0}^{N-1}x[r]y[(n-r)_{\bmod N}] \quad\Leftrightarrow\quad V[k]=X[k]Y[k]$

$v[n]=x[n]y[n] \quad\Leftrightarrow\quad V[k]=\frac{1}{N}X[k]\circledast_N Y[k]\triangleq\frac{1}{N}\sum_{r=0}^{N-1}X[r]Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega})=-\frac{d\angle H(e^{j\omega})}{d\omega}=\Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}}=\Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}(\,)$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M\approx\frac{a}{3.5\Delta\omega}$
2. $M\approx\frac{a-8}{2.2\Delta\omega}$
3. $M\approx\frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in normalized rad/s.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

********* Questions and Solutions *********

1. a) i) Explain what is meant by saying that a linear time-invariant system is "BIBO-stable". [1]

An LTI system is BIBO-stable if any bounded input sequence $x[n]$ always results in an output sequence $y[n]$ that is also bounded. A sequence $x[n]$ is bounded iff $\exists B < \infty$ such that $|x[n]| < B\ \forall n$.

Several people though that "bounded" meant $\sum x[n] < B$ instead of $x[n] < B\,\forall n$. A few thought it meant finite energy: $\sum |x[n]|^2 < B$. Even those that knew what bounded meant were often imprecise about their definition, for example defining BIBO as "$x[n] < \infty \Rightarrow y[n] < \infty$". Others said that a BIBO-stable system implied that the input was always bounded; no property of a system can ever impose a condition on its input signal.

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ii) The impulse response, $h[n]$, of a linear time-invariant system satisfies $\sum_{n=-\infty}^{\infty} |h[n]| = S$ where $S < \infty$. Prove that the system is BIBO-stable and also that $H(z)$ converges for $|z| = 1$. [2]

Suppose that $x[n]$ is any bounded sequence with $|x[n]| < B < \infty\ \forall n$. We need to show that the output, $y[n] = \sum_{r=-\infty}^{\infty} h[r]x[n-r]$ is also bounded. We have

$$\begin{aligned}
|y[n]| &= \left|\sum_{r=-\infty}^{\infty} h[r]x[n-r]\right| \\
&\le \sum_{r=-\infty}^{\infty} |h[r]x[n-r]| \\
&= \sum_{r=-\infty}^{\infty} |h[r]|\,|x[n-r]| \\
&< B\sum_{r=-\infty}^{\infty} |h[r]| = BS
\end{aligned}$$

Many people proved the converse of what was asked in the question: i.e. BIBO $\Rightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$. If you need to show something is true for any input $x[n]$, you cannot start by assuming one particular $x[n]$.

We have $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$. If $|z| = 1$ we can write

$$\begin{aligned}
|H(z)| &= \left|\sum_{n=-\infty}^{\infty} h[n]z^{-n}\right| \\
&\le \sum_{n=-\infty}^{\infty} |h[n]|\,|z^{-n}| \\
&= \sum_{n=-\infty}^{\infty} |h[n]| = S
\end{aligned}$$

Hence $H(z)$ is convergent for $|z| = 1$.

Some wrote $H(z) \leq \ldots$ instead of $|H(z)| \leq \ldots$; inequalities make no sense for complex numbers. Quite a few people proved the result for $z = 1$ and thought this was the same thing as proving it for $|z| = 1$ a few even assumed $|z| = 1 \Rightarrow z^{-n} = 1$. Others just restated the question, e.g. "Since $\sum_{n=-\infty}^{\infty} |h[n]| = S$ it follows that $H(z)$ converges for $|z| = 1$"; this does not constitute a proof.

Both of the proofs given above depend on the triangle inequality: $|x+y| \leq |x| + |y|$ (also called the "subadditivity" property); some people implicitly assumed that $|x+y| = |x| + |y|$ (or, equivalently $|\sum h[n]| = \sum |h[n]|$) which is not true in general.

---

b) A filter, with input $x[n]$ and output $y[n]$, is defined by the difference equation

$$y[n] = \alpha y[n-1] + (1-\alpha)x[n]$$

where $0 < \alpha < 1$ is a real constant.

i) Determine the system function of the filter, $H(z)$, and the impulse response, $h[n]$, for $n = -1, 0, 1, 2$. [2]

Taking z-transforms: $Y(z) = \alpha z^{-1} Y(z) + (1-\alpha)X(z)$ from which $H(z) = \frac{Y(z)}{X(z)} = \frac{1-\alpha}{1-\alpha z^{-1}}$.

$h[-1] = 0$, $h[0] = 1-\alpha$, $h[1] = (1-\alpha)\,\alpha$, $h[2] = (1-\alpha)\,\alpha^2$.

You can easily find the impulse response directly from the difference equation by setting $x[n] = \delta[n]$. Working out by taking the inverse z-transform or IDTFT is much more laborious and error prone since the inverse z-transform is not unique.

Surprisingly, very many people made the response non-causal, usually saying $h[-1] = (1-\alpha)\alpha^{-1}$. You can tell from the difference equation that $h[n]$ is causal since the output $y[n]$ depends only on past or present inputs and outputs. In contrast, since the inverse z-transform of $H(z)$ is not unique, you cannot tell from $H(z)$ whether $h[n] = (1-\alpha)\alpha^n u[n]$ (bounded, causal) or $h[n] = (\alpha-1)\alpha^n u[-n-1]$ (unbounded, anti-causal) unless you also known the region of convergence.

---

ii) State the values of z at which $H(z)$ has a pole or zero. [2]1

The system function is $H(z) = \frac{1-\alpha}{1-\alpha z^{-1}} = \frac{(1-\alpha)z}{z-\alpha}$. This has a pole at $z = \alpha$ and a zero at $z = 0$.

The zero at $z = 0$ is easily overlooked. It is easiest to find all the poles and zeros if you express the transfer function as a rational polynomial in z rather than in z^{-1}. Even when $H(z)$ is written in the form $H(z) = \frac{1-\alpha}{1-\alpha z^{-1}}$, it can easily be seen that $H(0) = 0$.

---

iii) Determine the frequency at which the filter has a gain of $-3\,\text{dB}$.[3]

We want $\left|H(e^{j\omega})\right|^2 = H(e^{j\omega}) \times H^*(e^{j\omega}) = 0.5$ *. Thus*

$$\begin{aligned} 0.5 &= \frac{1-\alpha}{1-\alpha e^{-j\omega}} \times \frac{1-\alpha}{1-\alpha e^{j\omega}} \\ &= \frac{(1-\alpha)^2}{1-2\alpha\cos\omega+\alpha^2} \end{aligned}$$

From this, $\cos\omega = \frac{1+\alpha^2-2(1-\alpha)^2}{2\alpha} = \frac{4\alpha-1-\alpha^2}{2\alpha} = 1-\frac{(1-\alpha)^2}{2\alpha}$ *and so* $\omega_{3\text{dB}} = \cos^{-1}\left(\frac{4\alpha-1-\alpha^2}{2\alpha}\right) = \cos^{-1}\left(1-\frac{(1-\alpha)^2}{2\alpha}\right)$.

Quite a few people wrote down an equation with modulus signs: $\left|H(e^{j\omega})\right| = \sqrt{0.5}$ *but did not known how to solve it. The standard procedure is to multiply both sides by the complex conjugate as we do above. Some people just quietly ignored the modulus signs instead and consequently got an imaginary value for* ω*. Several thought* $-3\,\text{dB}$ *meant a gain of* 0.5 *instead of* $\sqrt{0.5}$*. Several took the question to mean* $\left|H(e^{j\omega_{3\text{dB}}})\right| = \sqrt{0.5} \times \left|H(e^{j0})\right|$*; Luckily for them,* $\left|H(e^{j0}) = 1\right|$ *in this case.*

iv) If the sample frequency is f_s, show that, for $n \geq 0$, the impulse response, $h[n]$, is equal to a sampled version of $g(t) = Ae^{-\frac{t}{\tau}}$ and determine the values of the constants A and τ. [2]

For $n \geq 0$*, we have from part i) that* $h[n] = (1-\alpha)\,\alpha^n$*. This can also be proved by induction from the given recurrence relation assuming that* $x[n]$ *is an impulse at* $n = 0$*. If we now substitute* $t = \frac{n}{f_s}$ *(or, equivalently* $t = nT$ *where the sample period is* $T = \frac{1}{f_s}$*) into* $g(t) = Ae^{-\frac{t}{\tau}}$ *we obtain* $h[n] = g\left(\frac{n}{f_s}\right) = Ae^{-\frac{n}{\tau f_s}}$ *from which* $A = 1-\alpha$ *and* $\alpha = e^{-\frac{1}{\tau f_s}}$*. Rearranging the later equation gives* $\tau = \frac{-1}{f_s \ln\alpha}$.

Surprisingly many people couldn't do this. It really just involves making the substitution $t = nT = \frac{n}{f_s}$ *and then matching the coefficients of the resultant expressions. Some said* $t = nf_s$ *instead; this is not dimensionally consistent since* f_s *has dimensions* $\frac{1}{\text{time}}$.

c) Figure 1.1 shows the block diagram of a filter implementation comprising two delays, five multipliers with real-valued coefficients $c_1, \cdots, c_5$ and four adder elements.

i) Show that transfer function $\frac{Y(z)}{X(z)} = \frac{c_3+c_4z^{-1}+c_5z^{-2}}{1-c_1z^{-1}-c_2z^{-2}}$. [3]

The two delay elements form a shift register whose input is $u[n]$*. Hence the inputs to the* c_1 *and* c_2 *multipliers are* $u[n-1]$ *and* $u[n-2]$ *respectively. It follows directly from the diagram that* $u[n] = x[n] + c_1u[n-1] + c_2u[n-2]$*. Taking z-transforms gives* $U(z) = X(z) +$

$c_1z^{-1}U(z)+c_2z^{-1}U(z)$ from which $\frac{U(z)}{X(z)}=\frac{1}{1-c_1z^{-1}-c_2z^{-2}}$. From the diagram $y[n]=c_3u[n]+c_4u[n-1]+c_5u[n-2]$ from which $\frac{Y(z)}{U(z)}=c_3+c_4z^{-1}+c_5z^{-2}$. Combining this with the previous result gives $\frac{Y(z)}{X(z)}=\frac{Y(z)}{U(z)}\times\frac{U(z)}{X(z)}=\frac{c_3+c_4z^{-1}+c_5z^{-2}}{1-c_1z^{-1}-c_2z^{-2}}$.

Most got this right. A few had sign errors in the coefficients c_1 and/or c_2 even thought the correct answer was given in the question. Some just stated that it was a Direct Form II implementation of B(z)/A(z); this does not constitute a proof.

---

ii) Suppose that each multiplier introduces independent additive white noise at its output with power spectral density $S(\omega)=S_0$ and that the noise signals are uncorrelated with $x[n]$. Show that the combined effect of the five noise sources is equivalent to two additive white noise signals at $x[n]$ and $y[n]$ respectively. Hence determine the overall power spectral density, $N(\omega)$, of the noise at $y[n]$. [3]

The noise components added by multipliers c_1 and c_2 merely add onto the input and so add noise with power spectral density $2S_0$ onto the input (note that, since the noise sources are assumed independent, their power spectral densities add). Similarly, the remaining noise sources add noise with power spectral density $3S_0$ onto the output. Hence the overall noise power spectral density at $y[n]$ is

$$
\begin{aligned}
N(\omega) &= 3S_0+2S_0\times\left|H(e^{j\omega})\right|^2 \\
&= S_0\left(3+2\frac{\left(c_3+c_4z^{-1}+c_5z^{-2}\right)\left(c_3+c_4z^{1}+c_5z^{2}\right)}{\left(1-c_1z^{-1}-c_2z^{-2}\right)\left(1-c_1z^{1}-c_2z^{2}\right)}\right) \\
&= S_0\left(3+2\frac{c_3^2+c_4^2+c_5^2+2\left(c_3+c_5\right)c_4\cos\omega+2c_3c_5\cos2\omega}{1+c_1^2+c_2^2+2\left(c_2-1\right)c_1\cos\omega-2c_2\cos2\omega}\right).
\end{aligned}
$$

Note that the expression in the first line gets full marks.

Some mixed up the time domain and z-transform domain: you never get $x[n]$ and z in the same equation. Some just assumed the input noise was multiplied by c_3 instead of by $H(e^{j\omega})$ and most people forgot that the noise power spectrum was multiplied by the gain squared.

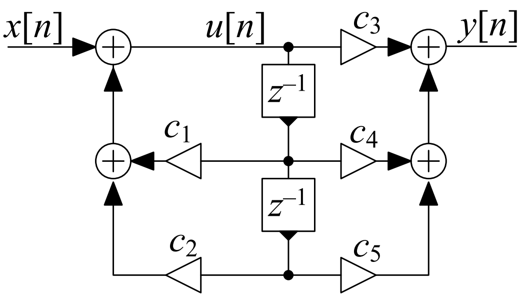


Figure 1.1

d) The impulse response of an antisymmetric FIR filter, $H(z)$, of order M satisfies the relation $h[n] = -h[M-n]$.

i) Show that the magnitude response $\left|H(e^{j\omega})\right|$ can be expressed as the absolute value of the sum of N sine waves where $N = \frac{M}{2}$ if M is even and $N = \frac{M+1}{2}$ if M is odd. [3]

If M is odd, there is an even number of coefficients so we can write

$$\begin{aligned}
H(z) &= \sum_{n=0}^{\frac{M-1}{2}} h[n]z^{-n} + \sum_{n=\frac{M+1}{2}}^{M} h[n]z^{-n} \\
&= \sum_{n=0}^{\frac{M-1}{2}} h[n]z^{-n} + \sum_{r=0}^{\frac{M-1}{2}} h[M-r]z^{-(M-r)} \\
&= \sum_{n=0}^{\frac{M-1}{2}} h[n]\left(z^{-n} - z^{n-M}\right) \\
&= z^{-0.5M} \sum_{n=0}^{\frac{M-1}{2}} h[n]\left(z^{-n+0.5M} - z^{n-0.5M}\right) \\
H(e^{j\omega}) &= -je^{-j0.5M\omega} \sum_{n=0}^{\frac{M-1}{2}} 2h[n]\sin\left((n-0.5M)\,\omega\right) \\
\left|H(e^{j\omega})\right| &= \left|\sum_{n=0}^{\frac{M-1}{2}} 2h[n]\sin\left((n-0.5M)\,\omega\right)\right|
\end{aligned}$$

The right side is the absolute value of the sum of $\frac{M+1}{2}$ sine waves as required.

If M is even, there is an odd number of coefficients but the central coefficient, $h[0.5M]$, must be zero since $h[0.5M] = -h[M-0.5M] = -h[0.5M]$. Hence

$$\begin{aligned}
H(z) &= \sum_{n=0}^{\frac{M}{2}-1} h[n]z^{-n} + h\left[\frac{M}{2}\right] z^{-0.5M} + \sum_{n=\frac{M}{2}+1}^{M} h[n]z^{-n} \\
&= 0 + \sum_{n=0}^{\frac{M-2}{2}} h[n]\left(z^{-n} - z^{n-M}\right) \\
\Rightarrow \left|H(e^{j\omega})\right| &= \left|\sum_{n=0}^{\frac{M-2}{2}} 2h[n]\sin\left((n-0.5M)\,\omega\right)\right|
\end{aligned}$$

The derivation is identical to that for odd M except for the upper summation limit. The right side is the absolute value of the sum of $\frac{M}{2}$ sine waves as required.

The coefficients, $h[n]$ go from $n = 0$ to $n = M$ and so if M is odd, there is an even number of coefficients. Many got the summation limits slightly wrong, sometimes because they assumed there were M coefficients rather than $M+1$ and sometimes because they assumed *the upper summation limit to be* $0.5M$*. Quite often the summation limits were not even integers. An easy way to check that the limits are correct is to try a very small value for* M *such as* $M=1$ *or* $M=2$*.Several did not notice that if* M *was even,* $h[0.5M]$ *must equal zero. for the even* M *case, a few people extracted the* $h[0]$ *term from the summation rather than the* $h[0.5M]$ *term; this is incorrect because* $h[0]=-h[M]$ *is a symmetric pair. Not everyone seemed completely familiar with* $e^{j\theta}-e^{-j\theta}=2j\sin\theta$*which can easily be derived from* $e^{j\theta}=\cos\theta+j\sin\theta$*.*

---

ii) Show that $H(e^{j\omega})$ is necessarily zero at $\omega=0$ but may be non-zero at $\omega=\pi$ if M is odd. Give an example of a filter for which this is the case. [2]

When $\omega=0$ *then* $\sin\alpha\omega=0$ *for any* α*. Hence all the summation terms are zero for both odd and even* M*.*

When $\omega=\pi$ *then* $\sin\alpha\omega=0$ *if* α *is an integer. So, if* M *is even,* $\sin((n-0.5M)\omega)$ *will always be* 0 *at* $\omega=\pi$*. However, if* M *is odd then this is not necessarily true.*

An example, for $M=1$*, is* $H(z)=1-z^{-1}$*. For this case,* $H(e^{j\omega})=1-e^{-j\omega}=2je^{-0.5j\omega}\sin 0.5\omega$*. When* $\omega=\pi$*,* $H(e^{j\omega})=2$ *which is non-zero.*

Most people correctly explained the case of $\omega=0$*. However the arguments for* $\omega=\pi$ *were often not precisely correct and, in deed, many people omitted this part entirely.*

---

iii) Derive an expression for the phase response, $\angle H(e^{j\omega})$, and determine the group delay, $\tau_H\left(e^{j\omega}\right)=-\frac{d\angle H(e^{j\omega})}{d\omega}$. [2]

From part d), the phase response is

$$\begin{aligned}\angle H(e^{j\omega}) &= \angle\left(-je^{-j0.5M\omega}\right)+\frac{\pi}{2}\left(\operatorname{sgn}\left(\sum_{n=0}^{\frac{M-1}{2}}2h[n]\sin((n-0.5M)\omega)\right)-1\right)\\ &= -0.5(M\omega+\pi)+\frac{\pi}{2}\left(\operatorname{sgn}\left(\sum_{n=0}^{\frac{M-1}{2}}2h[n]\sin((n-0.5M)\omega)\right)-1\right)\end{aligned}$$

Differentiating this gives $\tau_H\left(e^{j\omega}\right)=-\frac{d\angle H(e^{j\omega})}{d\omega}=0.5M$*. Note that the* $\operatorname{sgn}()$ *function is piecewise constant and so its derivative is zero.*

Almost everyone omitted the $\frac{\pi}{2}(\operatorname{sgn}()-1)$ *term completely and many omitted the* -0.5π *term arising from the factor* $-j$*. Some tried to derive* $\tau_H\left(e^{j\omega}\right)$ *from the formula sheet expression* $\tau_H\left(e^{j\omega}\right)=\Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}}$ *which, although correct, is not a good approach.*

---

e) Figure 1.2 shows the analysis and synthesis sections of a subband processing system. The input and output signals are $x[n]$ and $y[n]$ respectively and the intermediate signals are $v_m[n]$, $u_m[r]$ and $w_m[n]$ where $m = 0$ or 1 according to the subband. The corresponding z-transforms are $X(z)$, $Y(z)$ etc.

i) Show that it is possible to express the overall transfer function in the form $Y(z) = \begin{bmatrix} T(z) & A(z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$ and determine expressions for $T(z)$ and $A(z)$.

You may assume without proof that for $m = 0$ or 1, [3]

$$\begin{aligned} U_m(z) &= \frac{1}{2}\left\{V_m\left(z^{\frac{1}{2}}\right) + V_m\left(-z^{\frac{1}{2}}\right)\right\} \\ W_m(z) &= U_m\left(z^2\right). \end{aligned}$$

Combining the two given equations we get

$$\begin{aligned} W_m(z) = U_m\left(z^2\right) &= \frac{1}{2}\left\{V_m(z) + V_m(-z)\right\} \\ &= = \frac{1}{2}\left\{H_m(z)X(z) + H_m(-z)X(-z)\right\}. \end{aligned}$$

Hence

$$\begin{aligned} Y(z) &= G_0(z)W_0(z) + G_1(z)W_1(z) \\ &= \frac{1}{2}\left\{(G_0(z)H_0(z) + G_1(z)H_1(z))X(z) + (G_0(z)H_0(-z) + G_1(z)H_1(-z))X(-z)\right\} \\ &= \begin{bmatrix} T(z) & A(z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix} \end{aligned}$$

where $T(z) = \frac{1}{2}(G_0(z)H_0(z) + G_1(z)H_1(z))$ *and* $A(z) = \frac{1}{2}(G_0(z)H_0(-z) + G_1(z)H_1(-z))$.

Most people got this right.

---

ii) Explain why it is normally desirable to have $A(z) \equiv 0$. [2]

The factor $A(z)$ *multiplies* $X(-z)$. *The spectrum of* $X(-z)$ *is* $X(-e^{j\omega}) = X(e^{j(\omega+\pi)})$ *is an aliased version of the spectrum* $X(e^{j\omega})$ *in which the spectrum is shifted by* π *(or equivalently conjugated and reflected around* $\frac{\pi}{2}$ *for the case of a conjugate symmetric spectrum). These aliased spectral images are normally unwanted and so we would like* $A(e^{j\omega}) \equiv 0 \Leftrightarrow A(z) \equiv 0$ *(assuming* $A(z)$ *is analytic).*

Most people got this right although few described the relationship between $X(-z)$ *and* $X(z)$ *beyond saying that it was "aliased" and therefore undesirable.*

---

iii) Suppose that $H_0(z) = H_1(-z) = G_0(z) = -G_1(-z)$. Show that in this case $A(z) = 0$ and explain how the frequency responses $H_1(e^{j\omega})$,

$G_0(e^{j\omega})$ and $G_1(e^{j\omega})$ are related to $H_0(e^{j\omega})$ assuming that $H_0(z)$ is an FIR or IIR filter with real coefficients. [2]

From the relations given, we can express all the blocks in terms of $H_0(z)$ to obtain

$A(z) = G_0(z)H_0(-z) + G_1(z)H_1(-z) = H_0(z)H_0(-z) - H_0(-z)H_0(z) = 0.$

Our no-alias condition is therefore met.

Some people correctly derived $A(z) = G_0(z)H_0(-z) - G_0(-z)H_0(z)$ but went on to say that this must be zero because, they wrongly asserted, $G_0(-z) = G_0(z)$ and $H_0(z) = H_0(-z)$. This is the wrong pairing: in fact $G_0(-z) = H_0(-z)$ and $H_0(z) = G_0(z)$.

Clearly $G_0(e^{j\omega})$ is identical to $H_0(e^{j\omega})$.

$H_1(e^{j\omega}) = H_0(-e^{j\omega}) = H_0(e^{j(\omega-\pi)}) = H_0^*(e^{j(\pi-\omega)})$.

$H_1(e^{j\omega})$ is therefore the frequency response of $H_0(e^{j\omega})$ but conjugated and reflected around $\omega = \frac{\pi}{2}$.

$G_1(e^{j\omega})$ is the same as $H_1(e^{j\omega})$ but negated or, equivalently, with π added onto the phase response.

Some people only discussed the magnitude responses.

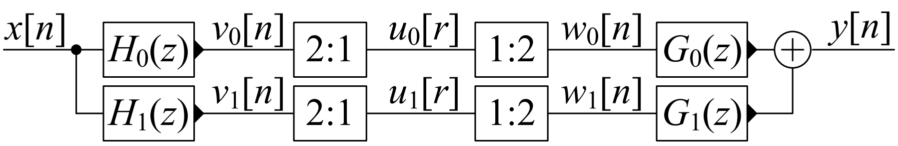


Figure 1.2

f) Figure 1.3 shows an upsampler with real-valued input $x[n]$ and output

$$y[r]=\begin{cases}x\left[\frac{r}{K}\right] & \text{if } K \mid r\\ 0 & \text{otherwise}\end{cases}$$

where $K \mid r$ means K is a factor of r.

i) Show that $Y(z)=X\left(z^K\right)$. [1]

$Y(z)=\sum_{r=-\infty}^{\infty} y[r]z^{-r}=\sum_{\{r:K|r\}} y[r]z^{-r}=\sum_{n=-\infty}^{\infty} y[nK]z^{-nK}=\sum_{n=-\infty}^{\infty} x[n]\left(z^K\right)^{-n}=X\left(z^K\right)$.

In order to prove this you need to change the summation variable using, in the expression above, $r=nK$. "Proofs" that did not make such a substitution were usually not correct. Many people omitted the stage $\sum_{\{r:K|r\}} y[r]z^{-r}$ in the above proof but instead wrote $\sum_{r=-\infty}^{\infty} x[\frac{r}{K}]z^{-r}$ which is invalid/meaningless when $\frac{r}{K}$ is not an integer.

---

ii) The energy and average power of $x[n]$ are defined respectively as

$$\begin{aligned}E_x &= \sum_{n=-\infty}^{\infty} |x[n]|^2\\ P_x &= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2.\end{aligned}$$

Give expressions for the energy and average power of $y[r]$ in terms of E_x and P_x. [2]

The non-zero samples of $y[n]$ are identical to the samples of $x[n]$ and so the energy of the two signals is the same: $E_Y=E_X$. However $y[n]$ has K times as many samples, so its power is $P_Y=\frac{1}{K}P_X$.

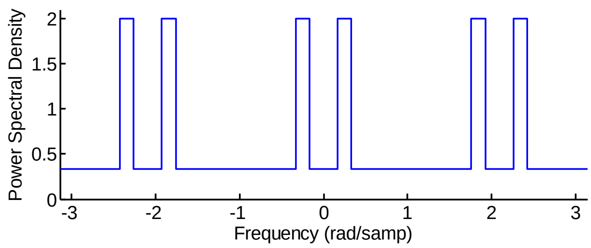
To show this algebraically (rather laborious), we write

$$\begin{aligned}P_x &\triangleq \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2\\ &= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|y[Kn]|^2\\ &= \lim_{N\to\infty}\left(\frac{2KN+1}{2N+1}\right)\frac{1}{2KN+1}\sum_{m=-KN}^{KN}|y[m]|^2\\ &= \lim_{N\to\infty}\left(K+\frac{K-1}{2N+1}\right)\times\lim_{N\to\infty}\left(\frac{1}{2KN+1}\sum_{m=-KN}^{KN}|y[m]|^2\right)\\ &= K\times P_y.\end{aligned}$$

Most people correctly said $E_y=E_x$ but quite a few also said $P_y=P_x$. Many people got this question wrong because they wrote down *complicated algebraic expressions which they were then unable to evaluate.*

iii) Figure 1.4 shows the power spectral density of $x[n]$ which comprises white noise of unit magnitude together with a bandpass signal component occupying the range $0.5 < |\omega| < 1$. Sketch the power spectral density of $y[r]$ when $K = 3$ and give the magnitude of its white noise component and the magnitude and frequency range of all bandpass components. [3]

The upsampling horizontally compresses the 2-sided spectrum by a factor of K and replicates it K times at increments of $\Delta\omega = \frac{2\pi}{K}$*. For the DTFT and the energy spectrum, each image is the same magnitude as the original. However, for the power spectrum, the magnitudes will be reduced by a factor of K.*



Thus the magnitude of the white component will be 0.33 *and that of the bandpass components will be* 2*.*

The frequency ranges of the bandpass components will be

$\left\{-\frac{2\pi}{3}, 0, \frac{2\pi}{3}\right\} \pm \frac{1}{4} \pm \frac{1}{12} = \left\{-\frac{2\pi}{3}, 0, \frac{2\pi}{3}\right\} \pm \left\{\frac{1}{6}, \frac{1}{3}\right\}$.

That is, each will be of width $\frac{1}{6}$ *and centered* $\pm\frac{1}{4}$ *either side of the image centre. The band edges are therefore at*

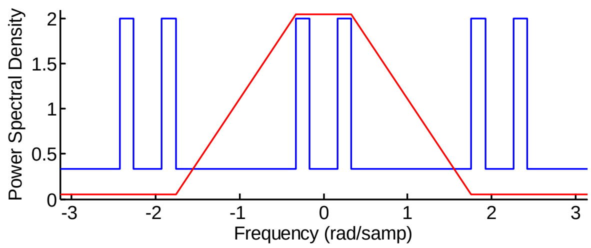
$\omega = \pm\{0.167, 0.333, 1.761, 1.928, 2.261, 2.428\}$.

Most people got the overall spectrum shape correct although a few expanded the spectrum horizontally by a factor of 3 instead of compressing it. Often the precise limits of the bandpass components were incorrectly calculated. Some made the bandpass regions even spaced apart. Very few reduced the magnitude by a factor of 3 even those that correctly said that $P_y = K^{-1}P_x$ *in the previous part.*

iv) The diagram of Fig. 1.3 is followed by a lowpass filter to remove spectral images. If $K = 3$ and $x[n]$ is as specified in part iii) above, determine the transition bandwidth and transition band centre frequency of a suitable lowpass filter and explain the reasons for your choices.

[2]

The required filter order is inversely proportional to the transition width, $\Delta\omega$ *which we therefore wish to make as wide as possible. We therefore make the transition width from the edge of the wanted* *signal component to the start of the first image component:* $\frac{1}{3} = 0.333$ *to* $\frac{2\pi-1}{3} = 1.761$ *which makes the total width equal to* $\frac{2\pi-2}{3} = 1.428$*. The centre of the transition band is the average of these two values and corresponds to the old Nyquist frequency, i.e.* $\frac{\pi}{3} = 1.047$*.*



Mostly OK.

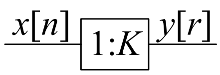


Figure 1.3

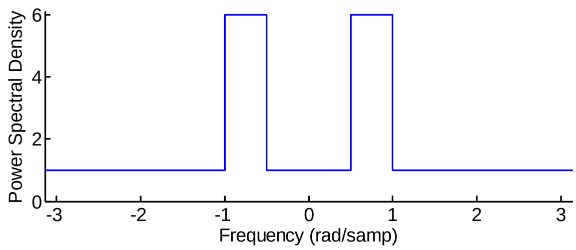


Figure 1.4

2. a) Suppose that $G_1(z) = 1 - pz^{-1}$ and $G_2(z) = 1 - qz^{-1}$ where the constants p and q may be complex. If $q = \frac{1}{p^*}$ show that $\left|G_1(e^{j\omega})\right| = \alpha \left|G_2(e^{j\omega})\right|$ for all ω and determine an expression for the constant α. [4]

For $z = e^{j\omega}$, we have $z^* = z^{-1}$ and so we can write

$$\begin{aligned}
|G_2(z)| &= \left|1 - (p^*)^{-1} z^{-1}\right| \\
|G_2(z)| = |G_2(z)^*| &= \left|1 - p^{-1} z\right| \\
&= \left|p^{-1} z\right| \left|pz^{-1} - 1\right| \\
&= |p|^{-1} |G_1(z)|
\end{aligned}$$

Thus the ratio $\frac{|G_1(z)|}{|G_2(z)|} = |p|$ is independent of ω and $\alpha = |p|$. Note the second line is the complex conjugate of the first and conjugation does not affect the magnitude of a complex number.

One of several alternative derivations is

$$\begin{aligned}
|G_1(z)|^2 = G_1(z) G_1^*(z) &= \left(1 - pz^{-1}\right)(1 - p^* z) \\
&= 1 - pz^{-1} - p^* z + pp^* \\
&= 1 - (q^*)^{-1} z^{-1} - q^{-1} z + (qq^*)^{-1} \\
&= (qq^*)^{-1} \left(qq^* - qz^{-1} - q^* z + 1\right) \\
&= (qq^*)^{-1} \left(1 - qz^{-1}\right)(1 - q^* z) \\
&= |q|^{-2} |G_2(z)|
\end{aligned}$$

from which $\alpha = \sqrt{|q|^{-2}} = |q|^{-1} = |p|$.

The question proved much harder than I expected; in part this was because of numerous algebraic errors involving complex numbers. Many people had difficulty with the algebra because of the modulus signs; some people just omitted the modulus signs completely which gives easy algebra but the wrong answer. Equations containing modulus signs are fine if you restrict yourself to multiplication and conjugation (as in the first method above) since $|xy| = |x|\,|y|$ and $|x^*| = |x|$. Alternatively, as in the second method, it can be a good idea to eliminate modulus signs from algebraic equations by squaring them (i.e. multiplying by the complex conjugate). This is usually a much better idea than using $|x| = \sqrt{\Re(x)^2 + \Im(x)^2}$ which is inevitably very messy. Some people decomposed $z = e^{j\omega} = \cos\omega + j\sin\omega$ and/or decomposed $p = a + jb$ which led to lots of algebra and occasionally the right answer. Several implicitly assumed p was real by writing $\left|1 - pz^{-1}\right| = \sqrt{(1 - p\cos\omega)^2 + p^2 \sin^2\omega}$.

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b) Suppose that $H_1(z) = 4 + 14z^{-1} - 8z^{-2}$. Determine the coefficients of $H_2(z)$ such that $\left|H_1(e^{j\omega})\right| = \left|H_2(e^{j\omega})\right|$ for all ω and that all the zeros of $H_2(z)$ lie inside the unit circle. [4]

To find the roots of $H_1(z)$ you can either treat it as a polynomial in z^{-1} or else treat $z^2 H(z)$ as a polynomial in z. These alternatives give $z^{-1} = \frac{-14 \pm \sqrt{14^2 + 128}}{-16} = \frac{-14 \pm 18}{-16} = \{2, -0.25\}$ or else $z = \frac{-14 \pm \sqrt{14^2 + 128}}{8} = \frac{-14 \pm 18}{8} = \{0.5, -0.4\}$. The *roots of* $H_1(z) = 4\left(1-0.5z^{-1}\right)\left(1+4z^{-1}\right)$ *are* $z = \{0.5, -4\}$ *and so the roots of* $H_2(z)$ *must be* $z = \{0.5, -0.25\}$ *which implies that* $H_2(z) = \alpha\left(1-0.5z^{-1}\right)\left(1+0.25z^{-1}\right)$. *To determine alpha, we substitute a suitable value of* z *that lies on the unit circle, e.g.* $z = 1$*, to obtain* $|H_2(1)| = |\alpha(1-0.5)(1+0.25)| = 0.625|\alpha| = |H_1(1)| = |4+14-8| = 10$. *From this we get* $\alpha = \pm\frac{10}{0.625} = \pm 16$ *and so* $H_2(z) = 16\left(1-0.25z^{-1}-0.125z^{-2}\right) = 16-4z^{-1}-2z^{-2}$ *or* $-16+4z^{-1}+2z^{-2}$.

We can check the scaling by evaluating at $z = 1$*:* $H_1(1) = 4+14-8 = 10$ *and* $H_2(1) = 16-4-2 = 10$*. These have the same magnitude so all is well.*

Several people just reversed all the coefficients (as in an allpass filter) to get $H_2(z) = -8+14z^{-1}+4z^{-2}$*. This inverts all the roots including the ones that were inside the unit circle to start with. Thus the roots of this polynomial are* 2 *and* -0.25*.*

---

c) When designing an IIR filter $H(e^{j\omega}) = \frac{B(e^{j\omega})}{A(e^{j\omega})}$ to approximate a complex target response $D(\omega)$ two error measures that may be used are the weighted solution error, $E_S(\omega)$, and the weighted equation error, $E_E(\omega)$, defined respectively by

$$\begin{aligned} E_S(\omega) &= W_S(\omega)\left(\frac{B(e^{j\omega})}{A(e^{j\omega})} - D(\omega)\right) \\ E_E(\omega) &= W_E(\omega)\left(B(e^{j\omega}) - D(\omega)A(e^{j\omega})\right). \end{aligned}$$

Explain the relative advantages of the two error measures and explain the purpose of the real-valued non-negative weighting functions $W_S(\omega)$ and $W_E(\omega)$.

[2]

The solution error is a direct measure of the error in the frequency response but has the disadvantage that it gives rise to a set of non-linear simultaneous equations which do not have a closed form solution. Although the equation error gives rise to a set of linear equations which are straightforward to solve, it multiplies the true frequency response errors by $A(e^{j\omega})$ *and so, unless* $W_E(\omega)$ *is adjusted accordingly, gives a higher weight to spectral regions in which* $\left|A(e^{j\omega})\right|$ *is large. Although the two errors are equivalent if there happens to be a solution that makes them zero, minimizing the equation error will not generally give the same solution as minimizing the solution error.*

The weighting functions, $W_S(\omega)$ *and* $W_E(\omega)$ *are used to control the relative importance of errors in different parts of the spectrum. A high weight will result in a lower error.*

Most understood the benefit of the equation error over the solution error. However, many people did not understand that the weighting functions allow you to decide that you would like higher accuracy in one part of the spectrum than another. For example, you might well want lower errors in the stop band than in the pass band.

---

d) Suppose that $0 \le \omega_1 < \omega_2 < \ldots < \omega_K \le \pi$ is a set of K frequencies and that $A(z) = 1 + [z^{-1}\ z^{-2}\ \cdots\ z^{-N}]\mathbf{a}$ and $B(z) = [1\ z^{-1}\ z^{-2}\ \cdots\ z^{-M}]\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are real-valued coefficient column vectors.

i) Show that it is possible to express the equations $E_E(\omega_k) = 0$ for $1 \le k \le K$ as a set of K simultaneous linear equations in the form

$$(\mathbf{P}\ \mathbf{Q})\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}=\mathbf{d}.$$

State the dimensions of the matrices **P** and **Q** and of the vector **d** and derive expressions for the elements of **P**, **Q** and **d**. [4]

If $E_E(\omega_k)=0$ then $W_E(\omega_k)\left(B(e^{j\omega_k})-D(\omega_k)A(e^{j\omega_k})\right)=0$. Substituting in the expressions for $A(z)$ and $B(z)$ gives

$$\begin{aligned}W_E(\omega_k)\,([1\ e^{-j\omega_k}\ e^{-j2\omega_k}\ \cdots\ e^{-jM\omega_k}]\mathbf{b}\\ -D(\omega_k)-D(\omega_k)[e^{-j\omega_k}\ e^{-j2\omega_k}\ \cdots\ e^{-jN\omega_k}]\mathbf{a}) &= 0.\end{aligned}$$

Rearranging this equation gives $\left(\mathbf{p}_k^T\ \mathbf{q}_k^T\right)\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}=W_E(\omega_k)D(\omega_k)$ *where* $\mathbf{p}_k^T=-W_E(\omega_k)D(\omega_k)[e^{-j\omega_k}\ e^{-j2\omega_k}\ \cdots\ e^{-jN\omega_k}]$ *and* $\mathbf{q}_k^T=W_E(\omega_k)[1\ e^{-j\omega_k}\ e^{-j2\omega_k}\ \cdots\ e^{-jM\omega_k}]$. *Note that there are $M+1$ coefficients in* **b** *but only N coefficients in* **a** *because the first coefficient of $A(z)$ is always equal to 1.*

Thus the dimensions of **P**, **Q** *and* **d** *are $K\times N$, $K\times(M+1)$ and $K\times 1$. and their k*[th] *rows are $\mathbf{p}_k^T$, $\mathbf{q}_k^T$ and $W_E(\omega_k)D(\omega_k)$ respectively.*

Note that finding a least squares solution to an equation is different from finding an exact solution. Thus if $f(x)g(x)=0$ has an exact solution, it must satisfy $f(x)=0$ or $g(x)=0$. However, if the original equation has no exact solution, its least-squares solution is not the same as the least squares solution to either f(x)=0 or g(x)=0. Thus in the equation $W_E(\omega_k)\left(B(e^{j\omega_k})-D(\omega_k)A(e^{j\omega_k})\right)=0$, you cannot just ignore the $W_E(\omega_k)$ term even if it is non-zero (several people did this). Many people got the dimensions wrong; some used M instead of $M+1$ and others said the number of columns of **P** *and/or* **Q** *was* $\max(M,N)$ *or something similar. The total number of columns of of* **P** *and* **Q** *must equal the number of unknowns $(N+M+1)$ and the number of rows of* **P**, **Q** *and* **d** *must be equal to the number of equations (K). Some assumed $k=0,\cdots,K$ even though the question says $k=1,\cdots,K$.*

---

ii) Explain how, by separating the real and imaginary parts of **P**, **Q** and **d**, it is possible to obtain a set of simultaneous linear equations for $\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}$ in which all coefficients are real-valued. Explain the circumstances under which some of the resultant equations will necessarily have all-zero coefficients. [4]

By treating the real and imaginary parts of each row as two separate equations, we can write $\begin{pmatrix}\Re(\mathbf{P}) & \Re(\mathbf{Q})\\ \Im(\mathbf{P}) & \Im(\mathbf{Q})\end{pmatrix}\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}=\begin{pmatrix}\Re(W_E(\omega_k)D(\omega_k))\\ \Im(W_E(\omega_k)D(\omega_k))\end{pmatrix}$ *in which all coefficients are real.*

If $W_E(\omega_k)=0$ for any particular k, then the corresponding pair of equations will have all-zero coefficients. In addition, if $\omega_k=0$ or π

then $\mathbf{p}_k^T$ and $\mathbf{q}_k^T$ will both be real-valued if, as is likely, $D(\omega_k)$ is real, so the equation formed from their imaginary parts will have all-zero coefficients.

Not everyone realized that this means that we will now have $2K$ equations (which is good: the more equations the better when you are trying to minimize errors). Some people suggested using only the real parts of the equations, but this is a bad idea as it leaves the imaginary parts unconstrained. Only a few people understood why the imaginary parts might be structurally zero.

---

iii) Explain why it may be desirable to apply the transformation of part b) after obtaining the solution to the equations of part d) ii). [2]

To obtain a stable filter, the zeros of $A(z)$ must lie inside the unit circle. There is no guarantee that the solutions of the equations will meet this requirement so the transformation of part b) may be applied to $A(z)$ to enforce stability. This will change the phase response but not the magnitude response of the resultant filter.

Some suggested that the transformation should also be applied to $B(z)$ in order to make it minimum phase. This is a possible goal but is not nearly as important as ensuring a stable filter.

---

iv) Assuming that $\omega_1 = 0$ and $\omega_K = \pi$, determine the minimum value of K to ensure that the equations of part d) ii) are not underdetermined. [4]

From part (ii) we get $2K$ real-valued equations. However, since $\mathbf{p}_k^T$ and $\mathbf{q}_k^T$ are real for $k = 1$ and $k = K$, we have a total of $2K - 2$ equations with $M + N + 1$ unknowns.

Therefore we require $2K - 2 \geq M + N + 1$ which implies that $K \geq \frac{M+N+3}{2}$.

Several said there were only $\max(M, N)$ unknowns but without any explanation.

---

e) Suppose now that $H(z) = \frac{b}{1+az^{-1}}$, that $K = 3$, that $\omega_k = \left\{0, \frac{\pi}{2}, \pi\right\}$, that

$$\begin{aligned} D(\omega) &= \begin{cases} 2 & \text{for } \omega \leq 0.25\pi \\ 1 & \text{for } \omega > 0.25\pi \end{cases}. \\ W_E(\omega) &\equiv 1 \end{aligned}$$

Determine the numerical values of the elements of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{d}$ and hence determine the numerical values of a and b that minimize $\sum_k |E_E(\omega_k)|^2$. [6]

You may assume without proof that the least squares solution to an overdetermined set of real-valued linear equations, $\mathbf{Rx} = \mathbf{q}$, is given by $\mathbf{x} = \left(\mathbf{R}^T\mathbf{R}\right)^{-1}\mathbf{R}^T\mathbf{q}$ assuming that $\mathbf{R}$ has full column rank.

Since $K=3$ the values of ω_k are $\{0\; 0.5\pi\; \pi\}$ and hence $e^{-j\omega_k}=\{1\; -j\; -1\}$. At these frequencies $D(\omega_k)=\{2\,1\,1\}$. The complex equations from part d)i) are therefore given by $\begin{pmatrix} -2 & 1 \\ j & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ *where the vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *each contain only a single element. Thus* $\mathbf{P}=\begin{pmatrix} -2 \\ j \\ 1 \end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ *and* $\mathbf{d}=\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

From this, applying the technique of d)ii), we get a set of four real equations $\begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

Hence, using the formula given in the question, we can write

$$\begin{aligned}
\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} &= \left(\begin{pmatrix} -2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\right)^{-1}\begin{pmatrix} -2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 6 & -1 \\ -1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \\
&= \frac{1}{17}\begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix}=\frac{1}{17}\begin{pmatrix} -5 \\ 21 \end{pmatrix}
\end{aligned}$$

Where, in the last line, we have used the formula for the inverse of a 2×2 matrix, $\begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1}=\frac{1}{ps-rq}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$.

From this $a=-0.294$ and $b=1.235$.

Equivalently, you can just solve the simultaneous equations $6a-b=3$ and $-a+3b=4$.

Surprisingly, many people did not convert the 3 complex-valued equations into 4 real-valued equations despite having given a correct answer to part (ii). As a result, they ended up with complex values for a and b.

3. a) Figure 3.1 shows the block diagram of a system that multiplies the input sample rate by $\frac{P}{Q}$ where P and Q are coprime with $P < Q$.

i) Explain why the cutoff frequency of the lowpass filter $H(z)$ should be placed at the Nyquist rate of the output signal, $y[m]$ and give the normalized cutoff frequency, ω_0, in rad/sample in terms of P and/or Q.

Using the approximation formula $M \approx \frac{a}{3.5\Delta\omega}$, determine the required filter order M in terms of P and/or Q if the stopband attenuation in dB is $a = 60$ and the normalized transition bandwidth is $\Delta\omega = 0.1\omega_0$. [4]

The lowpass filter must eliminate the images introduced by the up-sampler and the alias components introduced by the down sampler and must therefore eliminate all frequencies above the lower of the input and output Nyquist frequencies. Since $Q > P$,the output Nyquist frequency is $\frac{\pi}{Q}$. The normalized cutoff frequency is therefore $\omega_0 = \frac{\pi}{\max(P,Q)} = \frac{\pi}{Q}$.

We have $M = \frac{60}{3.5\times 0.1\omega_0} = \frac{60Q}{3.5\times 0.1\pi} = 54.6Q$.

Most correctly said that the filter cutoff should be the lower of the two Nyquist frequencies but several people calculated its value incorrectly. Some gave the Nyquist frequency as $\frac{2\pi}{Q}$, some used $\min(P,Q)$ instead of $\max(P,Q)$, and some normalized ω_0 by the input sample rate of $x[n]$ rather than by the sample rate of $v[r]$ to get $\frac{\pi P}{Q}$. The cutoff frequency of a filter is always normalized by the sampling frequency at the input and output of the filter itself. Some left the answer as $\omega_0 = \frac{\pi}{\max(P,Q)}$ even though the question explicitly said that $Q > P$; this makes the answers to the remaining parts of the question much messier.

---

ii) Using the value of M from part a) i), estimate the average number of multiplications per input sample, $x[n]$, needed to implement the system in the form of Figure 3.1. [2]

The filter requires $M + 1$ multiplications per filter output sample, $v[r]$, which equals $(M+1)P$ per input sample, $x[n]$ (since there are P times as many samples of $v[r]$ as there are of $x[n]$). Substituting $M = 54.6Q$ gives $(54.6Q+1)P \approx 54.6PQ$ multiplications per input sample.

Many people did not multiply by P and gave the answer as $(M+1)$. A few said $\frac{(M+1)}{P}$ instead of $(M+1)\times P$. In the diagram of Figure 3.1, the lowpass filter operates at P times times the sample rate of the input $x[n]$ and requires $M+1$ multiplications for each value of $v[r]$ even though most of the samples at the input of the filter are zero; it is this inefficiency that allows polyphase decomposition to gain an advantage. Some people included the number of additions as well (giving $2M+1$ instead of $M+1$ per $v[r]$) even though the question explicitly asked only for multiplications.

iii) The filter $H(z)$ has a symmetrical impulse response $h[r] = g[r]w[r]$ for $0 \le r \le M$ where $g[r]$ is the impulse response of an ideal lowpass filter with cutoff frequency ω_0 and $w[r]$ is a symmetrical window function.

Derive an expression for the ideal response, $g[r]$, in terms of ω_0, M and r. [4]

The ideal response (centered on $r = 0$) is $G(e^{j\omega}) = 1$ for $|\omega| < \omega_0$ and zero otherwise. To this ideal response, we need to add a delay of $\frac{M}{2}$ samples which corresponds to a phase shift of $e^{-j0.5M\omega}$ (or, equivalently, we can design a centred filter and then delay the coefficients by $0.5M$ samples to make it causal). Using the inverse DTFT (available in the formula sheet)

$$\begin{aligned} g[r] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{-j0.5M\omega}e^{j\omega r}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(r-0.5M)}d\omega = \frac{1}{j2(r-0.5M)\pi}\left[e^{j\omega(r-0.5M)}\right]_{-\omega_0}^{\omega_0} \\ &= \frac{1}{j2(r-0.5M)\pi} \times 2j\sin((r-0.5M)\omega_0) = \frac{\sin((r-0.5M)\omega_0)}{\pi(r-0.5M)} \end{aligned}$$

We can also write this as $g[r] = \frac{\omega_0}{\pi}\operatorname{sinc}((r-0.5M)\omega_0)$ where $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$.

Many people omitted the offset of $0.5M$. Instead of calculating it explicitly, some just stated $g[r] = \omega_0 \operatorname{sinc}(r\omega_0)$ which is correct except for the offset and the factor of $\frac{1}{\pi}$.

A few people said that $g[r] = \begin{cases} 1 & |r| < \omega_0 \\ 0 & \text{otherwise} \end{cases}$ which mixes up the time and frequency domains. A few wrote the correct integral but were unable to integrate $\int e^{j\omega r}d\omega$; converting it to $\int(\cos\omega r + j\sin\omega r)\,d\omega$ makes the integral slightly harder if anything. Quite a few people tried to calculate $H(e^{j\omega})$ instead of $g[r]$ which was what the question asked for.

b) The filter $H(z)$ is now implemented as a polyphase filter as shown in Fig. 3.2. The filter implementation uses a single set of delays and multipliers with commutated coefficients.

i) State the length of the filter impulse response $h_0[n]$ in terms of M, P and/or Q and give an expression for the coefficients $h_0[n]$ in terms of $h[r]$. [2]

The length of the filter $h_0[n]$ is $\frac{M+1}{P}$ rounded up to the nearest integer (the order of $h_0[n]$ is one less than this). The coefficients are given by $h_0[n] = h[nP]$.

Some said $\frac{M+1}{Q}$ instead of $\frac{M+1}{P}$. From the diagram, the sub-filters *are labelled $h_0[n], \cdots, h_{P-1}[n]$ and so, since there are P of them and the total number of coefficients stays the same, each one must be of length $\frac{M+1}{P}$. A few said $\frac{M}{P}$ because they forgot that an FIR filer of order M has $M+1$ coefficients.*

---

ii) If $x[n] = 0$ for $n < 0$, give expressions for $v[0]$, $v[1]$, $v[2P+1]$ in terms of the input $x[n]$ and the coefficients $h_p[n]$. [2]

Since $x[n]$ is causal, $v[0] = h_0[0]x[0]$, $v[1] = h_1[0]x[0]$ and $v[2P+1] = h_1[0]x[2] + h_1[1]x[1] + h_1[2]x[0]$.

For each input sample, n, there are P samples of $v[r]$, namely

$\{v[Pn+0], v[Pn+1], \cdots, v[Pn+P-1\},$

one coming from each of the filters $\{h_0[n], h_1[n], \cdots, h_{P-1}[n]\}$. If we just consider one of the sub-filters, the outputs from filter $h_0[n]$ are $v[Pn] = \sum_{s=0}^{S} h_0[s]x[n-s]$ and, in general, the outputs from filter $h_p[n]$ are $v[Pn+p] = \sum_{s=0}^{S} h_p[s]x[n-s]$ where $S = \lceil \frac{M+1}{P} - 1 \rceil$ is the order of the sub-filters.

Because $x[n]$ is causal, $v[Pn+p]$, has only $n+1$ non-zero terms. So, for example, $v[0], \cdots, v[P-1]$ all depend only on $x[0]$ and

$v[P], \cdots, v[2P-1]$ depend on $x[0]$ and $x[1]$ only.

Some said $v[1] = h_1[0]x[1]$ or else $v[1] = h_1[0]x[1] + h_1[1]x[0]$, the second expression is actually the formula for $v[P+1]$. A few wrote things like $v[0] = h_0[n]x[n]$; an expression like this makes no sense since "n" appears only on the right hand side.

---

iii) Explain how it is possible to eliminate the output decimator by changing both the sequence and rate at which the coefficient sets, $h_p[n]$ are accessed.

Determine the new coefficient set order for the case $P = 5$ and $Q = 7$. [3]

The output decimator selects every Q^{th} sample of $v[r]$ and discards the others. Therefore if we access the coefficient sets in the order $p = (mQ) \mod P$ for $m = 0, 1, \cdots$ and reduce the rate by a factor of Q will will generate only the wanted output samples.

For the specific values $P = 5$ and $Q = 7$, so the coefficient set sequence becomes

$h_0[n], h_{7 \mod 5}[n], h_{14 \mod 5}[n], h_{21 \mod 5}[n], h_{28 \mod 5}[n]$

which equals $h_0[n], h_2[n], h_4[n], h_1[n], h_3[n]$.

Since $Q \mod P = 2$ the value of p increments each time either by 2 or by $2 - P = -3$

Quite well done on the whole although not everyone got the correct sequence. The original question used "order" instead of "sequence" which several people interpreted as the order (i.e. length) of the *subfilters. Quite a few people suggested applying the Noble identity to swap $H_p(z)$ and $Q:1$ but this is only possible if the filter is $H_p(z^Q)$ which is not the case.*

iv) Determine the number of multiplications per input sample for the system of part b) iii) and the number of distinct coefficients that must be stored. You may assume that $M+1$ is a multiple of P. [2]

We require $\frac{M+1}{P}$ multiplications per output sample, $y[m]$, and therefore $\frac{M+1}{P} \times \frac{P}{Q} = \frac{M+1}{Q}$ multiplications per input sample, $x[n]$. Because of symmetry, the number of distinct coefficients is only $\frac{M+1}{2}$ although it is not necessarily easy to take advantage of this symmetry to reduce the storage requirements.

Although not requested in the question, it is interesting to note that since $M \approx 54.6Q$, the number of multiplications per input sample is approximately 54.6 independently of P or Q.

Although most got the number of multiplications per output sample correct (at $\frac{M+1}{P}$), not everyone multiplied by the correct factor to get the number of multiplications per input sample. Since the overall system reduces the sample rate, there must be fewer multiplications per input sample than per output sample; hence we multiply by $\frac{P}{Q}$ which is < 1.

c) Suppose now that the sample rate of the input, $x[n]$, is 18 kHz and that the system is implemented as in part b) iii) with the values of a and $\Delta\omega$ as given in part a) i).

Determine the values of P, Q and M when the sample rate of the output, $y[m]$, is (i) 10 kHz and (ii) 10.1 kHz [note that 101 is a prime number].

For each of these cases estimate the number of multiplications per input sample and the number of distinct coefficients that must be stored. [5]

(i) For an output sample rate of $10\,\text{kHz} = \frac{5}{9} \times 18\,\text{kHz}$, $P = 5$ and $Q = 9$. $M = 54.6Q = 491$. The number of multiplications per input sample is therefore $\frac{M+1}{Q} = 54.7$. The total number of distinct coefficients is $\frac{M+1}{2} = 246$.

(ii) For an output sample rate of $10.1\,\text{kHz} = \frac{101}{180} \times 18\,\text{kHz}$, $P = 101$ and $Q = 180$. $M = 54.6Q = 9822$. The number of multiplications per input sample is therefore $\frac{M+1}{Q} = 54.6$ (virtually unchanged). The total number of distinct coefficients is $\frac{M+1}{2} = 4912$ (much increased).

A small number of people did not know that "coprime" means "having no common factors" and therefore used $P = 10$ and $Q = 18$ for the first example.

d) In a Farrow filter, the coefficients, $h_p[n]$, are approximated by a low-order polynomial $f_n(t)$ where $t = \frac{p}{P}$ for $0 \le p \le P-1$.

i) Assuming that a rectangular window, $w[r] \equiv 1$, is used in the design of $H(z)$ and that $\omega_0 = \frac{\pi}{P}$, give an expression for the target value of $f_0(t)$ in terms of t, M, P and Q. [3]

From the answer to part a)iii) $h[r] = g[r] = \frac{\sin(r-0.5M)\omega_0}{\pi(r-0.5M)}$ *so* $h_p[0] = g[p] = \frac{\sin((p-0.5M)\omega_0)}{\pi(p-0.5M)} = f_0\left(\frac{p}{P}\right)$. *We now substitute* $p \to Pt$ *and* $\omega_0 = \frac{\pi}{Q}$ *to get* $f_0(t) = g[Pt] = \frac{\sin\left((Pt-0.5M)\frac{\pi}{Q}\right)}{\pi(Pt-0.5M)}$ *for* $0 \le t \le 1$.

Although not requested in the question, a more general formula is $h_p[n] = g[Pn+p] = \frac{\sin((Pn+p-0.5M)\omega_0)}{\pi(Pn+p-0.5M)} = f_n\left(\frac{p}{P}\right)$ *from which* $f_n(t) = g[P(n+t)] = \frac{\sin\left((P(t+n)-0.5M)\frac{\pi}{Q}\right)}{\pi(P(t+n)-0.5M)}$ *for* $0 \le t \le 1$ *and* $0 \le n \le \left\lceil \frac{M+1}{P} - 1 \right\rceil$.

Very few attempted this part.

---

ii) If the polynomials, $f_n(v)$, are of order $K = 5$, determine the number of coefficients that must be stored for each of the cases defined in part c). [3]

The $\frac{M+1}{P}$ *polynomials* $f_n(t)$ *each require* $K = 6$ *coefficients, so we require a total of* $\frac{6(M+1)}{P}$ *coefficients. For the two cases, this gives (i)* $\frac{6\times 492}{5} = 590$ *(somewhat larger than before) and (ii)* $\frac{6\times 9823}{101} = 584$ *(much less than before).*

Very few attempted this part.

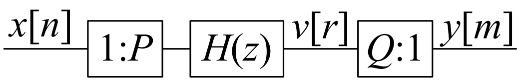


Figure 3.1

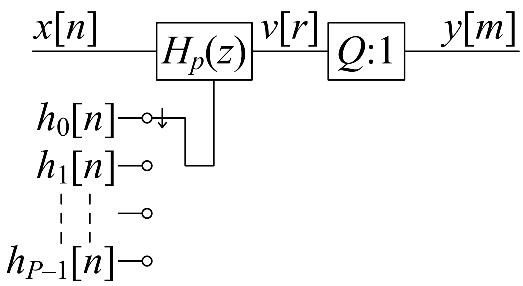


Figure 3.2

4. A complex-valued frequency-modulated signal, $x(t) = a(t)e^{j\phi(t)}$, has a $0\,Hz$ carrier frequency and a peak frequency deviation of $d = 75\,\text{kHz}$. The amplitude, $a(t)$, is approximately constant with $a(t) \approx 1$ and the phase is $\phi(t) = k\int_0^t m(\tau)d\tau$ where k is a constant and $m(t)$ is a baseband audio signal with bandwidth $b = 15\,\text{kHz}$. The signal $x(t)$ is sampled at $400\,\text{kHz}$ to obtain the discrete-time signal $x[n]$.

a) Carson's rule for the bandwidth of a double-sideband FM signal is $B = 2(d+b)$. Use this to determine the single-sided bandwidth, ω_0, of $x[n]$ in radians/sample. [2]

From Carson's rule, $B = 180\,\text{kHz}$. This bandwidth includes both sidebands, so $\omega_0 = 2\pi \times \frac{90\,\text{kHz}}{400\,\text{kHz}} = 0.45\pi = 1.41\,\text{rad/samp}$.

Mostly correct. A few people did not divide by $400\,\text{kHz}$ which gave an answer of $5.7 \times 10^5\,\text{rad/samp}$; except for very rare circumstances, frequencies in units of rad/samp always lie in the range $\pm\pi$.

---

b) Show that $m(t) = k^{-1}a^{-2}(t)\Im\left(x^*(t)\frac{dx(t)}{dt}\right)$ where $\Im(\,)$ denotes the imaginary part. [4]

From the definition of $\phi(t)$, $m(t) = k^{-1}\frac{d\phi}{dt}$. Differentiating $x(t)$ gives $\frac{dx}{dt} = \frac{da}{dt}e^{j\phi(t)} + ja(t)\frac{d\phi}{dt}e^{j\phi(t)}$ from which (since $a(t)$ is real-valued), $x^*(t)\frac{dx(t)}{dt} = a(t)\frac{da}{dt} + ja^2(t)\frac{d\phi}{dt}$ and hence $\Im\left(x^*(t)\frac{dx(t)}{dt}\right) = a^2(t)\frac{d\phi}{dt}$. It follows that $k^{-1}a^{-2}(t)\Im\left(x^*(t)\frac{dx(t)}{dt}\right) = m(t)$ as required.

Note that since $x(t) = a(t)e^{j\phi(t)}$, we could alternatively write $x^*(t) = a(t)e^{-j\phi(t)} = a^2(t)x^{-1}(t)$ which results in $m(t) = k^{-1}\Im\left(x^{-1}(t)\frac{dx(t)}{dt}\right)$. However this expression is harder to implement because it involves taking the reciprocal, $x^{-1}(t)$, of a rapidly varying complex number instead of the reciprocal, $a^{-2}(t)$, of a slowly varying real number whose value is always close to 1.

Some people derived instead the alternative expression involving $x^{-1}(t)$ by differentiating $\log x(t)$. Several assumed that $a(t) \equiv 1\,\forall t$ which stronger than the statement in the question.

---

c) Figure 4.1 shows a block diagram that implements the equation of part b) in discrete time. Complex-valued signals are shown as bold lines and are represented using their real and imaginary parts. The block labelled "Conj" takes the complex conjugate of its input. The differentiation block, $D(z)$, is designed as an FIR filter using the window method with a target response

$$\overline{D}(e^{j\omega}) = \begin{cases} jc\omega & \text{for } |\omega| \le \omega_1 \\ 0 & \text{otherwise} \end{cases}$$

where c is a scaling constant.

i) Determine the impulse response $\bar{d}[n]$ of $\overline{D}(z)$ in simplified form. [4]

From the inverse DTFT formula (included in the formula sheet) we *use integration by parts to obtain*

$$\begin{aligned}
\bar{d}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{D}\left(e^{j\omega}\right)e^{j\omega n}d\omega \\
&= \frac{jc}{2\pi}\int_{-\omega_1}^{\omega_1}\omega e^{j\omega n}d\omega \\
&= \frac{jc}{2\pi}\left[\frac{\omega}{jn}e^{j\omega n}-\frac{1}{(jn)^2}e^{j\omega n}\right]_{-\omega_1}^{\omega_1} \\
&= \frac{jc}{2\pi}\left(\frac{\omega_1}{jn}2\cos n\omega_1-\frac{1}{(jn)^2}2j\sin n\omega_1\right) \\
&= \frac{c}{\pi n^2}\left(n\omega_1\cos n\omega_1-\sin n\omega_1\right).
\end{aligned}$$

For interest only, the above formula gives $\bar{d}[-5:5]=[-0.07,\ 0.18,\ -0.19,\ -0.02,\ 0.71,\ 0,\ -$ *This is fairly similar to the simplest zero-phase differentiator which would be* $\bar{d}[-1:1]=[0.5,\ 0,\ -0.5]$.

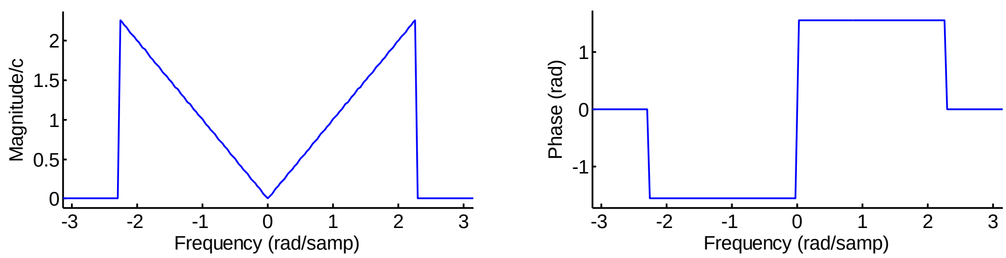
Some tried to use the inverse z-transform instead of the inverse DTFT but not everyone who chose this method realized that the inverse z-transform involves a contour integration in the complex plane. To use it, you have to choose an integration contour; if you choose the unit circle as your contour (a sensible choice), then the inverse z-transform is identical to the inverse DTFT.

---

ii) Assuming that $\omega_1=\frac{\omega_0+\pi}{2}$, draw dimensioned sketches showing the magnitude and phase responses of $\overline{D}(e^{j\omega})$ over the range $-\pi\le\omega\le\pi$.

[3]

Since $\omega_0=0.45\pi=1.41$, $\omega_1=\frac{\omega_0+\pi}{2}=2.28$. *For* $\omega>\omega_1$, $\overline{D}(e^{j\omega})=0$ *and so the phase is indeterminate (shown here as zero). For* $|\omega|\le\omega_1$, *we can write* $\left|\overline{D}(e^{j\omega})\right|=|jc\omega|=c\times|\omega|$ *and* $\angle\overline{D}(e^{j\omega})=\angle jc\omega=\frac{\pi}{2}[-\pi\ if\ \omega<0]=\frac{\pi}{2}\operatorname{sgn}(\omega)$ *assuming* $c>0$.



Surprisingly a large number of people made the magnitude gain, $\left|\overline{D}(e^{j\omega})\right|$, *negative for* $\omega<0$. *Some thought that the phase too was* $\propto\omega$ *or that it was always* $+\frac{\pi}{2}$.

---

iii) Assume that the DTFT of the window function used when designing $D(z)$ has a main lobe width of $\omega=\pm\frac{18}{M+1}$ for a window of length

$M+1$. If ω_1 is chosen as $\omega_1 = \frac{\omega_0+\pi}{2}$, determine the smallest value of M that will ensure that the transition in the response of $D(e^{j\omega})$ near $\omega = \omega_1$ lies completely within the range (ω_0, π). [3]

The transition in the response of $D(e^{j\omega})$ near the discontinuity in $\overline{D}(e^{j\omega})$ near $\omega = \omega_1 = 2.28$ will extend for $\frac{18}{M+1}$ either side of the discontinuity for a total width of $\frac{36}{M+1}$. So we need $\frac{36}{M+1} \le \pi - \omega_0 = 0.55\pi$ from which $M \ge \frac{36}{\pi-\omega_0} - 1 = \frac{36}{0.55\pi} - 1 = \frac{36}{1.728} - 1 = 20.83 - 1 = 19.83 \approx 20$.

Some took the width to be $\frac{18}{M+1}$ rather than $\frac{36}{M+1}$.

---

iv) Stating any assumptions, determine the maximum value of c that will ensure $|s[n]| \le 1$ where $s[n]$ is the output of the differentiation block, $D(z)$, as shown in Figure 4.1. [4]

We assume that $a(t) \equiv 1$ and that $D(e^{j\omega}) = \overline{D}(e^{j\omega})$. Then, at the maximum frequency deviation of 75 kHz, $x[n] = e^{j\omega_f n}$ where $\omega_f = 2\pi \times \frac{75}{400} = 1.18$. To ensure $|s[n]| \le 1$, we require $\left|D(e^{j\omega_f})\right| = c\omega_f \le 1$. Hence $c \le \frac{1}{\omega_f} = \frac{400}{2\pi\times 75} = 0.849$.

Many people chose instead to make $c \le \frac{1}{\omega_1} = 0.44$ which is over conservative.

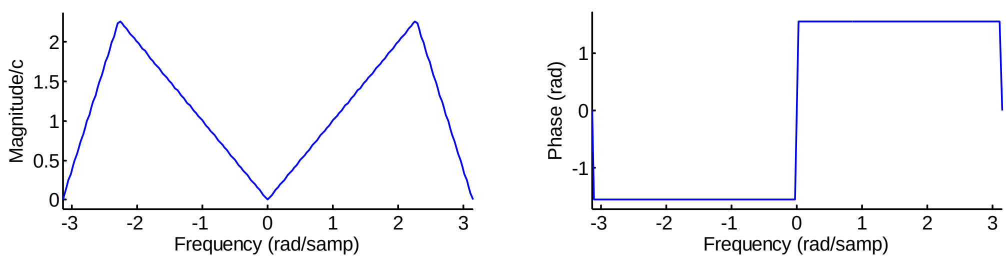
---

d) An alternative choice for the target response is

$$\widetilde{D}(e^{j\omega}) = \begin{cases} \frac{-jc\omega_1(\pi+\omega)}{\pi-\omega_1} & \text{for } -\pi < \omega \le -\omega_1 \\ jc\omega & \text{for } |\omega| \le \omega_1 \\ \frac{jc\omega_1(\pi-\omega)}{\pi-\omega_1} & \text{for } \omega_1 < \omega \le \pi \end{cases}$$

i) Assuming that $\omega_1 = \frac{\omega_0+\pi}{2}$, draw dimensioned sketches showing the magnitude and phase responses of $\widetilde{D}(e^{j\omega})$ over the range $-\pi \le \omega \le \pi$.

[4]



Again, quite a few people made the magnitude gain negative for $\omega < 0$.

---

ii) Outline the relative advantages and disadvantages of using $\widetilde{D}(e^{j\omega})$

rather than $\overline{D}(e^{j\omega})$ as the target response when designing $D(e^{j\omega})$.

[2]

The advantage is that since the $\widetilde{D}(e^{j\omega})$ is a continuous function of ω it will not be affected by Gibbs phenomenon and the coefficients will decay $\propto n^{-2}$ instead of $\propto n^{-1}$. For any given filter length, $M+1$, the errors will be much smaller for $\widetilde{D}(e^{j\omega})$ than for $\overline{D}(e^{j\omega})$.

The disadvantage is that the gain is no longer approximately zero for frequencies above ω_1 and so it may be necessary to include additional filtering in the channel selection process in order to remove frequency components between ω_0 and π.

Many people said correctly that a smooth target response was good, but few were precise about why this was so.

---

e) An alternative structure that avoids any divisions is shown in Fig. 4.2 where the polynomial $f(v)$ is the truncated Taylor series for v^{-1} expanded around $v = 1$. Determine $f(v)$ for the cases when it is (i) a linear expression and (ii) a quadratic expression. In each case determine the gain error (expressed in dB) resulting from the approximation when $a(t) = 1.1$. [4]

If we write $f(v) = v^{-1}$*,we can write* $f(v) = f(1) + (v-1)f'(1) + \frac{1}{2!}(v-1)^2 f''(1) + \ldots = 1 + (v-1) \times -1 + \frac{1}{2!}(v-1)^2 \times 2 + \ldots$ *The Taylor series can also be obtained easily from the geometric progression formula by writing* $w = 1 - v$*. Then* $v^{-1} = (1-w)^{-1} = 1 + w + w^2 + w^3 + \ldots = 1 + (1-v) + (1-v)^2 + \ldots$*.*

(i) Linear case: $f(v) = 1 + (1-v) = 2 - v$*. Thus we multiply by* $2 - a^2$ *instead of* a^{-2}*. The error is therefore found by substituting* $a = 1.1$ *in the ratio of the actual gain* $(2-a^2)$ *to the ideal gain* (a^{-2}) *to obtain* $\frac{2-a^2}{a^{-2}} = 0.959 = -0.392\,\text{dB}$.

(ii) Quadratic case: $f(v) = 1 + (1-v) + (1-v)^2 = 3 - 3v + v^2$*. The error is therefore found by substituting* $a = 1.1$ *in the ratio* $\frac{3-3a^2+a^4}{a^{-2}} = 1.009 = +0.0801\,\text{dB}$.

Few attempted this part. Many of those who did attempt it, had difficulty in writing down the Taylor series even for the linear case.

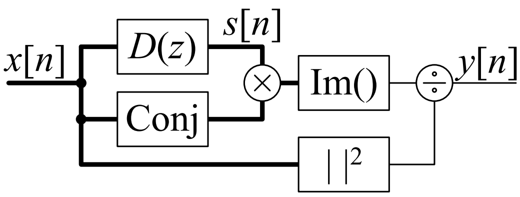


Figure 4.1

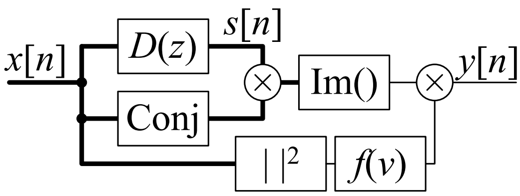


Figure 4.2