IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2016

MSc and EEE/EIE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Friday, 29 April 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible   First Marker(s) :   D.M. Brookes

                             Second Marker(s) :   P.T. Stathaki

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Information for Candidates:

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.

- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.

- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.

- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.

- Where necessary, the sample rate of a signal in a block diagram is indicated in the form "$@ f$".

Abbreviations

BIBO	Bounded Input, Bounded Output		IIR	Infinite Impulse Response
CTFT	Continuous-Time Fourier Transform		LTI	Linear Time-Invariant
DCT	Discrete Cosine Transform		MDCT	Modified Discrete Cosine Transform
DFT	Discrete Fourier Transform		PSD	Power Spectral Density
DTFT	Discrete-Time Fourier Transform		SNR	Signal-to-Noise Ratio
FIR	Finite Impulse Response			

A datasheet is included at the end of the examination paper.

1. a) The signals $x[n]$ and $y[n]$ are defined as

$$x[n] = \begin{cases} 2^{-n} & n > 0 \\ 0 & n \le 0 \end{cases}, \quad \text{and} \quad y[n] = \begin{cases} 0 & n > 0 \\ 5^{-n} & n \le 0 \end{cases}.$$

   i) Determine the z-transform of $x[n]$ and its region of convergence. [3]

   ii) Determine the z-transform of $y[n]$ and its region of convergence.[3]

   You may assume without proof that $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ provided that $|\alpha| < 1$.

   b) Consider the convolution $y[n] = h[n] * x[n] = \sum_{m=0}^{M} h[m]x[n-m]$ where $h[n]$ is the impulse response of an FIR filter of order M (i.e. $n \in [0, M]$) and $x[n]$ is a signal defined for $n \in [-\infty, \infty]$.

   In the overlap-save method of convolution, $y[n]$ is divided into blocks of length K and a circular convolution of length $K + M$ is used to calculate each block. To calculate block b, the circular convolution evaluates

$$y[bK + n] = \sum_{m=0}^{M} h[m]x[(n - m + M)_{\mathrm{mod}(K+M)} - M + bK]$$

   for $n \in [0, K - 1]$. The notation $P_{\mathrm{mod}\,Q}$ denotes the remainder when P is divided by Q and satisfies $0 \le P_{\mathrm{mod}\,Q} < Q$.

   i) Show that the expression given above is equivalent to the direct convolution given by $y[bK + n] = \sum_{m=0}^{M} h[m]x[bK + n - m]$. [3]

   ii) Suppose that a circular convolution of length R requires approximately $5R \log_2 R$ multiplications. If $M = 200$, estimate the number of multiplications per output sample required by the overlap-save method when $K = 20$, 1500 and 10^4 and compare these results with the number of multiplications required for implementing the direct convolution. [4]

c)   The filter $H(z)$ is given by

$$H(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.8z^{-2}}.$$

The pole-zero diagram and magnitude response (in dB) of $H(z)$ are shown in Figures 1.1 and 1.2 respectively.

    i)   Determine the transfer function $F(z) = H(z^2)$ and sketch its pole-zero diagram and magnitude response in dB. It is not necessary to determine exact values of the magnitude response.   [4]
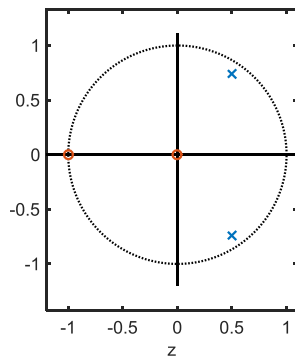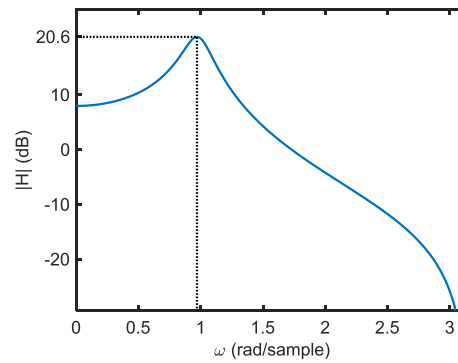
    ii)   Determine the transfer function $G(z) = H(1.25z)$ and sketch its pole-zero diagram and magnitude response in dB. It is not necessary to determine exact values of the magnitude response.   [4]

Figure 1.1

Figure 1.2

d)   A bilinear transformation, $s = \alpha \frac{z-1}{z+1}$, is used to convert a continuous-time filter into a discrete-time filter.

    i)   Show that if $\alpha = \frac{\Omega_0}{\tan(0.5\omega_0)}$ then $z = e^{j\omega_0} \Leftrightarrow s = j\Omega_0$.   [3]

    ii)   A continuous-time highpass filter with a cutoff frequency of $1\,\text{kHz}$ is given by $H(s) = \frac{s}{s + \Omega_0}$ where $\Omega_0 = 2000\pi\,\text{rad/s}$. Using the bilinear transformation given above, determine the coefficients (to 3 decimal places) of a discrete-time filter having an unnormalized cutoff frequency of $1\,\text{kHz}$. The sample frequency is $8\,\text{kHz}$.   [4]

e) i) Explain why the average power of a discrete time signal (i.e. the average energy per sample) is always decreased by upsampling but is normally unchanged by downsampling. Give an example of a signal for which the latter statement is untrue. [3]

ii) Figure 1.3 shows the power spectral density (PSD) of a real-valued signal, $x[n]$; the horizontal portions of the PSD have values 1 or 4. The signal $y[m]$ is obtained by upampling $x[n]$ by a factor of 3 as shown. Draw a dimensioned sketch of the PSD of $y[m]$ giving the values of all horizontal portions of the graph and the values of all frequencies at which there is a discontinuity in the PSD. [3]
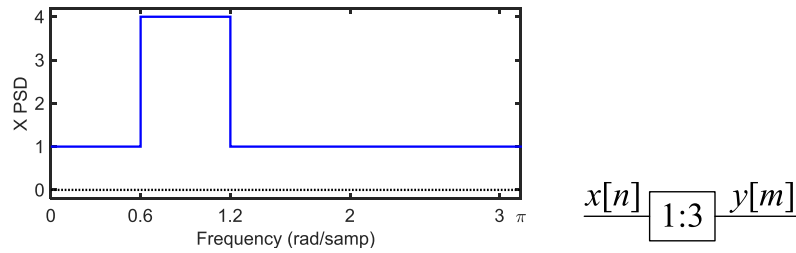
Figure 1.3

f) Figure 1.4 shows the block diagram of a two-band analysis processor. The inputs to the adder/subtractor blocks, $\oplus$, are additive unless labelled with a minus sign in which case they are subtractive.

i) By using the Noble identities or otherwise, determine $H_0(z)$ and $H_1(z)$ so that Figure 1.5 is equivalent to Figure 1.4. [3]

ii) Assuming that $P_0(z)$ and $P_1(z)$ are FIR filters with real-valued coefficients, show that $\left|H_1\left(e^{j\omega}\right)\right| = \left|H_0\left(e^{j(\pi-\omega)}\right)\right|$ and explain the significance of this relationship. [3]
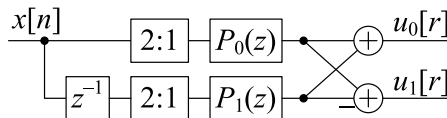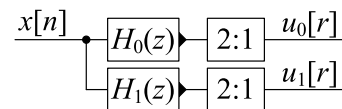

Figure 1.4

Figure 1.5

2. In the block diagram of Figure 2.1 the outputs of all adders and delay elements are on the right and solid arrows indicate the direction of information flow. The real-valued gain of each multiplier is written adjacent to its triangular symbol.
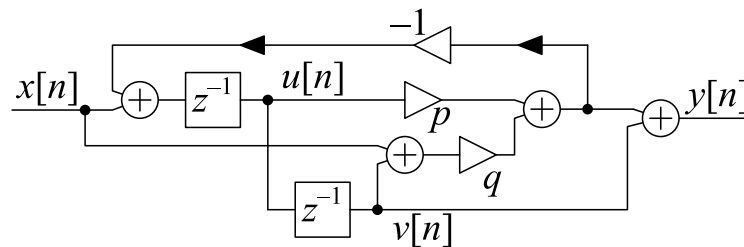

Figure 2.1

a) Show that [8]

$$G(z) = \frac{Y(z)}{X(z)} = \frac{q + pz^{-1} + z^{-2}}{1 + pz^{-1} + qz^{-2}}.$$

b) Prove that $\left|G(e^{j\omega})\right| = 1$ for all ω. [6]

c) Figure 2.2 shows a graph of $\angle G(e^{j\omega})$ when $p = -1.2$ and $q = 0.8$. The dotted lines indicate the frequencies, $\omega = \{a, b, c\}$, at which $G(e^{j\omega}) = \{-j, -1, +j\}$ respectively. Given that $G(e^{jb}) = -1$, derive the general formula $\cos b = \frac{-p}{q+1}$ and find the numerical value of b for the given values of p and q. [6]
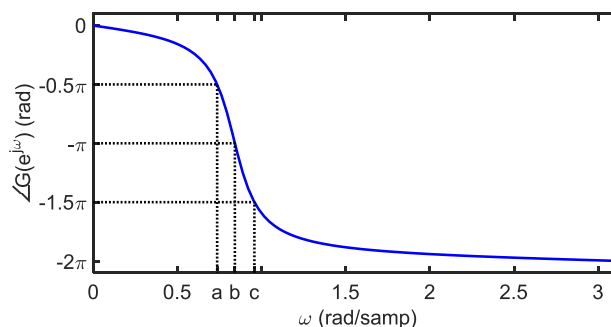

Figure 2.2

d) The filter $H(z)$ is defined as $H(z) = \frac{1}{2}(1 + G(z))$.

   i) Determine the value of $H(e^{j\omega})$ for each of $\omega = \{a, b, c\}$ defined above.
   Hence, for $p = -1.2$ and $q = 0.8$, sketch a graph of $\left|H(e^{j\omega})\right|$ for $\omega \in [0, \pi]$ using linear scales for both axes. [6]

   ii) For $p = -1.2$ and $q = 0.8$, determine the poles and zeros of $H(z)$ in polar form and sketch a diagram of the complex plane that includes the unit circle and the poles and zeros of $H(z)$ (indicated by $\times$ and $\circ$ respectively). [4]

3.   a)   A symmetric Hanning window of odd length $M+1$ is defined as

$$w[n] = 0.5 + 0.5\cos\omega_M n,$$

where $\omega_M = \frac{2\pi}{(M+1)}$ and $-0.5M \le n \le 0.5M$.

   i)   Show that the DTFT of $w[n]$ is given by

$$\begin{aligned}
W(e^{j\omega}) \;=\; & 0.5\frac{\sin 0.5\,(M+1)\,\omega}{\sin 0.5\omega} \\
& +0.25\frac{\sin 0.5\,(M+1)\,(\omega-\omega_M)}{\sin 0.5\,(\omega-\omega_M)} \\
& +0.25\frac{\sin 0.5\,(M+1)\,(\omega+\omega_M)}{\sin 0.5\,(\omega+\omega_M)}.
\end{aligned}$$

You may assume without proof that $\sum_{n=-0.5M}^{0.5M} e^{j\alpha n} = \frac{\sin 0.5(M+1)\alpha}{\sin 0.5\alpha}$ provided that $\alpha \ne 0$.   [5]

   ii)   We define $S(\omega)$ to be the integrated spectrum

$$S(\omega) = \frac{1}{2\pi}\int_{\theta=0}^{\omega} W(e^{j\theta})d\theta.$$

Using the inverse DTFT formula or otherwise, show that $S(\pi) = 0.5$.   [3]

   b)   i)   Show that, if $G(z)$ is an ideal lowpss filters with

$$G(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & \omega_0 < |\omega| \le \pi \end{cases}, \quad \text{then} \quad g[n] = \frac{\sin \omega_0 n}{\pi n}.$$

[4]

   ii)   By combining the ideal response from part b) i) with the Hanning window from part a), use the window method to design a causal lowpass filter, $H(z)$, of order M with a cutoff frequency of ω_0.

Give a formula for the impulse response, $h[n]$, of the filter where $0 \le n \le M$.   [3]

   iii)   Show that $H(e^{j\omega}) = e^{-0.5j\omega M}\left(S(\omega + \omega_0) - S(\omega - \omega_0)\right)$.   [5]

c)   Figures 3.1 and 3.2 show $W(e^{j\omega})$ and $S(\omega)$ for a Hanning window of length $M+1 = 41$. The first few values of ω for which $S(\omega)$ either equals 0.5 or has a turning point are listed in the following table:

ω	0.2565	0.3065	0.4003	0.4598	0.5490
$S(\omega)$	0.5	0.5064	0.5	0.4981	0.5

For the case $\omega_0 = 1$, the magnitude response, $\left|H(e^{j\omega})\right|$, of the resultant lowpass FIR filter from part b) is shown in Figure 3.3 plotted in dB. The ideal response, $G(e^{j\omega})$, is shown on the graph as a dashed line. Using appropriate values from the table given above,

    i)   estimate the smallest positive ω (marked "a" in Figure 3.3) for which $H(e^{j\omega}) = 0$;   [3]

    ii)   estimate the magnitude in dB (marked "b" in Figure 3.3) of the first peak in the stopband;   [3]

    iii)   estimate the peak passband gain in dB.   [4]
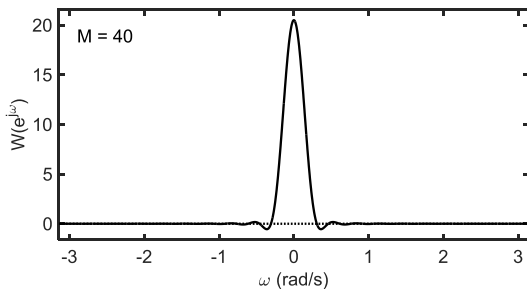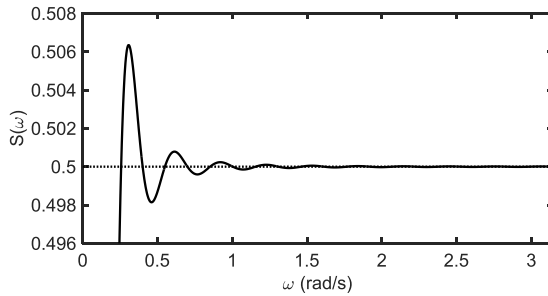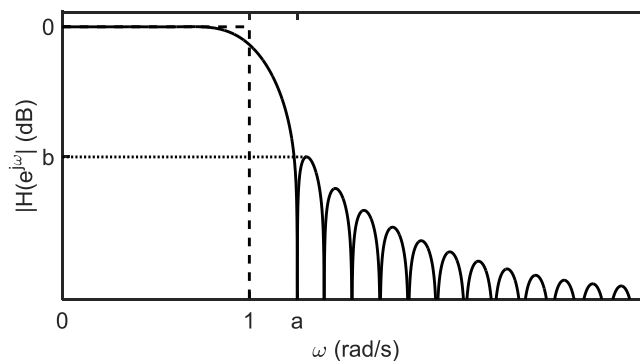

Figure 3.1

Figure 3.2

Figure 3.3

4. Figure 4.1 shows a system intended to apply a fractional-sample delay to its input signal, $x[n]$, where the delay is an integer multiple of $\frac{1}{P}$ samples.
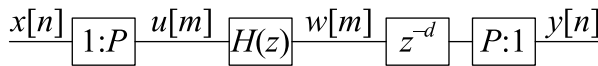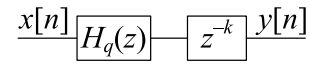

Figure 4.1   Figure 4.2

a) i) Explain the purpose of the lowpass filter, $H(z)$, in Figure 4.1. [2]

ii) The input signal, $x[n]$, contains frequency components in the range $0 \le \omega \le 0.8\pi$. Using the datasheet formula, $M \approx \frac{a}{3.5\Delta\omega}$, estimate the order required for $H(z)$ to give a stopband attenuation of 60 dB. [3]

iii) For a direct implementation of Figure 4.1, estimate as a function of P the number of multiplications required per input sample, $x[n]$. [3]

iv) If $H(z)$ is a causal symmetric FIR filter of order M, determine the delay of $y[n]$ relative to $x[n]$ as a function of M and d. [3]

b) i) The signal $w[m]$ in Figure 4.1 is given by $w[m] = \sum_{s=0}^{M} h[s]u[m-s]$ where $h[s]$ is the impulse response of $H(z)$.
If $m = Pn + p$ where $0 \le p < P$, show that $w[m]$ may be written in the form $w[m] = \sum_{r=0}^{R} h_p[r]x[n-r]$. Determine the value of R and give an expression for $h_p[r]$ in terms of $h[s]$. [6]

ii) Derive expressions for q and k in Figure 4.2 as functions of d in Figure 4.1 so that that the two figures are equivalent. You may assume that $y[n] = w[Pn - d]$ in Figure 4.1. [4]

c) Suppose now that, for each r, the coefficients $h_q[r]$ may be closely approximated using a polynomial of order T as $h_q[r] \approx \sum_{t=0}^{T} f_t[r] \left(\frac{q}{P}\right)^t$ where the polynomial argument, $\frac{q}{P}$, lies in the range $0 \le \frac{q}{P} < 1$.

i) The Farrow filter shown in Figure 4.3 calculates its output, $y[n]$, from $y[n+k] = \sum_{t=0}^{T} \left(\frac{q}{P}\right)^t v_t[n]$ where each of the signals $v_t[n]$ is obtained from $x[n]$ by applying a filter, $G_t(z)$, whose coefficients do not depend on q. Derive an expression for the coefficients of $G_t(z)$ so that Figure 4.3 is approximately equivalent to Figure 4.2. [4]

ii) Suppose that $P = 20$, $M = 199$ and $T = 4$. For each of Figure 4.2 and Figure 4.3, estimate the number of multiplications required per input sample, $x[n]$. [3]

iii) Explain why the implementation of Figure 4.3 may be preferable to that of Figure 4.2 under some circumstances. [2]
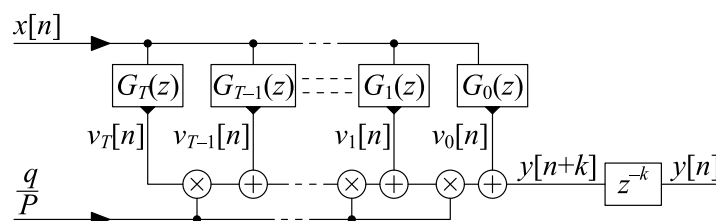

Figure 4.3

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.

- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.

- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1 - \alpha^{r+1} z^{-r-1}}{1 - \alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.

- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1 - \alpha z^{-1}}$ provided that $\left| \alpha z^{-1} \right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$	$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$	$x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N} \sum_{n=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \qquad \Leftrightarrow \qquad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$

$v[n] = x[n] y[n] \qquad \Leftrightarrow \qquad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r)_{\bmod N}] \qquad \Leftrightarrow \qquad V[k] = X[k] Y[k]$

$v[n] = x[n] y[n] \qquad \Leftrightarrow \qquad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N} \sum_{r=0}^{N-1} X[r] Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$

2. $M \approx \frac{a-8}{2.2\Delta\omega}$

3. $M \approx \frac{a-1.2-20\log_{10}b}{4.6\Delta\omega}$

where $a =$ stop band attenuation in dB, $b =$ peak-to-peak passband ripple in dB and $\Delta\omega =$ width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

| Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

$$\boxed{Q{:}1}-\boxed{H(z)} \;=\; \boxed{H(z^Q)}-\boxed{Q{:}1}$$

$$\boxed{H(z)}-\boxed{1{:}Q} \;=\; \boxed{1{:}Q}-\boxed{H(z^Q)}$$

Multirate Spectra

Upsample: $\quad v[n] \to \boxed{1{:}Q} \to x[r]$

$$x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$$

Downsample: $\quad v[n] \to \boxed{Q{:}1} \to y[m]$

$$y[m] = v[Qm] \qquad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$$

Multirate Commutators

Input Commutator	Output Commutator
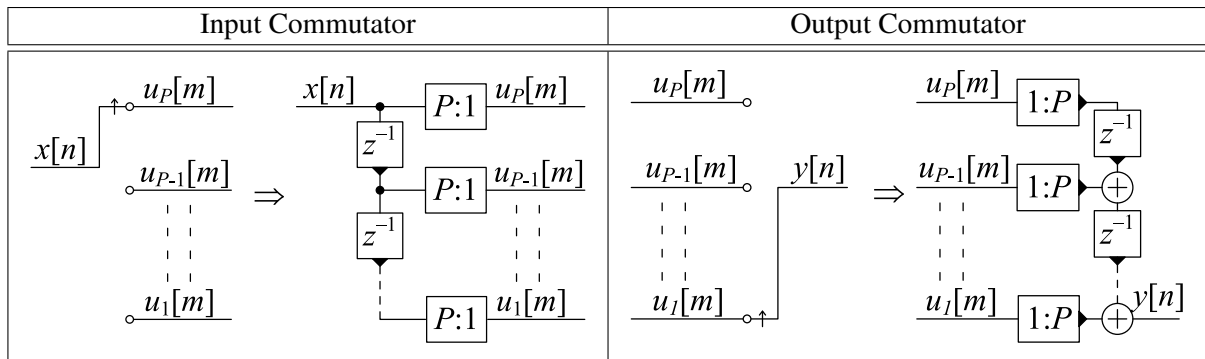
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********* Solutions *********

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A datasheet is included at the end of the examination paper.

1.   a)   The signals $x[n]$ and $y[n]$ are defined as

$$x[n] = \begin{cases} 2^{-n} & n > 0 \\ 0 & n \le 0 \end{cases}, \quad \text{and} \quad y[n] = \begin{cases} 0 & n > 0 \\ 5^{-n} & n \le 0 \end{cases}.$$

   i) Determine the *z*-transform of $x[n]$ and its region of convergence. [3]

$$X(z) = \sum_{n=1}^{\infty} 2^{-n} z^{-n} = 2^{-1} z^{-1} \sum_{n=0}^{\infty} 2^{-n} z^{-n} = \frac{0.5 z^{-1}}{1 - 0.5 z^{-1}} = \frac{1}{2z - 1}$$

provided that $\left\| 2^{-1} z^{-1} \right\| < 1 \Leftrightarrow |z| > 0.5.$

Many people started the sum at $n = 0$ instead of $n = 1$. Several people omitted the modulus sign and gave the ROC as $z > 0.5$; since z is complex, this makes no sense ("$>$" must always have real arguments).

---

   ii) Determine the *z*-transform of $y[n]$ and its region of convergence.[3]

$$Y(z) = \sum_{n=-\infty}^{0} 5^{-n} z^{-n} = \sum_{m=0}^{\infty} 5^{m} z^{m} = \frac{1}{1 - 5z}$$

provided that $\|5z\| < 1 \Leftrightarrow |z| < 0.2.$

Some negated the answer (perhaps because they were interchanging the summation limits and drew an incorrect analogy with integrals).

You may assume without proof that $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ provided that $|\alpha| < 1$.

b) Consider the convolution $y[n] = h[n] * x[n] = \sum_{m=0}^{M} h[m] x[n-m]$ where $h[n]$ is the impulse response of an FIR filter of order M (i.e. $n \in [0, M]$) and $x[n]$ is a signal defined for $n \in [-\infty, \infty]$.

In the overlap-save method of convolution, $y[n]$ is divided into blocks of length K and a circular convolution of length $K + M$ is used to calculate each block. To calculate block b, the circular convolution evaluates

$$y[bK + n] = \sum_{m=0}^{M} h[m] x[(n - m + M)_{\mathrm{mod}(K+M)} - M + bK]$$

for $n \in [0, K-1]$. The notation $P_{\mathrm{mod}\,Q}$ denotes the remainder when P is divided by Q and satisfies $0 \le P_{\mathrm{mod}\,Q} < Q$.

   i) Show that the expression given above is equivalent to the direct convolution given by $y[bK + n] = \sum_{m=0}^{M} h[m] x[bK + n - m]$.   [3]

For $m \in [0, M]$ and $n \in [0, K-1]$, the value of $n-m+M$ lies in the range $0-M+M=0$ to $K-1-0+M=K+M-1$ (where the first expression takes the lowest value of n and the highest of m and the second expression takes the opposite). It follows that, since $(n-m+M) \in [0, K+M-1]$, the modulo operation has no effect and $(n-m+M)_{\bmod (K+M)} = n-m+M$. Thus we can write

$$
\begin{aligned}
y[bK+n] &= \sum_{m=0}^{M} h[m]x[(n-m+M)_{\bmod (K+M)} - M + bK] \\
&= \sum_{m=0}^{M} h[m]x[(n-m+M) - M + bK] \\
&= \sum_{m=0}^{M} h[m]x[bK+n-m]
\end{aligned}
$$

Not everyone's handwriting makes a clear distinction between "m" and "M". Several people showed that $(n-m+M) < K+M$ but not that $(n-m+M) \geq 0$; both these conditions must be true for " $\bmod (K+M)$" to have no effect.

---

ii) Suppose that a circular convolution of length R requires approximately $5R\log_2 R$ multiplications. If $M = 200$, estimate the number of multiplications per output sample required by the overlap-save method when $K = 20$, 1500 and 10^4 and compare these results with the number of multiplications required for implementing the direct convolution. [4]

The overlap-save method uses a circular convolution of length $M+K$ to calculate K values of $y[n]$. The number of multiplications per output sample is therefore $5\frac{M+K}{K}\log_2(M+K)$. For $M = 200$ and $K = \{20, 1500, 10000\}$ this gives $\{428.0, 60.8, 67.9\}$ per output sample or $\{8560, 91220, 679130\}$ per entire block. A direct convolution requires $M+1 = 201$ multiplications per output sample or $\{4020, 301500, 2010000\}$ per entire block, so the overlap save method is more efficient for moderate K. For large K the number of multiplications is approximately $5\log_2 K$ which increases slowly with K. Although not requested in the exam, the optimum value of K is the root of $M\ln(M+K) = K$ which, for $M = 200$, is $K = 1486$.

Many people calculated the number of multiplications per block of K outputs rather than the number for each output sample as the question asked. Several people doubled the number of multiplications needed for the overlap-save mehod (possibly remembering that a circular convolution requires two DFTs) even though the question explicitly says how many multiplications are needed. Others doubled the number needed for the direct convolution (perhaps including the additions as well). Very few people comment on the slightly surprising fact that the benefit decreases for very large K meaning that there is an optimum value of K to use.

---

c)   The filter $H(z)$ is given by

$$H(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.8z^{-2}}.$$

The pole-zero diagram and magnitude response (in dB) of $H(z)$ are shown in Figures 1.1 and 1.2 respectively.

   i)   Determine the transfer function $F(z) = H(z^2)$ and sketch its pole-zero diagram and magnitude response in dB. It is not necessary to determine exact values of the magnitude response.   [4]

   _____

The transfer function is $F(z) = \frac{1+z^{-2}}{1-z^{-2}+0.8z^{-4}}$. The poles and zeros of $F(z)$ are the square root of the poles and zeros of $H(z)$ since $H(z_0) = 0 \Leftrightarrow F(z_0^{\frac{1}{2}}) = 0$. Therefore the zero at $z = -1$ becomes a zero pair at $\pm j$ and the pole pair at $0.5 \pm 0.74j$ becomes two pole pairs with half the argument at $\pm 0.835 \pm 0.444j$ (candidates are not required to calculate the exact values).These poles move closer to the unit circle and the peak bandwidths are correspondingly halved. The two square roots of a complex number have the same magnitude but are 180° apart. The magnitude response is shrunk horizontally by a factor of 2 and replicated but is otherwise unchanged.
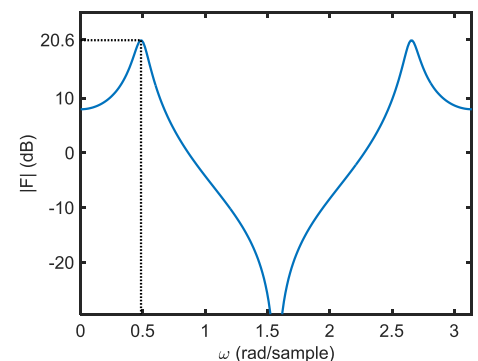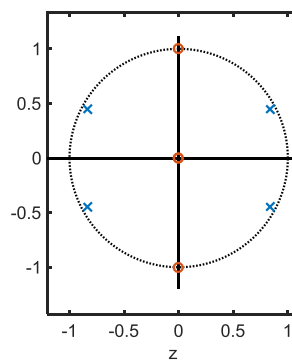
A sketched graph needs a scale on each axis; many people omitted these. Many people put the zeros at ± 1 instead of $\pm j$. Quite a few had the gain only falling to ≈ 10 dB at $\omega = \frac{\pi}{2}$ rather than to $-\infty$ dB as is correct. A few people had the second half of the response flipped horizontally making it an exact repetition of the first half, i.e. the response jumped from $-\infty$ dB to 8 dB at $\omega = \frac{\pi}{2}$. Several people added two new poles at the reciprocal positions of the original ones. A few people flipped the response so that it went to $-\infty$ dB at $\omega = 0$ instead of at $\omega = \frac{\pi}{2}$.

   _____

   ii)   Determine the transfer function $G(z) = H(1.25z)$ and sketch its pole-zero diagram and magnitude response in dB. It is not necessary to determine exact values of the magnitude response.   [4]

   _____

The coefficients of $H(z)$ are multiplied by $1.25^{-n} = \{1, 0.8, 0.64\}$ so the transfer function is $G(z) = \frac{1+0.8z^{-1}}{1-0.8z^{-1}+0.512z^{-2}}$. The poles and zeros of $F(z)$ are the poles and zeros of $H(z)$ but multiplied by $\frac{1}{1.25} =$
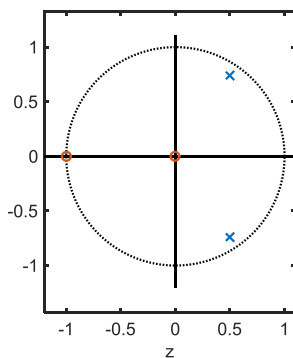
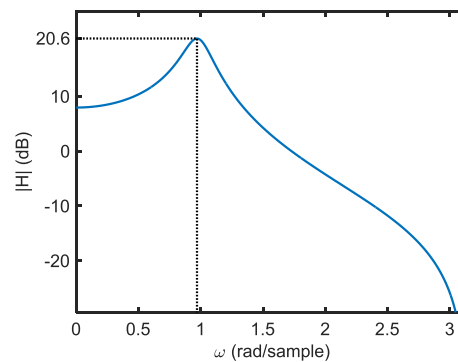Figure 1.1   Figure 1.2

d)   A bilinear transformation, $s = \alpha \frac{z-1}{z+1}$, is used to convert a continuous-time filter into a discrete-time filter.

      i)   Show that if $\alpha = \frac{\Omega_0}{\tan(0.5\omega_0)}$ then $z = e^{j\omega_0} \Leftrightarrow s = j\Omega_0$.   [3]

We will calculate $s \times \frac{z+1}{z-1}$ and show that it equals α.

$$s \times \frac{z+1}{z-1} = j\Omega_0 \times \frac{e^{j\omega_0}+1}{e^{j\omega_0}-1} = j\Omega_0 \times \frac{e^{j0.5\omega_0}\left(e^{0.5j\omega_0}+e^{-0.5j\omega_0}\right)}{e^{j0.5\omega_0}\left(e^{0.5j\omega_0}-e^{-0.5j\omega_0}\right)}$$

$$= j\Omega_0 \times \frac{e^{0.5j\omega_0}+e^{-0.5j\omega_0}}{e^{0.5j\omega_0}-e^{-0.5j\omega_0}}$$

$$= j\Omega_0 \times \frac{2\cos(0.5\omega_0)}{2j\sin(0.5\omega_0)} = \frac{\Omega_0}{\tan(0.5\omega_0)}$$

ii)   A continuous-time highpass filter with a cutoff frequency of $1\,\mathrm{kHz}$ is given by $H(s) = \frac{s}{s+\Omega_0}$ where $\Omega_0 = 2000\pi\,\mathrm{rad/s}$. Using the bilinear transformation given above, determine the coefficients (to 3 decimal places) of a discrete-time filter having an unnormalized cutoff frequency of $1\,\mathrm{kHz}$. The sample frequency is $8\,\mathrm{kHz}$.   [4]

The normalized cutoff frequency of the discrete-time filter is $\omega_0 = \frac{\Omega_0}{f_s} = \frac{2000\pi}{8000} = \frac{\pi}{4} = 0.785$. The filter is

$$
\begin{aligned}
H(z) &= \alpha \frac{z-1}{z+1} \div \left(\alpha \frac{z-1}{z+1} + \Omega_0 \right) \left(\alpha\,(z-1) + \Omega_0\,(z+1) \right) \\
&= \alpha\,(z-1) \div \left(\alpha\,(z-1) + \Omega_0\,(z+1) \right) \\
&= \frac{z-1}{\left(\frac{\Omega_0}{\alpha} + 1 \right) z + \left(\frac{\Omega_0}{\alpha} - 1 \right)} \\
&= \frac{1 - z^{-1}}{\left(\frac{\Omega_0}{\alpha} + 1 \right) + \left(\frac{\Omega_0}{\alpha} - 1 \right) z^{-1}}
\end{aligned}
$$

Since $\alpha = \frac{\Omega_0}{\tan(0.5\omega_0)} = \frac{6283.2}{0.41421} = 15168.95$, $\frac{\Omega_0}{\alpha} = \tan(0.5\omega_0) = 0.414$. Thus the filter is

$$
H(z) = \frac{1 - z^{-1}}{\left(\frac{\Omega_0}{\alpha} + 1 \right) + \left(\frac{\Omega_0}{\alpha} - 1 \right) z^{-1}} = \frac{1 - z^{-1}}{1.414 - 0.586 z^{-1}} = \frac{0.707 - 0.707 z^{-1}}{1 - 0.414 z^{-1}}
$$

e)   i)   Explain why the average power of a discrete time signal (i.e. the average energy per sample) is always decreased by upsampling but is normally unchanged by downsampling. Give an example of a signal for which the latter statement is untrue.   [3]

Upsampling by Q inserts $Q-1$ zero-valued samples between each of the original samples. Thus in any given time interval, the energy stays the same but the number of samples is multiplied by Q. It follows that the power of the signal has been reduced by a factor of Q.

Downsampling by Q removes $Q-1$ out of every Q samples but the average energy of the remaining smples will be unchanged. Therefore, if the average energy of the samples that are removed is the same as the average energy of the retained samples, the signal power will be unchanged.

If we define a signal $x[n] = \begin{cases} 0 & n\,\text{even} \\ 1 & n\,\text{odd} \end{cases}$ then the average power of $x[n]$ is 0.5. However, if we downsample by 2 then the resultant signal is always 0 and its average power is 0.

Some chose white noise as their example signal; however the power of white noise is unchanged by downsampling. For the example signal, you need one in which the average power of the odd and even samples is different (assuming $Q=2$). Several said that this was related to aliasing which is only half right: aliasing is a necessary but insufficient condition for the average power to change because it depends on the relative phases of the aliasing components.

---

   ii)   Figure 1.3 shows the power spectral density (PSD) of a real-valued signal, $x[n]$; the horizontal portions of the PSD have values 1 or 4. The signal $y[m]$ is obtained by upampling $x[n]$ by a factor of 3 as shown. Draw a dimensioned sketch of the PSD of $y[m]$ giving the values of all horizontal portions of the graph and the values of all frequencies at which there is a discontinuity in the PSD.   [3]
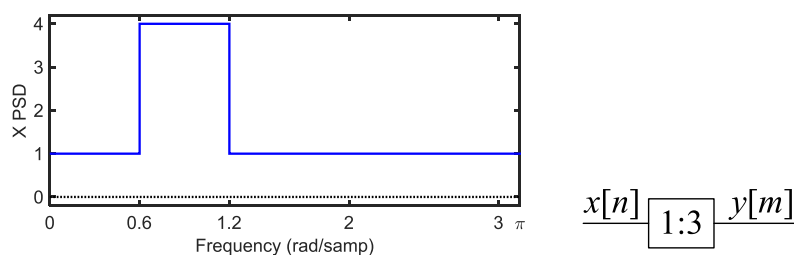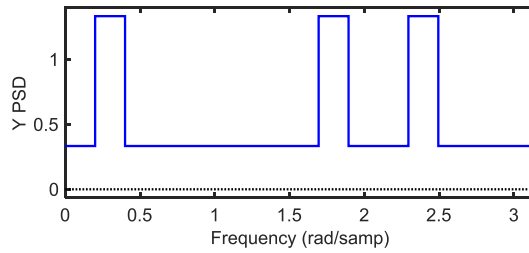

Figure 1.3

The upsampled PSD is shrunk horizontally by a factor of 3 and replicated at intervals of $\frac{2\pi}{3}$. The total power, which equals $\frac{1}{2\pi}\int S_{xx}(e^{j\omega})d\omega$,

f) Figure 1.4 shows the block diagram of a two-band analysis processor. The inputs to the adder/subtractor blocks, $\oplus$, are additive unless labelled with a minus sign in which case they are subtractive.

   i) By using the Noble identities or otherwise, determine $H_0(z)$ and $H_1(z)$ so that Figure 1.5 is equivalent to Figure 1.4. [3]

   ii) Assuming that $P_0(z)$ and $P_1(z)$ are FIR filters with real-valued coefficients, show that $\left|H_1\left(e^{j\omega}\right)\right| = \left|H_0\left(e^{j(\pi - \omega)}\right)\right|$ and explain the significance of this relationship. [3]

We can see that $H_1(z) = P_0(z^2) - z^{-1}P_1(z^2) = P_0((-z)^2) + -(-z)P_1((-z)^2) = H_0(-z)$.

It follows that $H_1\left(e^{j\omega}\right) = H_0\left(-e^{j\omega}\right) = H_0\left(e^{j(\omega-\pi)}\right) = H_0^*\left(e^{j(\pi-\omega)}\right)$ where the final equivalence uses the fact that all the coefficients of $H_0(z)$ are real-valued. It follows that $\left|H_1\left(e^{j\omega}\right)\right| = \left|H_0\left(e^{j(\pi-\omega)}\right)\right|$. Thus the magnitude response of $H_1(z)$ is the same as that of $H_0(z)$ but reflected around $\omega = \frac{\pi}{2}$.

Some completely ignored the modulus signs (but the relationship without these is false since $H_1\left(e^{j\omega}\right) = H_0^*\left(e^{j(\pi-\omega)}\right)$).
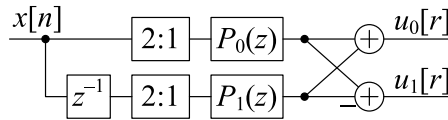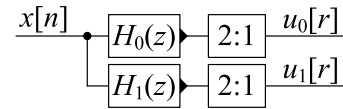

Figure 1.4

Figure 1.5

2. In the block diagram of Figure 2.1 the outputs of all adders and delay elements are on the right and solid arrows indicate the direction of information flow. The real-valued gain of each multiplier is written adjacent to its triangular symbol.
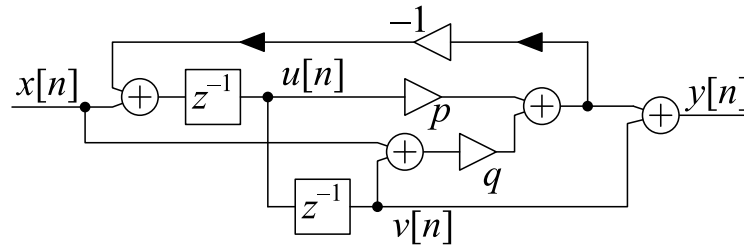

Figure 2.1

a) Show that   [8]

$$G(z) = \frac{Y(z)}{X(z)} = \frac{q + pz^{-1} + z^{-2}}{1 + pz^{-1} + qz^{-2}}.$$

From the diagram, we can write

$$
\begin{aligned}
U &= z^{-1}(X - (pU + q(X + V))) = z^{-1}((1 - q)X - pU - qV) \\
&\Rightarrow (1 - q)X - (z + p)U - qV = 0 \\
V &= z^{-1}U \\
Y &= V + pU + q(X + V) = qX + pU + (1 + q)V.
\end{aligned}
$$

We now need to eliminate U and V from these equations. Substituting for V gives

$$
\begin{aligned}
0 &= (1 - q)X - (z + p)U - qz^{-1}U \\
&\Rightarrow U = \frac{1 - q}{z + p + qz^{-1}}X \\
Y &= qX + pU + (1 + q)z^{-1}U \\
&= qX + \frac{(p + z^{-1} + qz^{-1})(1 - q)}{z + p + qz^{-1}}X \\
&= \frac{qz + pq + q^2z^{-1} + p + z^{-1} + qz^{-1} - pq - qz^{-1} - q^2z^{-1}}{z + p + qz^{-1}}X \\
&= \frac{qz + p + z^{-1}}{z + p + qz^{-1}}X = \frac{q + pz^{-1} + z^{-2}}{1 + pz^{-1} + qz^{-2}}X
\end{aligned}
$$

Mostly done OK. Some people got confused between their own handwritten symbols for "U" and "V".

b) Prove that $\left| G(e^{j\omega}) \right| = 1$ for all ω.   [6]

If $z = e^{j\omega}$, then $z^{-1} = z^*$ and $|z| = 1$. We can write

$$
\begin{aligned}
|G(z)| &= \frac{|q + pz^{-1} + z^{-2}|}{|z^{-2}(z^2 + pz + q)|} \\
&= \frac{|q + pz^* + (z^*)^2|}{|z^{-2}||z^2 + pz + q|} \\
&= \frac{|(z^2 + pz + q)^*|}{1 \times |z^2 + pz + q|} \\
&= \frac{|z^2 + pz + q|}{|z^2 + pz + q|} = 1.
\end{aligned}
$$

where the second line relies on $z^{-1} = z^*$ and the third line on the fact that p and q are real-valued.

Some multiplied the fraction by its complex conjugate which gives the correct result after quite a lot of algebra. Others substituted $z = \cos\omega + j\sin\omega$ early on which also results in a lot of algebra (especially if you then multiply the numerator and denominator by their complex conjugates). Many people did not explicitly state where they were relying on p and q being real and/or that $z^{-1} = z^*$ which is only true for z on the unit circle. One or two people said that $|z^2 + pz + q| = |z^2| + |pz| + |q|$ which is only a valid step if all the terms involved are real and positive (not true in this case). One or two people just said that $|G(e^{j\omega})| = 1$ must be true because the numerator and denminator coefficients were reflected copies of each other; this is not a proof.

---

c)   Figure 2.2 shows a graph of $\angle G(e^{j\omega})$ when $p = -1.2$ and $q = 0.8$. The dotted lines indicate the frequencies, $\omega = \{a, b, c\}$, at which $G(e^{j\omega}) = \{-j, -1, +j\}$ respectively. Given that $G(e^{jb}) = -1$, derive the general formula $\cos b = \frac{-p}{q+1}$ and find the numerical value of b for the given values of p and q.   [6]

If $z = e^{jb}$, then $z + z^{-1} = 2\cos b$ (used in line 5 below). From $G(z) = -1$, we can write

$$
\begin{aligned}
G(z) = -1 &= \frac{qz^2 + pz + 1}{z^2 + pz + q} \\
-(z^2 + pz + q) &= qz^2 + pz + 1 \\
(q+1)z^2 + (1+q) + 2pz &= 0 \\
(q+1)z + (q+1)z^{-1} + 2p &= 0 \\
2(q+1)\cos b + 2p &= 0 \\
\cos b &= \frac{-p}{q+1}.
\end{aligned}
$$

For the given values of p and q,

$$
\begin{aligned}
b &= \cos^{-1}\frac{-p}{q+1} \\
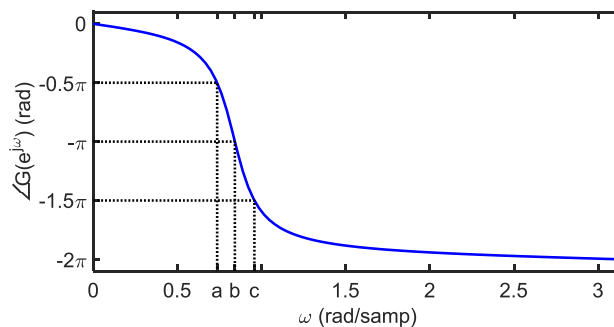&= \cos^{-1} 0.667 \\
&= 0.8411
\end{aligned}
$$

Figure 2.2

d) The filter $H(z)$ is defined as $H(z) = \frac{1}{2}(1 + G(z))$.

   i) Determine the value of $H(e^{j\omega})$ for each of $\omega = \{a, b, c\}$ defined above.

      Hence, for $p = -1.2$ and $q = 0.8$, sketch a graph of $\left|H(e^{j\omega})\right|$ for $\omega \in [0, \pi]$ using linear scales for both axes. [6]

   For $\omega = \{a, b, c\}$ we know that $G(e^{j\omega}) = \{-j, -1, +j\}$ and so $H(e^{j\omega}) = \frac{1}{2}\{1 - j, 1 - 1, 1 + j\} = \{0.5 - 0.5j, 0, 0.5 + 0.5j\}$. From this, $\left|H(e^{j\omega})\right| = \{0.707, 0, 0.707\}$. In addition, for $\omega = \{0, \pi\}$, $z = \pm 1$ and $G(z) = \frac{q \pm p + 1}{1 \pm p + q} = 1$ and so $H(z) = \frac{1}{2}(1 + G(z)) = 1$ as well. We also have $G(e^{j0}) = G(e^{j\pi}) = 1$ which means that $H(e^{j0}) = H(e^{j\pi}) = 1$ Thus we have a notch filter
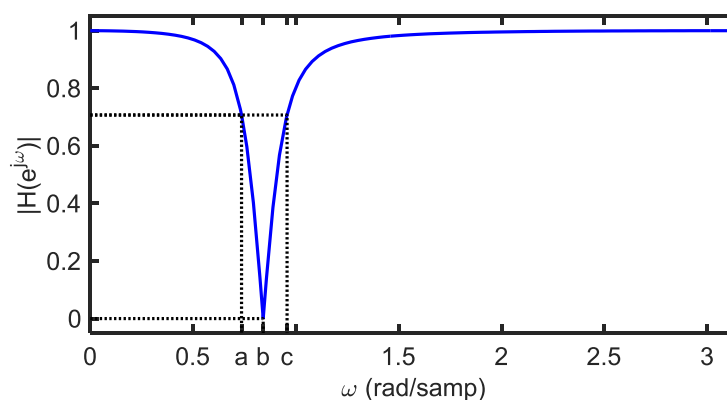
Surprisingly, many people correctly calculated $H(e^{jb}) = 0$ but then plotted its gain as some other value, e.g. 0.5, on the graph. Also, many people did not have the correct gain of 1 at $\omega = 0$ and $\omega = \pi$.

   ii) For $p = -1.2$ and $q = 0.8$, determine the poles and zeros of $H(z)$ in

polar form and sketch a diagram of the complex plane that includes the unit circle and the poles and zeros of $H(z)$ (indicated by $\times$ and $\circ$ respectively). [4]

We have

$$H(z) = \frac{1}{2}(1 + G(z)) = \frac{1}{2}\left(1 + \frac{qz^2 + pz + 1}{z^2 + pz + q}\right)$$

$$= \frac{(q+1)z^2 + 2pz + (q+1)}{2(z^2 + pz + q)}$$

Thus the poles are the roots of $z^2 + pz + q = 0$, i.e. $z = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = 0.6 \pm j\sqrt{0.44} = 0.6 \pm 0.6633j = 0.8944\angle \pm 0.8355 = 0.8944\angle \pm 47.9°$

The zeros are the roots of $(q+1)z^2 + 2pz + (q+1) = 0$, i.e. $z = \frac{-p \pm \sqrt{p^2 - (q+1)^2}}{q+1} = \frac{1.2 \pm j\sqrt{1.8}}{1.8} = \frac{1.2 \pm 1.342}{1.8} = 0.667 \pm 0.745j = 1\angle \pm 0.841 = 1\angle \pm 48.2°$.

The pole-zero diagram is therefore
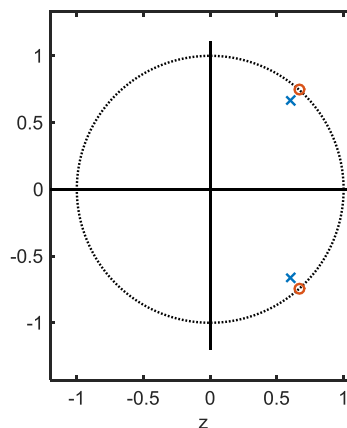
Many people did not place the zeros exactly on the unit circle; if the gain is zero at ω_0 then there must be a zero at $e^{j\omega_0}$.

---

3. a) A symmetric Hanning window of odd length $M+1$ is defined as

$$w[n] = 0.5 + 0.5\cos\omega_M n,$$

where $\omega_M = \frac{2\pi}{(M+1)}$ and $-0.5M \le n \le 0.5M$.

  i) Show that the DTFT of $w[n]$ is given by

$$
\begin{aligned}
W(e^{j\omega}) \;=\; & 0.5\frac{\sin 0.5\,(M+1)\,\omega}{\sin 0.5\omega} \\
& +0.25\frac{\sin 0.5\,(M+1)\,(\omega-\omega_M)}{\sin 0.5\,(\omega-\omega_M)} \\
& +0.25\frac{\sin 0.5\,(M+1)\,(\omega+\omega_M)}{\sin 0.5\,(\omega+\omega_M)}.
\end{aligned}
$$

You may assume without proof that $\sum_{n=-0.5M}^{0.5M} e^{j\alpha n} = \frac{\sin 0.5(M+1)\alpha}{\sin 0.5\alpha}$ provided that $\alpha \ne 0$.

[5]

From the DTFT formula in the datasheet. we have

$$
\begin{aligned}
W(e^{j\omega}) \;=\; & \sum_{n=-0.5M}^{0.5M} w[n]e^{-jn\omega} \\
=\; & \sum_{n=-0.5M}^{0.5M} (0.5 + 0.5\cos\omega_M n)\,e^{-jn\omega} \\
=\; & \sum_{n=-0.5M}^{0.5M} \left(0.5 + 0.25e^{j\omega_M n} + 0.25e^{-j\omega_M n}\right)e^{-jn\omega} \\
=\; & \sum_{n=-0.5M}^{0.5M} 0.5e^{-jn\omega} + 0.25e^{-j(\omega-\omega_M)n} + 0.25e^{-j(\omega+\omega_M)n}
\end{aligned}
$$

The summation formula given in the question can now be applied to each of these terms to obtain the desired result. The first term, for example, has $\alpha = -\omega$ so that the sum is $0.5\frac{\sin -0.5(M+1)\omega}{\sin -0.5\omega} = 0.5\frac{-\sin 0.5(M+1)\omega}{-\sin 0.5\omega} = 0.5\frac{\sin 0.5(M+1)\omega}{\sin 0.5\omega}$. Note that the minus sign in the exponent of the summand cancels out and so has no effect.

Most people did this OK. A few people did not think to express the $\cos\omega_M n$ term as a sum of exponentials and, instead, expressed the $e^{-jn\omega}$ factor as $\cos n\omega - j\sin n\omega$. In most instances, manipulating exponentials is much easier than manipulating trigonometric functions. A few people did not notice that the formula for $\sum_{n=-0.5M}^{0.5M} e^{j\alpha n}$ was given in the question and tried, often unsuccessfully, to sum the geometric progression from first principles. The sum can be converted to a more familiar form by taking out a factor:

$$
\begin{aligned}
\sum_{n=-0.5M}^{0.5M} e^{j\alpha n} \;=\; & e^{-0.5j\alpha M}\sum_{n=0}^{M} e^{j\alpha n} = e^{-0.5j\alpha M}\left(\frac{1 - e^{j\alpha(M+1)}}{1 - e^{j\alpha}}\right) \\
=\; & e^{-0.5j\alpha M}\left(\frac{e^{0.5j\alpha(M+1)}\left(e^{-0.5j\alpha(M+1)} - e^{0.5j\alpha(M+1)}\right)}{e^{0.5j\alpha}(e^{-0.5j\alpha} - e^{0.5j\alpha})}\right) \\
=\; & \frac{e^{-0.5j\alpha(M+1)} - e^{0.5j\alpha(M+1)}}{e^{-0.5j\alpha} - e^{0.5j\alpha}} \\
=\; & \frac{-2j\sin 0.5\,(M+1)\,\alpha}{-2j\sin 0.5\alpha} = \frac{\sin 0.5\,(M+1)\,\alpha}{\sin 0.5\alpha}.
\end{aligned}
$$

ii) We define $S(\omega)$ to be the integrated spectrum

$$S(\omega) = \frac{1}{2\pi} \int_{\theta=0}^{\omega} W(e^{j\theta}) d\theta.$$

Using the inverse DTFT formula or otherwise, show that $S(\pi) = 0.5$.

[3]

Since $W(e^{j\omega})$ is even [because (a) $w[n]$ is real and even or alternatively (b) the expression given in part (i) is the sum of three even terms], we can write

$$
\begin{aligned}
S(\pi) &= \frac{1}{2\pi} \int_{\theta=0}^{\pi} W(e^{j\theta}) d\theta \\
&= \frac{1}{2} \times \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} W(e^{j\omega}) d\omega \\
&= \frac{1}{2} \times \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} W(e^{j\omega}) e^{j\omega 0} d\omega \\
&= \frac{1}{2} \times w[0] = \frac{1}{2} \times 1 = 0.5
\end{aligned}
$$

where we use the fact that $W(e^{j\omega})$ is even in line 2 and the inverse DTFT formula for $w[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\omega}) e^{j\omega n} d\omega$ with $n = 0$ in line 4.

Many found this difficult and did not realize that it is possible to eliminate the factor $e^{j\omega n}$ in the IDTFT formula by setting $n = 0$.

b) i) Show that, if $G(z)$ is an ideal lowpss filters with

$$G(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & \omega_0 < |\omega| \leq \pi \end{cases}, \quad \text{then} \quad g[n] = \frac{\sin \omega_0 n}{\pi n}.$$

[4]

From the inverse DTFT formula

$$
\begin{aligned}
g[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) e^{j\omega n} d\omega \\
&= \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi jn} \left[e^{j\omega n} \right]_{-\omega_0}^{\omega_0} \\
&= \frac{1}{2\pi jn} \left(e^{j\omega_0 n} - e^{-j\omega_0 n} \right) \\
&= \frac{1}{2\pi jn} \times 2j \sin \omega_0 n = \frac{\sin \omega_0 n}{\pi n}.
\end{aligned}
$$

Almost everyone did this OK.

ii) By combining the ideal response from part b) i) with the Hanning window from part a), use the window method to design a causal lowpass filter, $H(z)$, of order M with a cutoff frequency of ω_0.

Give a formula for the impulse response, $h[n]$, of the filter where $0 \le n \le M$. [3]

We multiply (in the time domain) the ideal impulse response by the window to obtain the FIR impulse response as $h[n] = g\left[n - \frac{M}{2}\right] w\left[n - \frac{M}{2}\right]$ where the time-shift of $\frac{M}{2}$ samples is needed to make a causal filter. As an explicit formula, this is

$$h[n] = \frac{\sin \omega_0 \left(n - \frac{M}{2}\right)}{\pi \left(n - \frac{M}{2}\right)} \times \left(0.5 + 0.5\cos \omega_M \left(n - \frac{M}{2}\right)\right) \qquad \text{for } 0 \le n \le M.$$

Surprisingly a few people tried to do a convolution in the frequency domain instead of the much easier multiplication in the time domain. To do the time-shifting, you need to replace all three occurences of n by $n - \frac{M}{2}$; a few people only replaced one or two of them. Some people omitted the time shifting altogether giving a non-causal filter.

---

iii) Show that $H(e^{j\omega}) = e^{-0.5j\omega M} \left(S(\omega + \omega_0) - S(\omega - \omega_0)\right)$. [5]

From the convolution section of the datasheet, the DTFT of $h[n + \frac{M}{2}] = g[n] w[n]$ is

$$
\begin{aligned}
e^{0.5j\omega M} H(e^{j\omega}) &= \frac{1}{2\pi} \int_{\theta=-\pi}^{\pi} G(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta \\
&= \frac{1}{2\pi} \int_{\theta=-\omega_0}^{\omega_0} W(e^{j(\omega-\theta)}) d\theta \\
&= \frac{-1}{2\pi} \int_{\phi=\omega+\omega_0}^{\omega-\omega_0} W(e^{j\phi}) d\phi \\
&= -\left[S(\phi)\right]_{\omega+\omega_0}^{\omega-\omega_0} \\
&= S(\omega + \omega_0) - S(\omega - \omega_0)
\end{aligned}
$$

where in the second line we restrict the integral to the support of $G(e^{j\omega})$ and in the third line we substitute $\theta = \omega - \phi$; performing this substitution multiplies the integral by $\frac{d\theta}{d\phi} = -1$.

Since convolution is commutative, you can also write the convolution as the integral of $W(e^{j\theta}) G(e^{j(\omega-\theta)})$; however, if you do this, is is somewhat more complicated to determine the integration limits that impose the support of $G(e^{j\omega})$ to be $[-\omega_0, \omega_0]$.

---

c) Figures 3.1 and 3.2 show $W(e^{j\omega})$ and $S(\omega)$ for a Hanning window of length $M+1 = 41$. The first few values of ω for which $S(\omega)$ either equals 0.5 or has a turning point are listed in the following table:

ω	0.2565	0.3065	0.4003	0.4598	0.5490
$S(\omega)$	0.5	0.5064	0.5	0.4981	0.5

.

For the case $\omega_0 = 1$, the magnitude response, $|H(e^{j\omega})|$, of the resultant lowpass FIR filter from part b) is shown in Figure 3.3 plotted in dB. The ideal response, $G(e^{j\omega})$, is shown on the graph as a dashed line. Using appropriate values from the table given above,

i) estimate the smallest positive ω (marked "a" in Figure 3.3) for which $H(e^{j\omega}) = 0$; [3]

From Figure 3.3, the answer is in the region of $\omega \approx 1.2$. Thus $\omega + \omega_0 \approx 2.2$ and, from 3.2, $S(\omega + \omega_0) \approx 0.5$. We therefore need to find the smallest ω for which $S(\omega - \omega_0) = S(\omega + \omega_0) - H(e^{j\omega}) = 0.5 - 0 = 0.5$. From the table, this happes when $\omega - \omega_0 = 0.2565$ i.e. when $\omega = 1.2565$. Although not requested in the exam, the figure below shows the integration range $\phi = (\omega - \omega_0, \omega + \omega_0)$ when $\omega = 1.2565$:
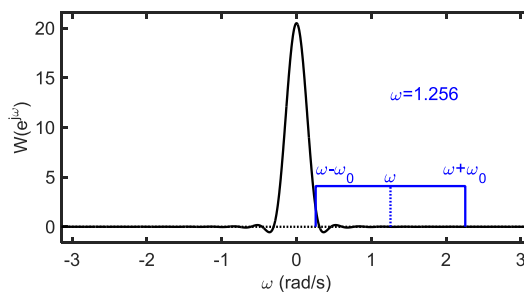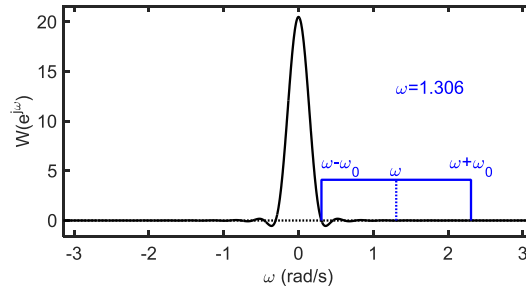

This value of ω gives $H(e^{j\omega}) = 0$; the range of integration includes a small part of the main lobe which exactly cancels out the negative integral of the rest of the range.

Many people did not answer part (c) at all. Most of those who tried it however, got this part correct although a few took the value of ω when $W(e^{j\omega}) = 0$(i.e. $\omega = 1.3065$).

---

ii) estimate the magnitude in dB (marked "b" in Figure 3.3) of the first peak in the stopband; [3]

As in the previous part, we can assume that $S(\omega + \omega_0) \approx 0.5$ which means that $H(e^{j\omega}) = 0.5 - S(\omega - \omega_0)$. First peak in $\|H(e^{j\omega})\|$ in the stopband will therefore be at the peak of $S(\omega - \omega_0)$ when $\omega - \omega_0 = 0.3065$, i.e. $\omega = 1.3065$. At this frequency, we have $H(e^{j\omega}) = 0.5 - 0.5064 = -0.0064$; note that the peaks in Figure 3.3 correspond to values of $H(e^{j\omega})$ that are alternately negative and positive. It follows that $|H(e^{j\omega})| = 20\log_{10} 0.0064 = -43.9\,\text{dB}$. Although not requested in the exam, the figure below shows the integration range $\phi = (\omega - \omega_0, \omega + \omega_0)$ when $\omega = 1.3065$:
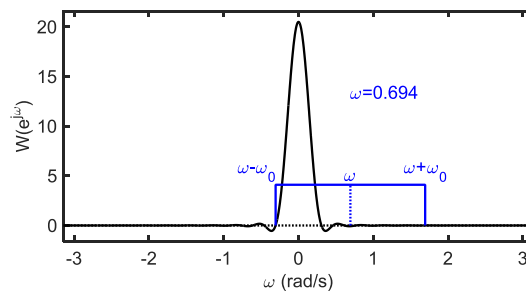
This value of ω gives the most negative value of $H(e^{j\omega})$ because the range of integration excludes the main lobe entirely but includes the negative lobe that follows it.

Most of those who tried this got it correct,

---

iii)   estimate the peak passband gain in dB.   [4]

Since $S(\omega)$ is an odd function, we can write $H(e^{j\omega}) = S(\omega + \omega_0) + S(\omega_0 - \omega)$ when $\omega < \omega_0$. If we still assume that $S(\omega + \omega_0) \approx 0.5$, we have the peak gain when $\omega_0 - \omega = 0.3065$ i.e. when $\omega = 0.6935$. At this frequency, $H(e^{j\omega}) = 0.5 + 0.5064 = 1.0064 = +0.055\,\mathrm{dB}$. Although not requested in the exam, the figure below shows the integration range $\phi = (\omega - \omega_0, \omega + \omega_0)$ when $\omega = 0.6935$:

This value of ω gives the most positive value of $H(e^{j\omega})$ because the range of integration includes all of the the main lobe but excludes the negative lobe that precedes it.

Mostly OK (for those who attempted it). Not everyone realized how to calculate $S(\omega - \omega_0)$ when $\omega < \omega_0$.
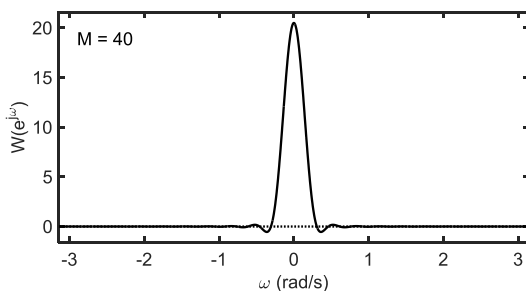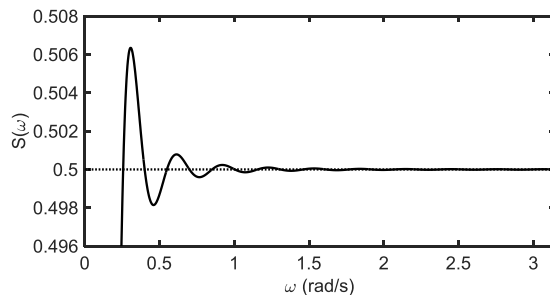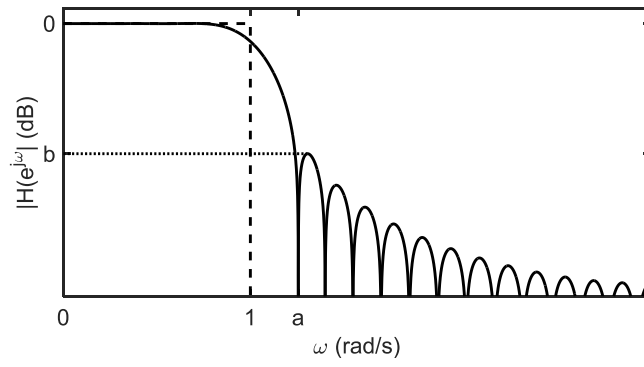

Figure 3.1

Figure 3.2

Figure 3.3

4. Figure 4.1 shows a system intended to apply a fractional-sample delay to its input signal, $x[n]$, where the delay is an integer multiple of $\frac{1}{P}$ samples.
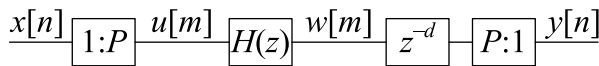
$$x[n] \rightarrow \boxed{1:P} \xrightarrow{u[m]} \boxed{H(z)} \xrightarrow{w[m]} \boxed{z^{-d}} \rightarrow \boxed{P:1} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{H_q(z)} \rightarrow \boxed{z^{-k}} \rightarrow y[n]$$

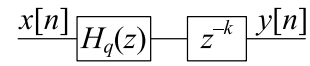Figure 4.1   Figure 4.2

a)   i)   Explain the purpose of the lowpass filter, $H(z)$, in Figure 4.1.   [2]

The filter removes frequency components above $\frac{\pi}{P}$ from the signal. It is needed because the upsampler, $1:P$, introduces $P-1$ images of the baseband whch must be removed before downsampling, $P:1$, to prevent aliasing.

Mostly correct although the explanations given by some people were too brief e.g. "Remove aliasing".

    ii)   The input signal, $x[n]$, contains frequency components in the range $0 \leq \omega \leq 0.8\pi$. Using the datasheet formula, $M \approx \frac{a}{3.5\Delta\omega}$, estimate the order required for $H(z)$ to give a stopband attenuation of $60\,\text{dB}$.[3]

The passband edge for $x[n]$ is given as 0.8π. After upsampling, this becomes $\frac{0.8\pi}{P}$ and the edge of the first image is at $\frac{2\pi-0.8\pi}{P} = \frac{1.2\pi}{P}$. Thus the transition bandwidth for the filter is $\Delta\omega = \frac{0.4\pi}{P}$. Substituting this into the formula, $M \approx \frac{60P}{3.5 \times 0.4\pi} = 13.6P$.

Many people used a transition bandwidth of 0.4π instead of $\frac{0.4\pi}{P}$. Some used the passband width instead of $\frac{0.8\pi}{P}$ or, even worse, 0.8π.

    iii)   For a direct implementation of Figure 4.1, estimate as a function of P the number of multiplications required per input sample, $x[n]$.[3]

The number of multiplications per sample of $v[m]$ is $M+1 = 13.6P+1$. The number per input sample, $x[n]$, is therefore $(13.6P+1)P = 13.6P^2 + P$.

Some people gave $\frac{M+1}{P}$ instead of $(M+1) \times P$.

    iv)   If $H(z)$ is a causal symmetric FIR filter of order M, determine the delay of $y[n]$ relative to $x[n]$ as a function of M and d.   [3]

If $H(z)$ is a symmetric causal filter of order M, it will introduce a delay of $\frac{M}{2}$ samples and the z^{-d} delay will introduce a further d samples giving $\frac{M+2d}{2}$ samples delay in all at the high sample rate. Thus, at the input sampling rate, we have a delay of $\frac{M+2d}{2P}$ samples.

Many gave the delay of the symmetric filter as M or $M+1$ instead of $\frac{M}{2}$. To see why it must be $\frac{M}{2}$, consider a filter zero except for

b)   i)   The signal $w[m]$ in Figure 4.1 is given by $w[m] = \sum_{s=0}^{M} h[s]u[m-s]$ where $h[s]$ is the impulse response of $H(z)$.

If $m = Pn + p$ where $0 \leq p < P$, show that $w[m]$ may be written in the form $w[m] = \sum_{r=0}^{R} h_p[r]x[n-r]$. Determine the value of R and give an expression for $h_p[r]$ in terms of $h[s]$.   [6]

We have

$$
\begin{aligned}
w[m] &= \sum_{s=0}^{M} h[s]u[m-s] \\
&= \sum_{s=0}^{M} h[s]u[Pn+p-s].
\end{aligned}
$$

However, we know that

$$
u[Pn+p-s] = \begin{cases} x[n+\frac{p-s}{P}] & P \mid (p-s) \\ 0 & otherwise \end{cases}.
$$

Hence, $u[Pn+p-s]$ is zero unless s is of the form $s = Pr+p$ (to ensure that $p-s$ is a multiple of P). The summation must cover the range $0 \leq s = Pr+p \leq M$ so r must cover the range $0 \leq r \leq \left\lceil \frac{M+1-P}{P} \right\rceil$ where the last inequality arise from the condition that $\max(Pr+p) = P\max(r) + (P-1) \geq M$. We can therefore write

$$
\begin{aligned}
w[m] &= \sum_{s=0}^{M} h[s]u[Pn+p-s] \\
&= \sum_{r=0}^{R} h[Pr+p]u[Pn+p-Pr-p] \\
&= \sum_{r=0}^{R} h[Pr+p]u[Pn-Pr] \\
&= \sum_{r=0}^{R} h[Pr+p]x[n-r] \\
&= \sum_{r=0}^{R} h_p[r]x[n-r]
\end{aligned}
$$

where $h_p[r] = h[Pr+p]$ if $0 \leq Pr+p \leq M$ and 0 otherwise. In line 2 above, we restrict the summation to the values of $u[\,]$ that are non-zero, i.e. those whose index is a multiple of P. The summation limit is $R = \left\lceil \frac{M+1-P}{P} \right\rceil$ where $\lceil \ldots \rceil$ denotes the ceiling function (i.e. the smallest integer greater than or equal to its argument). This is the polyphase decomposition of $h[s]$. The value of R can also be deduced from the total number of polyphase coefficients $P(R+1)$ which must be at least as big as the number of coefficients, $M+1$, in the original filter.

ii)   Derive expressions for q and k in Figure 4.2 as functions of d in Figure 4.1 so that that the two figures are equivalent. You may assume that $y[n] = w[Pn - d]$ in Figure 4.1.   [4]

From Figure 4.1, using the expression given in the question, we can write

$$y[n] = w[Pn - d] \quad = \quad w[P\ell + q]$$

where $q = (-d) \mod P$ and $Pn - d = P\ell + q \Rightarrow \ell = n - \frac{d+q}{P}$. We now have an expression of the form $y[n] = w[P\ell + q]$ which allows us to use the result given in part i) and write

$$y[n] = w[P\ell + q] \quad = \quad \sum_{r=0}^{R} h_q[r]x[\ell - r]$$

$$= \quad \sum_{r=0}^{R} h_q[r]x[n - \frac{d+q}{P} - r].$$

From Figure 4.2 we can write

$$y[n+k] \quad = \quad \sum_{r=0}^{R} h_q[r]x[n - r]$$

$$\Rightarrow y[n] \quad = \quad \sum_{r=0}^{R} h_q[r]x[n - k - r]$$

Equating this with the expression based on Figure 4.2, we must have $k = \frac{d+q}{P}$ which, since $0 \le q < P$, is equivalent to $k = \left\lceil \frac{d}{P} \right\rceil$ where $\lceil \cdots \rceil$ denotes the smallest integer that is greater than or equal to its argument (the "ceiling" function). Thus $q = (-d) \mod P$ and $k = \frac{d+q}{P} = \left\lceil \frac{d}{P} \right\rceil$.

c)   Suppose now that, for each r, the coefficients $h_q[r]$ may be closely approximated using a polynomial of order T as $h_q[r] \approx \sum_{t=0}^{T} f_t[r] \left(\frac{q}{P}\right)^t$ where the polynomial argument, $\frac{q}{P}$, lies in the range $0 \le \frac{q}{P} < 1$.

i) The Farrow filter shown in Figure 4.3 calculates its output, $y[n]$, from $y[n+k] = \sum_{t=0}^{T} \left(\frac{q}{P}\right)^t v_t[n]$ where each of the signals $v_t[n]$ is obtained from $x[n]$ by applying a filter, $G_t(z)$, whose coefficients do not depend on q. Derive an expression for the coefficients of $G_t(z)$ so that Figure 4.3 is approximately equivalent to Figure 4.2. [4]

Figure 4.2 calculates $y[n+k] = \sum_{r=0}^{R} h_q[r]x[n-r]$ so we can write

$$
\begin{aligned}
y[n+k] &= \sum_{r=0}^{R} h_q[r]x[n-r] \\
&= \sum_{r=0}^{R}\sum_{t=0}^{T} f_t[r]\left(\frac{q}{P}\right)^t x[n-r] \\
&= \sum_{t=0}^{T}\left(\frac{q}{P}\right)^t \sum_{r=0}^{R} f_t[r]x[n-r] \\
&\triangleq \sum_{t=0}^{T}\left(\frac{q}{P}\right)^t v_t[n]
\end{aligned}
$$

where $v_t[n] \triangleq \sum_{r=0}^{R} f_t[r]x[n-r]$. In line 2, we substituted the expression for $h_q[r]$ given in the question.

It follows that, for each t in the range $0 \le t \le T$, the coefficients of $G_t(z)$ are given by $g_t[n] = f_t[n]$ for $0 \le r \le R$.

Very few people gave a valid derivation of $g_t[n]$.

ii) Suppose that $P = 20$, $M = 199$ and $T = 4$. For each of Figure 4.2 and Figure 4.3, estimate the number of multiplications required per input sample, $x[n]$. [3]

For Figure 4.2, we require $R+1 = \frac{M+1}{P} = 10$ multiplications per input sample. For Figure 4.3, we have $T+1 = 5$ filters of length $R+1 = 10$ and in addition, we have $T = 4$ mutiplications per input sample to evaluate the polynomial using the multipiers at the bottom of the diagram. This gives a total of 54 multiplications per input sample which is much more than Figure 4.2 which only requires $R+1 = 10$.

Most people got the correct answer for Figure 4.2 but many used the wrong length for the filters $G_t(z)$ (assuming it was P or even $M+1$).

iii) Explain why the implementation of Figure 4.3 may be preferable to that of Figure 4.2 under some circumstances. [2]

For a fixed fractional delay, Figure 4.2 is clearly preferable since it entails far fewer multiplications. However Figure 4.3 has the advantage that the fractional part of the delay, q/P can be varied on a sample-by-sample basis without needing to change any filter coefficients. Also, q is not restricted to being an integer and so the delay can be varied continuously. The number of coefficients that

must be stored is $(T+1)(R+1)$ whereas for the implementation of Figure 4.2, we must store $P(R+1)$. Thus if $P > T+1$, we have fewer coefficients to store.

Only a few people gave the correct answer to this part. Several said that Figure 4.3 required less computation whereas in fact it requires much more (by a factor of around $T+1$).
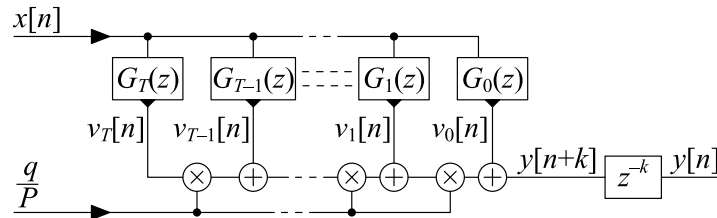

Figure 4.3

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.

- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.

- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.

- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

$$z: \qquad X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n} \qquad\qquad x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$\text{CTFT:} \qquad X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad\qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

$$\text{DTFT:} \qquad X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n} \qquad\qquad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$\text{DFT:} \qquad X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi\frac{kn}{N}} \qquad\qquad x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi\frac{kn}{N}}$$

$$\text{DCT:} \qquad X[k] = \sum_{n=0}^{N-1} x[n] \cos\frac{2\pi(2n+1)k}{4N} \qquad\qquad x[n] = \frac{X[0]}{N} + \frac{2}{N}\sum_{n=1}^{N-1} X[k] \cos\frac{2\pi(2n+1)k}{4N}$$

$$\text{MDCT:} \qquad X[k] = \sum_{n=0}^{2N-1} x[n] \cos\frac{2\pi(2n+1+N)(2k+1)}{8N} \qquad\qquad y[n] = \frac{1}{N}\sum_{0}^{N-1} X[k] \cos\frac{2\pi(2n+1+N)(2k+1)}{8N}$$

Convolution

$$\text{DTFT:} \qquad v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \qquad \Leftrightarrow \qquad V(e^{j\omega}) = X(e^{j\omega}) Y(e^{j\omega})$$

$$v[n] = x[n] y[n] \qquad \Leftrightarrow \qquad V(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) \circledast Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta$$

$$\text{DFT:} \qquad v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r)_{\bmod N}] \qquad \Leftrightarrow \qquad V[k] = X[k] Y[k]$$

$$v[n] = x[n] y[n] \qquad \Leftrightarrow \qquad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N}\sum_{r=0}^{N-1} X[r] Y[(k-r)_{\bmod N}]$$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\bigg|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$

2. $M \approx \frac{a-8}{2.2\Delta\omega}$

3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

where $a =$ stop band attenuation in dB, $b =$ peak-to-peak passband ripple in dB and $\Delta\omega =$ width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1} - \lambda}{1 - \lambda \hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0 - \hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0 + \hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1} + \lambda}{1 + \lambda \hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0 + \hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0 - \hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1) - 2\lambda\rho\hat{z}^{-1} + (\rho+1)\hat{z}^{-2}}{(\rho+1) - 2\lambda\rho\hat{z}^{-1} + (\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2 + \hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho) - 2\lambda\hat{z}^{-1} + (\rho+1)\hat{z}^{-2}}{(\rho+1) - 2\lambda\hat{z}^{-1} + (1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2 + \hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2 - \hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

Multirate Spectra

Upsample:

$$x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$$

Downsample:

$$y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$$

Multirate Commutators

Input Commutator	Output Commutator