EE4-12
EE9-AO2
EE9-SC1
EE9-FPN2-03

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MSc and EEE/EIE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Wednesday, 3 May 10:00 am

Time allowed: 3:00 hours

There are FOUR questions on this paper.

Answer Question 1 and any TWO other questions

Question 1 is worth 40% of the marks and other questions are worth 30%

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	D.M. Brookes
	Second Marker(s) :	P.T. Stathaki

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Information for Candidates:

Where a question requires a numerical answer, it must be given as a fully evaluated decimal number and not as an unevaluated arithmetic expression.

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.
- The expected value of x is denoted $E\{x\}$.
- In block diagrams: solid arrows denote the direction of signal flow; an open triangle denotes a gain element with the gain indicated adjacently; a "+" in a circle denotes an adder/subtractor whose inputs may be labelled "+" or "−" according to their sign; the sample rate of a signal may be indicated in the form "@f".

Abbreviations

BIBO	Bounded Input, Bounded Output
CTFT	Continuous-Time Fourier Transform
DCT	Discrete Cosine Transform
DFT	Discrete Fourier Transform
DTFT	Discrete-Time Fourier Transform
FIR	Finite Impulse Response

IIR	Infinite Impulse Response
LTI	Linear Time-Invariant
MDCT	Modified Discrete Cosine Transform
PSD	Power Spectral Density
SNR	Signal-to-Noise Ratio

A datasheet is included at the end of the examination paper.

1. a) A finite-length complex exponential signal is given by $x[n] = e^{j\omega n}$ for $n \in [0, N-1]$. The DFT of $x[n]$ satisfies

$$|X[k]| = \frac{\left|\sin \frac{2\pi k - N\omega}{2}\right|}{\left|\sin \frac{2\pi k - N\omega}{2N}\right|}.$$

i) By using the approximation $\sin\theta \approx \theta$ for $|\theta| < 0.2\,\text{rad}$, show that $|X[k]|$ is approximately bounded by $2\left(\frac{2\pi k}{N} - \omega\right)^{-1}$ for a suitable range of k. Give the range of k for which this bound applies and explain the significance of the term: $\left(\frac{2\pi k}{N} - \omega\right)$. [4]

ii) Explain why it is customary to multiply a signal, $x[n]$, by a window before performing a DFT and explain the tradeoffs that affect the choice of window function. [3]

b) i) Explain what is meant by saying that a linear time invariant system is "BIBO stable". [2]

ii) Prove that if a linear time invariant system is BIBO stable, then its impulse response, $h[n]$, satisfies $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$. [3]

c) A first-order FIR filter is given by $H(z) = 1 - 0.5z^{-1}$.

i) Determine a simplified expression for the squared magnitude response, $\left|H(e^{j\omega})\right|^2$, and sketch its graph for $\omega \in [0, \pi]$. [4]

ii) Using the formula in the datasheet, or otherwise, determine the group delay of the filter, $\tau_H(e^{j\omega})$, and sketch its graph for $\omega \in [0, \pi]$. [4]

d) In the block diagram of Figure 1.1, all elements are drawn with their outputs on the right. The input and output signals are $x[n]$ and $y[n]$ respectively.

i) Determine the transfer function of the system, $H(z) = \frac{Y(z)}{X(z)}$. [3]

ii) Draw the transposed form of the block diagram. [4]

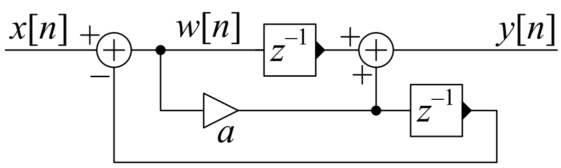


Figure 1.1

e) i) If a bounded discrete-time signal, $x[n]$, is stationary ergodic then $E\{x^2[n]\}$ for any n is equal to the average power of $x[n]$ (i.e. the average energy per sample). Explain why the average power of such a signal is unchanged by downsampling. [3]

ii) Figure 1.2 shows the power spectral density (PSD) of a real-valued stationary ergodic signal, $x[n]$; the horizontal portions of the PSD have values 1 or 4.

The signal $y[m] = x[3m]$ is obtained by downsampling $x[n]$ by a factor of 3. Draw a dimensioned sketch of the PSD of $y[m]$ giving the values of all horizontal portions of the graph and the values of all frequencies at which there is a discontinuity in the PSD. [4]

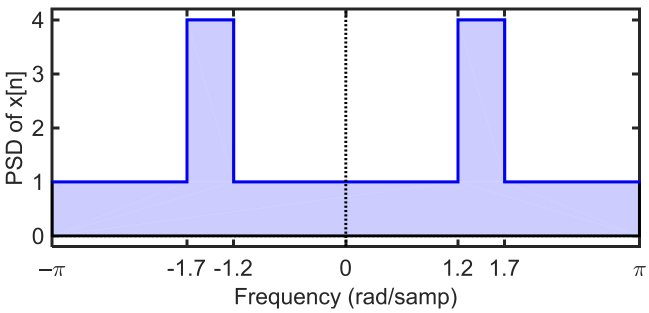


Figure 1.2

f) i) In the block diagram of Figure 1.3 the input is $x[m]$ and the output is $y[n]$. Determine a simplified expression for $Y(z)$ in terms of $X(z)$ and the filters $H_p(z)$ for $p \in [0, 2]$. [3]

ii) If $H_p(z) = \sum_{m=0}^{M} h_p[m]z^{-m}$, derive an expression for $g[n]$ in terms of the $h_p[m]$ so that the block diagram of Figure 1.4 is equivalent to that of Figure 1.3. [3]

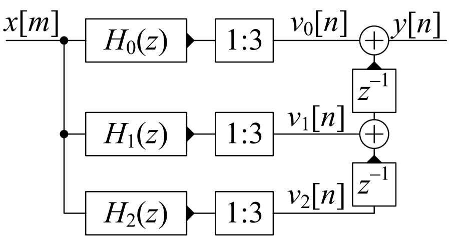


Figure 1.3

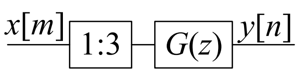


Figure 1.4

2. a) Outline the relative advantages of the bilinear and impulse-invariant transformations for converting a continuous-time filter into a discrete-time filter. [2]

b) If p is a complex-valued constant, show that the z-transform of the causal sequence $v[n] = e^{pn}$ is given by $V(z) = \left(1 - e^{p}z^{-1}\right)^{-1}$ and give its region of convergence. [3]

c) For $t \geq 0$, the impulse response of the causal continuous-time filter $H(s) = \frac{\Omega_0^2}{(s+\alpha)^2+\Omega_0^2}$ is given by

$$\begin{aligned} h(t) &= \Omega_0 e^{-\alpha t}\sin(\Omega_0 t) \\ &= -0.5j\Omega_0 e^{-\alpha t}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right). \end{aligned}$$

i) Use the result of part b) to find a simplified expression for the z-transform, $G(z)$, of the causal sequence given by $g[n] = T \times h(nT)$ where T is the sample period. Express $G(z)$ as a ratio of polynomials in z^{-1}. [7]

ii) If $T = 10^{-4}\,\mathrm{s}$, $\Omega_0 = 5000\,\mathrm{rad/s}$ and $\alpha = 800\,\mathrm{s}^{-1}$, give the numerical values of the coefficients of $G(z)$ to 3 decimal places after normalizing to make the leading denominator coefficient unity. [3]

d) i) Show that, under the mapping $s = \kappa\frac{z-1}{z+1}$, the value $s = j\Omega_0$ corresponds to $z = e^{j\omega_0}$ where $\Omega_0 = \kappa\tan(0.5\omega_0)$. Determine the numerical value of κ such that $\omega_0 = \Omega_0 T$ when T and Ω_0 have the values given in part c)ii). [4]

ii) Use the bilinear mapping from part d)i) to transform the filter $H(s)$ from part c) into a discrete time filter, $F(z)$, and give the numerical values of its coefficients to 3 decimal places after normalizing to make the leading denominator coefficient unity. [7]

e) Using the values given in part c)ii), determine the pole and zero positions of $H(s)$, $G(z)$ and $F(z)$ and comment on their relationship to the properties of the three filters. [4]

3. The FM radio baseband spectrum shown in Figure 3.1 comprises (i) a mono signal (L+R) with a bandwidth of 15 kHz, (ii) a 19 kHz pilot tone and (iii) stereo information (L–R) modulated on a suppressed 38 kHz subcarrier. To demodulate the stereo component it is necessary to regenerate the 38 kHz subcarrier by isolating the 19 kHz pilot tone and multiplying its frequency by 2. The baseband signal is sampled at $f_s = 200\,\text{kHz}$.

All filters in this question have real coefficients and are lowpass FIR filters with a stopband attenuation of 60 dB whose order may be estimated using the datatsheet formula $M = \frac{60}{3.5\Delta\omega}$ where $\Delta\omega$ is the transition bandwidth in rad/sample.

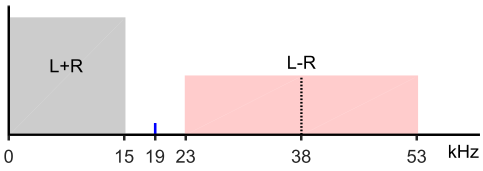


Figure 3.1

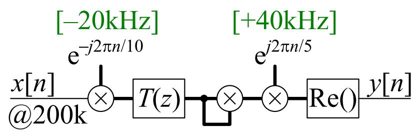


Figure 3.2

a) A block diagram for obtaining the 38 kHz subcarrier, $y[n]$, is shown in Figure 3.2 in which complex-valued signal paths are shown as bold lines. The baseband FM signal, $x[n]$, is translated down in frequency by 20 kHz and lowpass filtered by $T(z)$ to isolated the pilot tone component. The output of $T(z)$ is squared and translated up in frequency by 40 kHz and then the subcarrier, $y[n]$, is obtained by taking the real part of the signal.

i) The pilot tone component of $x[n]$ is given by $x_p[n] = \cos \omega_p n$ and has a frequency of $\omega_p = 2\pi \times \frac{19}{200} = 0.597\,\text{rad/sample}$.

Give the signed frequencies, in rad/sample, of the complex exponential components of the pilot tone signal at each stage of the processing, i.e. for each horizontal line segment in Figure 3.2. [3]

ii) Determine the passband edge frequency and the width of the transition band, $\Delta\omega$, for the lowpass filter, $T(z)$. Hence determine the required FIR filter order using the formula given at the start of the question. [3]

iii) Explain why squaring the output of $T(z)$ doubles the frequency of the pilot tone component. [3]

iv) Estimate the number of real multiplications per second needed to implement the block diagram of Figure 3.2. [3]

[This question is continued on the next page]

b) An alternative block diagram for generating the 38 kHz subcarrier is shown in Figure 3.3 in which $T(z)$ has been replaced by a lowpass filter, $G(z)$, operating at a sample frequency of 10 kHz.

i) Explain the reason that the lowpass filters $F(z)$ and $H(z)$ are needed. [2]

ii) Determine the passband edge, transition band width and filter order for each of the lowpass filters $F(z)$, $G(z)$ and $H(z)$. [6]

iii) Estimate the number of real multiplications per second needed to implement the block diagram assuming that $F(z)$ and $H(z)$ both use a polyphase implementation that incorporates the associated upsampler/downsampler. You may assume without proof that a polyphase filter of order M acting on a complex-valued signal requires $(2M+2)$ multiplications per sample at the lower of the two sample rates. [3]

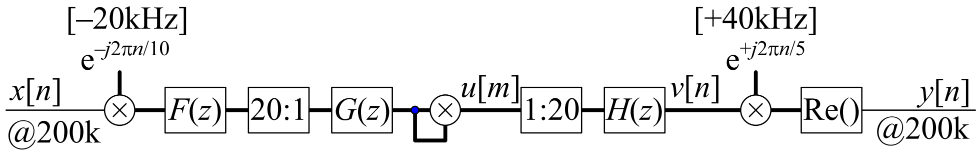


Figure 3.3

c) Suppose now that the upsampling is performed in two stages as illustrated in Figure 3.4 which replaces the blocks "1 : 20" and "$H(z)$" in Figure 3.3.

i) Determine the cutoff frequency, transition bandwidth and filter order for each of the lowpass filters $P(z)$ and $Q(z)$. [4]

ii) Estimate the number of real multiplications per second needed to implement the block diagram of Figure 3.4 assuming that a polyphase implementation is used for $P(z)$ and $Q(z)$. Compare this with the number of multiplications needed for the corresponding part of Figure 3.3. [3]

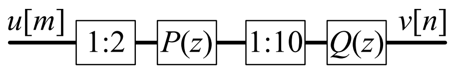


Figure 3.4

4. a) Explain briefly the advantages of processing signals in subbands. [2]

b) Figure 4.1 shows the analysis and synthesis stages of a 2-subband system. Show that $Y(z) = T(z)X(z)$ where $T(z) = \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z))$. [4]

For $p \in [0, 1]$ you may assume without proof that $W_p(z) = U_p\left(z^2\right)$ and that $U_p(z) = \frac{1}{2}\left\{V_p\left(z^{\frac{1}{2}}\right) + V_p\left(-z^{\frac{1}{2}}\right)\right\}$.

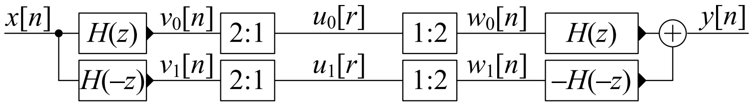


Figure 4.1

c) Given that the impulse response, $h[n]$, is causal and of odd order M, we define

$$t[n] = \frac{1}{2}\left(h[n] + (-1)^n h[n]\right) * \left(h[n] - (-1)^n h[n]\right)$$

where * denotes convolution.

i) Show that the z-transform of $t[n]$ is [3]

$$T(z) = \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z)).$$

ii) Show that, if $h[n]$ satisfies the symmetry condition $h[M-n] = h[n]$, then $t[n]$ satisfies the condition $t[2M-n] = t[n]$. [3]

iii) Deduce the group delay function, $\tau_T\left(e^{j\omega}\right)$, of the filter $T(z)$ from the symmetry condition of part ii). [2]

[This question is continued on the next page]

d) i) By using the inverse DTFT, show that the impulse response of an ideal lowpass filter whose frequency response is

$$G\left(e^{j\omega}\right) = \begin{cases} e^{-j0.5M\omega} & |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$$

is given by [3]

$$g[n] = \frac{\sin\left(0.5\pi\left(n-0.5M\right)\right)}{\pi\left(n-0.5M\right)}.$$

ii) A causal Hamming window of length $M+1$ is given by

$$w[n] = 0.54 - 0.46\cos\left(\frac{2n\pi}{M}\right)$$

for $n \in [0, M]$. Using the window design method with $g[n]$ and $w[n]$, design a causal FIR filter, $H(z)$, of order $M = 7$ with a cutoff frequency of $\omega = \frac{\pi}{2}$. Determine the numerical values of the filter coefficients, $h[n]$, to 3 decimal places. [4]

iii) The filter, $H(z)$, from part ii) is used in the block diagram shown in Figure 4.1. If $T(z) = \frac{Y(z)}{X(z)}$, determine the magnitude gain, $\left|T\left(e^{j\omega}\right)\right|$ for $\omega = 0$, $\frac{\pi}{2}$ and π. [4]

e) A "Johnston half-band filter" selects the coefficients, $h[n]$, to minimize the cost function

$$\alpha \int_{\frac{\pi}{2}+\Delta}^{\pi} \left|H(e^{j\omega})\right|^2 d\omega + (1-\alpha)\int_0^{\pi}\left(\left|H^2(e^{j\omega}) - H^2(-e^{j\omega})\right| - 1\right)^2 d\omega$$

for suitable choices of α and Δ.

i) Explain the significance of the two integrals in the cost function and hence explain the effect of reducing the value of α. [2]

ii) For $M = 7$, $\alpha = 0.5$ and $\Delta = 0.07$, the $h[n]$ are given by

$$h[0] = h[7] = 0.009, \qquad h[1] = h[6] = -0.071$$
$$h[2] = h[5] = 0.069, \qquad h[3] = h[4] = 0.490$$

Determine the magnitude gain, $\left|T\left(e^{j\omega}\right)\right|$ for $\omega = 0$, $\frac{\pi}{2}$ and π. [3]

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$	$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$	$x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N} \sum_{n=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$

$v[n] = x[n]y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r)_{\bmod N}] \quad \Leftrightarrow \quad V[k] = X[k]Y[k]$

$v[n] = x[n]y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N} \sum_{r=0}^{N-1} X[r] Y[(k-r)_{\bmod N}]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}(\,)$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, $b =$ peak-to-peak passband ripple in dB and $\Delta\omega =$ width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \nless \hat{\omega} \nless \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

$$-\boxed{Q{:}1}-\boxed{H(z)}- \;=\; -\boxed{H(z^Q)}-\boxed{Q{:}1}-$$

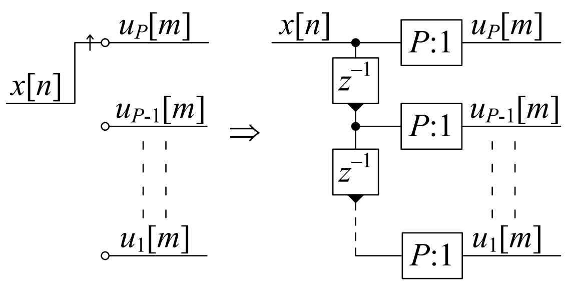
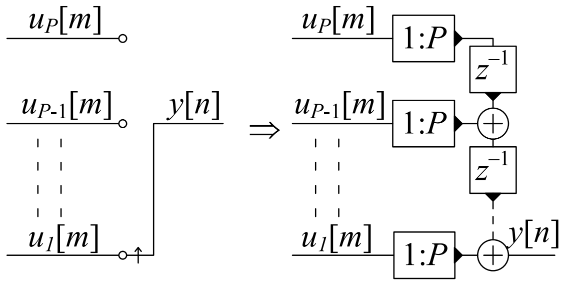
$$-\boxed{H(z)}-\boxed{1{:}Q}- \;=\; -\boxed{1{:}Q}-\boxed{H(z^Q)}-$$

Multirate Spectra

Upsample: $\xrightarrow{v[n]}\boxed{1{:}Q}\xrightarrow{x[r]}$ $\quad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if} Q \mid r \\ 0 & \text{if} Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample: $\xrightarrow{v[n]}\boxed{Q{:}1}\xrightarrow{y[m]}$ $\quad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators

Input Commutator	Output Commutator
	

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Key: B=bookwork, U=unseen example, T=Novel theory

********* Questions and Solutions *********

1. a) A finite-length complex exponential signal is given by $x[n] = e^{j\omega n}$ for $n \in [0, N-1]$. The DFT of $x[n]$ satisfies

$$|X[k]| = \frac{\left|\sin\frac{2\pi k - N\omega}{2}\right|}{\left|\sin\frac{2\pi k - N\omega}{2N}\right|}.$$

i) By using the approximation $\sin\theta \approx \theta$ for $|\theta| < 0.2\,\text{rad}$, show that $|X[k]|$ is approximately bounded by $2\left(\frac{2\pi k}{N} - \omega\right)^{-1}$ for a suitable range of k. Give the range of k for which this bound applies and explain the significance of the term: $\left(\frac{2\pi k}{N} - \omega\right)$. [4]

[T] The argument of $\sin$ *in the denominator is "small" if* $\frac{2\pi k - N\omega}{2N} < 0.2 \Leftrightarrow \left|k - \frac{\omega N}{2\pi}\right| < \frac{0.4N}{2\pi} = \frac{N}{15.7} = 0.0637N$. *Within this range,* $|X[n]| \approx \left|\sin\frac{2\pi k - N\omega}{2}\right| \times \frac{2N}{2\pi k - N\omega} = \left|\sin\frac{2\pi k - N\omega}{2}\right| \times 2\left(\frac{2\pi k}{N} - \omega\right)^{-1}$. *This proves the required result since the* $\sin$ *term is bounded by* 1. *Since* $X[k]$ *corresponds to a frequency of* $\frac{2\pi k}{N}$*, the term in parentheses gives the distance that a spectral component, k, is away from the frequency* ω *in rad/sample. Thus, we have shown that, when using a rectangular window, the spectral leakage falls as* $|k - k_0|^{-1}$ *where* $k_0 = \frac{\omega N}{2\pi}$ *over the range* $k \in (k_0 - 0.0637N, k_0 + 0.0637N)$.

Many people used the small angle approximation for both the numerator and the denominator which gives $|X[k]| \approx N$ *for* $\left|k - \frac{\omega N}{2\pi}\right| < \frac{0.4}{2\pi} = \frac{1}{15.7} = 0.0637$. *This is correct but not very helpful since the limits on the integer k restrict it to at best a single value and at worst no values at all. Very few people understood the significance of the term* $\left(\frac{2\pi k}{N} - \omega\right)$. *For some reason, several people re-derived the expression given in the question (not always correctly).*

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ii) Explain why it is customary to multiply a signal, $x[n]$, by a window before performing a DFT and explain the tradeoffs that affect the choice of window function. [3]

[B] The finite-length signal $x[n]$*, has often been extracted from an extended signal that is longer than the DFT length. The spectrum obtained from the DFT is the convolution of the spectrum of the unwindowed* $x[n]$ *convolved with the spectrum of the window (which is, if no other window is used, that of a rectangular window). By multiplying* $x[n]$ *by a window other than the rectangular window before taking the DFT, we can reduce the spectral leakage either by making it decay faster or by reducing the amplitude of the sidelobes. The principal tradeoffs are between, (a) the width of the main lobe (which determines how much spectral components are broadened), (b) the amplitude of the maximum sidelobe (which determines the energy of any spurious frequency components) and (c) the rate at which the sidelobe peaks decay with* $|k - k_0|$ *(which determines the range of frequencies affected by a strong spectral component).*

Rather few mentione that the effect of a broad sidelobe is to smooth the spectrum and reduce the spectral resolution. Many people dis-

cussed the tradeoffs involved in the window method of designing FIR filters (e.g. transition widths and stopband ripple) but the question was not asking about this.

b) i) Explain what is meant by saying that a linear time invariant system is “BIBO stable”. [2]

[B] An LTI system is BIBO stable if a bounded input sequence, $x[n]$, always gives a bounded output sequence, $y[n]$. That is,

$$|x[n]| \le B\ \forall n \ \Rightarrow\ |y[m]| \le f(B) < \infty\ \forall m$$

for some function $f(B)$.

Most people got this right. However quite a few incorrectly said that BIBO meant that “$x[n]$ and $y[n]$ are both bounded” instead of expressing the condition as an implication: “$x[n]$ bounded implies $y[n]$ bounded”.

ii) Prove that if a linear time invariant system is BIBO stable, then its impulse response, $h[n]$, satisfies $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$. [3]

[B] Define

$$x[n] = \begin{cases} +1 & h[-n] \ge 0 \\ -1 & h[-n] < 0 \end{cases}.$$

This clearly satisfies $|x[n]| \le 1\ \forall n$, so it follows from the BIBO condition that $|y[m]| \le K < \infty\ \forall m$ for some fixed K. In particular, $y[0] = \sum_{r=-\infty}^{+\infty} h[r]x[0-r] = \sum_{r=-\infty}^{+\infty} |h[r]| \le K < \infty$.

Quite a few people proved the converse of the question: that an absolutely summable impulse response implies a BIBO system. This is true but not what the question asked for. It involves assuming that $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$ which is exactly what you are asked to prove. Many peaople thought that $|y[m]| \le K$ and $|y[m]| \le B \sum_{r=-\infty}^{+\infty} |h[r]|$ together implied that $B \sum_{r=-\infty}^{+\infty} |h[r]| \le K$ which is not logically correct. Many people tried to use an arbitrary bounded signal as the input, $x[n]$ but this approach cannot work; the only way to use the knowledge that the system is BIBO, is to choose a specific bounded input and then use the knowledg that the output must be bounded.

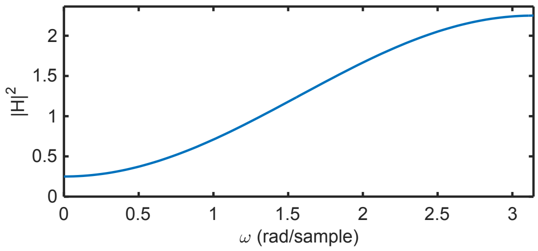
c) A first-order FIR filter is given by $H(z) = 1 - 0.5z^{-1}$.

i) Determine a simplified expression for the squared magnitude response, $\left|H(e^{j\omega})\right|^2$, and sketch its graph for $\omega \in [0, \pi]$. [4]

[U] Since $H(z) = 1 - 0.5z^{-1}$, $H(e^{j\omega}) = 1 - 0.5e^{-j\omega}$ *and*

$$\begin{aligned}\left|H(e^{j\omega})\right|^2 &= H(e^{j\omega})H(e^{-j\omega}) = \left(1 - 0.5e^{-j\omega}\right)\left(1 - 0.5e^{j\omega}\right)\\ &= 1 - \cos\omega + 0.25 = 1.25 - \cos\omega.\end{aligned}$$

Its graph is



Most people got the correct formula. However surprisingly many people drew the graph incorrectly. Quite a few people calculated the graph value at $\omega = \{0, 0.5\pi, \pi\}$ *and then drew a striaght line. Several gave the answer as* $|1.25 - \cos\omega|$ *which is correct although the magnitude signs are redundant; this usually arose because they wrote* $\left|H(e^{j\omega})\right|^2 = \left|H(e^{j\omega})H(e^{-j\omega})\right|$ *which is also correct but over-complicated.*

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ii) Using the formula in the datasheet, or otherwise, determine the group delay of the filter, $\tau_H(e^{j\omega})$, and sketch its graph for $\omega \in [0, \pi]$. [4]

[U] From the datasheet,

$$\begin{aligned}\tau_H(e^{j\omega}) &= \Re\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Bigg|_{z=e^{j\omega}}\\ &= \Re\left(\frac{-z \times 0.5z^{-2}}{1 - 0.5z^{-1}}\right)\Bigg|_{z=e^{j\omega}}\\ &= \Re\left(\frac{-0.5e^{-j\omega}}{1 - 0.5e^{-j\omega}}\right) = \Re\left(\frac{1}{1 - 2e^{j\omega}}\right)\\ &= \Re\left(\frac{1 - 2e^{-j\omega}}{1 - 4\cos\omega + 4}\right) = \Re\left(\frac{1 - 2\cos\omega + 2j\sin\omega}{5 - 4\cos\omega}\right)\\ &= \frac{1 - 2\cos\omega}{5 - 4\cos\omega}\end{aligned}$$

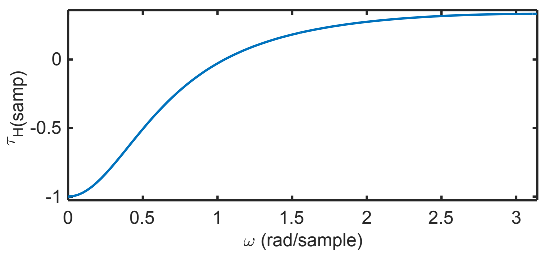
Using an alternative formula

$$\begin{aligned}
\tau_H(e^{j\omega}) &= \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right) \\
&= \Re\left(\frac{0-0.5e^{-j\omega}}{1-0.5e^{-j\omega}}\right) = \Re\left(\frac{-e^{-j\omega}}{2-e^{-j\omega}}\right) \\
&= \Re\left(\frac{-e^{-j\omega}\left(2-e^{j\omega}\right)}{\left(2-e^{-j\omega}\right)\left(2-e^{j\omega}\right)}\right) \\
&= \Re\left(\frac{1-2e^{-j\omega}}{4-4\cos\omega+1}\right) \\
&= \frac{1-2\cos\omega}{5-4\cos\omega}
\end{aligned}$$

For completeness, a bad choice of method uses the quite well known formula $\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}$ *to say:*

$$\begin{aligned}
\tau_H(e^{j\omega}) &= -\frac{d\angle H(e^{j\omega})}{d\omega} \\
&= -\frac{d\angle\left(1-0.5e^{-j\omega}\right)}{d\omega} = -\frac{d\left(\tan^{-1}\left(\frac{\sin\omega}{2-\cos\omega}\right)\right)}{d\omega} \\
&= -\frac{1}{1+\left(\frac{\sin\omega}{2-\cos\omega}\right)^2} \times \frac{d\left(\frac{\sin\omega}{2-\cos\omega}\right)}{d\omega} \\
&= -\frac{1}{1+\left(\frac{\sin\omega}{2-\cos\omega}\right)^2} \times \frac{(2-\cos\omega)\cos\omega-\sin^2\omega}{(2-\cos\omega)^2} \\
&= \frac{\sin^2\omega-2\cos\omega+\cos^2\omega}{4-4\cos\omega+\cos^2\omega+\sin^2\omega} = \frac{1-2\cos\omega}{5-4\cos\omega}
\end{aligned}$$

The graph is



with particular points of interest at $\tau_H(e^{j0}) = -1$, $\tau_H(e^{j\frac{\pi}{3}}) = 0$, $\tau_H(e^{j\frac{\pi}{2}}) = 0.2$ *and* $\tau_H(e^{j\pi}) = 0.333$.

Surprisingly many people could not differentiate $H(z) = 1-0.5z^{-1}$ *correctly to obtain* $\frac{dH(z)}{dz} = 0.5z^{-2}$. *Some people differentiated* $|H(z)|^2$ *instead. Several people tried using the formula* $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega}$ *directly (see above for how to do this): this is much messier than the other expressions given in the datasheet and no one used this method successfully. Several people thought that* $\angle\left(1-0.5e^{-j\omega}\right) = -\omega$ *instead of* $\tan^{-1}\left(\frac{\sin\omega}{2-\cos\omega}\right)$. *Several people also thought that* $\Re\left(\frac{x}{y}\right) = \frac{\Re(x)}{\Re(y)}$ *instead of* $\frac{\Re(xy^*)}{|y|^2}$.

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d) In the block diagram of Figure 1.1, all elements are drawn with their outputs on the right. The input and output signals are $x[n]$ and $y[n]$ respectively.

i) Determine the transfer function of the system, $H(z) = \frac{Y(z)}{X(z)}$. [3]

[U] We can write

$$W = X - z^{-1}aW \Rightarrow \left(1 + az^{-1}\right)W = X \Rightarrow W = \frac{1}{1+az^{-1}}X$$

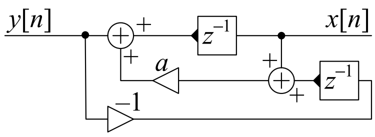
$$Y = z^{-1}W + aW = \left(a + z^{-1}\right)W = \frac{a+z^{-1}}{1+az^{-1}}X$$

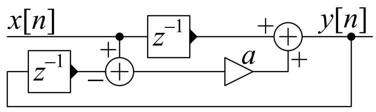
Although not requested, this is a first order allpass filter.

Most people got this right.

ii) Draw the transposed form of the block diagram. [4]

[U] The transposed block diagram is obtained by reversing the direction of all elements and interchanging junctions and adders. This gives the left diagram which can be re-drawn to give the right diagram (in which the gain of -1 has been absorbed into the adder that follows).





Most people either got this completely right or else were not abe to do it at all. Instead of drawing the transpose of the original block diagram, very many people drew the equivalent filter in a completely different form (e.g. a direct form or a transposed direct form) which uses two multiplication blocks. An advantage of the block diagram given in the question is that, since there is only one multiplier, the filter is bound to be allpass even if the value of a is incorrect (e.g. rounded to a small number of bits). Several people omitted the minus sign on the adder input or else put it in the wrong place.

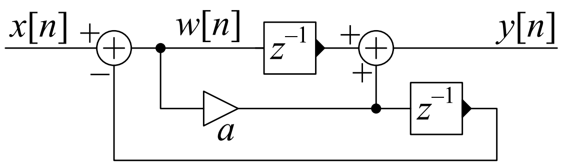


Figure 1.1

e) i) If a bounded discrete-time signal, $x[n]$, is stationary ergodic then $E\{x^2[n]\}$ for any n is equal to the average power of $x[n]$ (i.e. the average energy per sample). Explain why the average power of such a signal is unchanged by downsampling. [3]

[T] Downsampling by K retains only every K^{th} sample. If the signal is stationary ergodic, then all samples have the same average power,. $E\{x^2[n]\}$, and so retaining only every K^{th} sample leaves the average power unchanged.

Many people did not make clear that the property that $E\{x^2[n]\}$ for any n is equal to the average power of $x[n]$ is essential for the downsampling to leave the average power unchanged. An obvious counterexample is if $x[n] = 0$ for even n and 1 for odd n. Downsampling by a factor of 2 will change the average power from 0.5 to 0.

ii) Figure 1.2 shows the power spectral density (PSD) of a real-valued stationary ergodic signal, $x[n]$; the horizontal portions of the PSD have values 1 or 4.

The signal $y[m] = x[3m]$ is obtained by downsampling $x[n]$ by a factor of 3. Draw a dimensioned sketch of the PSD of $y[m]$ giving the values of all horizontal portions of the graph and the values of all frequencies at which there is a discontinuity in the PSD. [4]

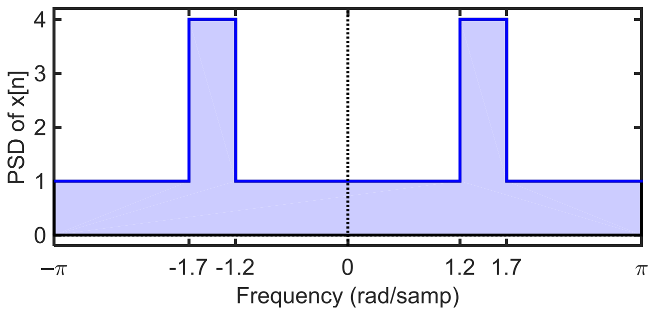
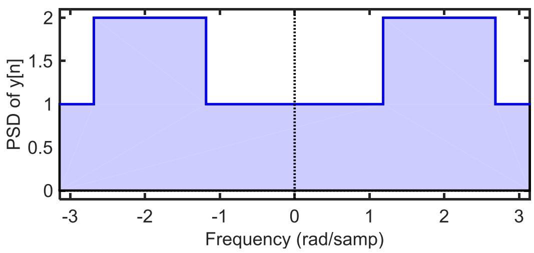


Figure 1.2

[U] Downsampling by 3 multiplies all the frequencies by 3 and so $\{-1.7, -1.2, 1.2, 1.7\}$ becomes $\{-5.1, -3.6, 3.6, 5.1\}$. Since these values are outside the range $\pm\pi$, we add/subtract 2π to alias them into the correct range. Thus they become $\{1.18, 2.68, -2.68, -1.18\}$.

We can regard the original signal as the sum of a broadband signal with a PSD of 1 and a band-limited signal with a PSD of 3. The PSD of the broadband component will remain unchanged while that of the bandlimited component will be divided by 3 (so that its total power remains unchanged). Thus the resultant PSD is



The horizontal portions of the graph have values of 1 or 2 and the discontinuities are at $\omega = \{\pm 1.18, \pm 2.68\}$.

Several people added/subtracted π instead of 2π to the normalized frequencies. Many people reduced the PSD of the broadband component to 0.333; however this reduces its total power (i.e. the integral of the PSD) by a factor of 3 which contradicts the first part of the question.

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f) i) In the block diagram of Figure 1.3 the input is $x[m]$ and the output is $y[n]$. Determine a simplified expression for $Y(z)$ in terms of $X(z)$ and the filters $H_p(z)$ for $p \in [0, 2]$. [3]

[BU] From the datasheet, $V_p(z) = H_p(z^3)X(z^3)$. So we can write

$$
\begin{aligned}
Y(z) &= V_0(z) + z^{-1}V_1(z) + z^{-2}V_2(z) \\
&= H_0(z^3)X(z^3) + z^{-1}H_1(z^3)X(z^3) + z^{-2}H_2(z^3)X(z^3) \\
&= \left(H_0(z^3) + z^{-1}H_1(z^3) + z^{-2}H_2(z^3)\right)X(z^3) = X(z^3)\sum_{p=0}^{2} z^{-p}H_p(z^3)
\end{aligned}
$$

Most people got this right

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ii) If $H_p(z) = \sum_{m=0}^{M} h_p[m]z^{-m}$, derive an expression for $g[n]$ in terms of the $h_p[m]$ so that the block diagram of Figure 1.4 is equivalent to that of Figure 1.3. [3]

[BU] From the diagram (and using the datasheet) $Y(z) = G(z)X(z^3)$. Thus we need to have $G(z) = H_0(z^3) + z^{-1}H_1(z^3) + z^{-2}H_2(z^3)$.

By writing $n = 3m + p$ where $m = \left\lfloor \frac{n}{3} \right\rfloor$ and $p = n - 3m \in [0, 2]$, we *can write*

$$
\begin{aligned}
G(z) &= \sum_{n=0}^{3M+2} g[n]z^{-n} \\
&= \sum_{p=0}^{2}\sum_{m=0}^{M} g[3m+p]z^{-3m-p} \\
&= \sum_{p=0}^{2} z^{-p}\sum_{m=0}^{M} g[3m+p]z^{-3m} \\
&= \sum_{p=0}^{2} z^{-p}\sum_{m=0}^{M} h_p[m]z^{-3m} \\
&= \sum_{p=0}^{2} z^{-p}H_p(z^3)
\end{aligned}
$$

This is of the required form with $g[3m+p] = h_p[m]$ for $m \geq 0$ and $0 \leq p \leq 2$ or, equivalently, $g[n] = h_{n \bmod 3}\left[\frac{n-n \bmod 3}{3}\right] = h_{n \bmod 3}\left[\left\lfloor \frac{n}{3} \right\rfloor\right]$ where $\lfloor\,\rfloor$ denotes the floor function. For example, $g[0] = h_0[0]$, $g[1] = h_1[0]$, $g[2] = h_2[0]$, $g[3] = h_0[1]$, $g[4] = h_1[1]$, etc.

Surprisingly few people realized that the g coefficients cycle through the h_p coefficients in sequence. Many people gave an expression for $g[n]$ that involved z and quite often one that did not involve n at all. An expression for $g[n]$ needs to be a function only of n. Quite a common wrong answer was $g[n] = h_0[3m] + h_1[3m-1] + h_2[3m-2]$ which makes no sense at all since the right hand side does not depend on n.

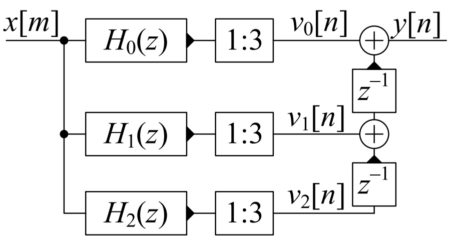

Figure 1.3

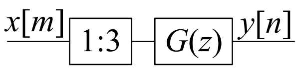

Figure 1.4

2. a) Outline the relative advantages of the bilinear and impulse-invariant transformations for converting a continuous-time filter into a discrete-time filter. [2]

[B] The bilinear mapping preserves both the magnitude and phase of the frequency response exactly but at the expense of a non-linear transformation of the frequency axis. In contrast, the impulse-invariant transformation preserves an undistorted frequency axis but introduces aliasing into the frequency response.

Most people got this approximately right although often the details were rather vague.

b) If p is a complex-valued constant, show that the z-transform of the causal sequence $v[n] = e^{pn}$ is given by $V(z) = \left(1 - e^{p}z^{-1}\right)^{-1}$ and give its region of convergence. [3]

[B] From the datasheet we have (summing from $n = 0$ since $v[n]$ is causal),

$$
\begin{aligned}
V(z) &= \sum_{n=0}^{\infty} v[n]z^{-n} \\
&= \sum_{n=0}^{\infty} e^{pn}z^{-n} \\
&= \frac{1}{1 - e^{p}z^{-1}}
\end{aligned}
$$

where, from the datasheet, the last line is true provided that $\left|e^{p}z^{-1}\right| < 1$ which is equivalent to $|z| > |e^{p}| = e^{\Re(p)}$ for the ROC.

Quite a few people omitted the modulus signs and said $|z| > e^{p}$ which makes no sense if p is complex (since "$>$" only applies to real numbers).

c) For $t \geq 0$, the impulse response of the causal continuous-time filter $H(s) = \frac{\Omega_0^2}{(s+\alpha)^2+\Omega_0^2}$ is given by

$$
\begin{aligned}
h(t) &= \Omega_0 e^{-\alpha t}\sin(\Omega_0 t) \\
&= -0.5j\Omega_0 e^{-\alpha t}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right).
\end{aligned}
$$

i) Use the result of part b) to find a simplified expression for the z-transform, $G(z)$, of the causal sequence given by $g[n] = T \times h(nT)$ where T is the sample period. Express $G(z)$ as a ratio of polynomials in z^{-1}. [7]

[U] We have

$$
\begin{aligned}
g[n] &= T \times h(nT) \\
&= -0.5jT\Omega_0 e^{-\alpha Tn}\left(e^{j\Omega_0 Tn} - e^{-j\Omega_0 Tn}\right) \\
&= -0.5jT\Omega_0\left(e^{(-\alpha T+j\Omega_0 T)n} - e^{(-\alpha T-j\Omega_0 T)n}\right).
\end{aligned}
$$

From the result of part b), we can write

$$\begin{aligned}
G(z) &= -0.5jT\Omega_0 \left(\frac{1}{1-e^{-\alpha T + j\Omega_0 T} z^{-1}} - \frac{1}{1-e^{-\alpha T - j\Omega_0 T} z^{-1}} \right) \\
&= -0.5jT\Omega_0 \left(\frac{e^{-\alpha T + j\Omega_0 T} z^{-1} - e^{-\alpha T - j\Omega_0 T} z^{-1}}{\left(1-e^{-\alpha T + j\Omega_0 T} z^{-1}\right)\left(1-e^{-\alpha T - j\Omega_0 T} z^{-1}\right)} \right) \\
&= -0.5jT\Omega_0 z^{-1} e^{-\alpha T} \left(\frac{e^{j\Omega_0 T} - e^{-j\Omega_0 T}}{1-e^{-\alpha T}\left(e^{j\Omega_0 T}+e^{-j\Omega_0 T}\right) z^{-1} + e^{-2\alpha T} z^{-2}} \right) \\
&= \frac{\Omega_0 T e^{-\alpha T} \sin\left(\Omega_0 T\right) z^{-1}}{1-2e^{-\alpha T}\cos\left(\Omega_0 T\right) z^{-1} + e^{-2\alpha T} z^{-2}}
\end{aligned}$$

or, defining $\omega_0 = \Omega_0 T$,

$$G(z) = \frac{\omega_0 e^{-\alpha T} \sin\left(\omega_0\right) z^{-1}}{1-2e^{-\alpha T}\cos\left(\omega_0\right) z^{-1} + e^{-2\alpha T} z^{-2}}.$$

Many people left the expression for $G(z)$ as in one of lines one to three of the above derivation; however the question asked for a simplified expression that was a ratio of polynomials in z^{-1}. Since the impulse reponse is real, the polynomial coefficients must also be real so an answer that includes "j" has not been simplified enough.

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ii) If $T = 10^{-4}$ s, $\Omega_0 = 5000\,\text{rad/s}$ and $\alpha = 800\,\text{s}^{-1}$, give the numerical values of the coefficients of $G(z)$ to 3 decimal places after normalizing to make the leading denominator coefficient unity. [3]

[U] Substituting the given values into the above formula gives

$$\begin{aligned}
\Omega_0 T &= 0.5 \\
e^{-\alpha T} &= 0.923 \\
G(z) &= \frac{0.221 z^{-1}}{1-1.620 z^{-1} + 0.852 z^{-2}}.
\end{aligned}$$

So the numerator and denominator coefficients are $[0, 0.221]$ *and* $[1, -1.620, 0.852]$.

Some people obtained complex coefficients; this is a sure sign of a calculation error if the impulse response is real. Others used $\sin(\Omega_0) = \sin(5000)$ *instead of* $\sin(\Omega_0 T) = \sin(0.5)$*; it normally indicates an error if the arguments to* sin *or* cos *are very large or very small.*

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d) i) Show that, under the mapping $s = \kappa\frac{z-1}{z+1}$, the value $s = j\Omega_0$ corresponds to $z = e^{j\omega_0}$ where $\Omega_0 = \kappa \tan\left(0.5\omega_0\right)$. Determine the numerical value of κ such that $\omega_0 = \Omega_0 T$ when T and Ω_0 have the values given in part c)ii). [4]

[B] Substituting $\Omega_0 = \kappa \tan(0.5\omega_0)$ into the mapping equation gives

$$\begin{aligned}
j\Omega_0 &= j\kappa \frac{\sin 0.5\omega_0}{\cos 0.5\omega_0} \\
&= \kappa \frac{e^{j0.5\omega_0} - e^{-j0.5\omega_0}}{e^{j0.5\omega_0} + e^{-j0.5\omega_0}} \\
&= \kappa \frac{e^{j\omega_0} - 1}{e^{j\omega_0} + 1} = \kappa \frac{z-1}{z+1}.
\end{aligned}$$

[U] Substituting the given values to determine κ gives

$$\begin{aligned}
\kappa &= \frac{\Omega_0}{\tan(0.5\Omega_0 T)} \\
&= \frac{5000}{\tan 0.25} \\
&= \frac{5000}{0.255} = 19582 = 1.9582 \times 10^4.
\end{aligned}$$

Some had their calculators set to degrees and calculated $\tan(0.25°) = \tan(0.0044) = 1.15 \times 10^6$ *(an easy mistake to make).*

---

ii) Use the bilinear mapping from part d)i) to transform the filter $H(s)$ from part c) into a discrete time filter, $F(z)$, and give the numerical values of its coefficients to 3 decimal places after normalizing to make the leading denominator coefficient unity. [7]

[U] Writing $\bar{\kappa} = \frac{\kappa}{\Omega_0}$ and $\bar{\alpha} = \frac{\alpha}{\Omega_0}$, we have

$$\begin{aligned}
F(z) &= \frac{\Omega_0^2}{(\bar{\kappa}\Omega_0 \frac{z-1}{z+1} + \bar{\alpha}\Omega_0)^2 + \Omega_0^2} \\
&= \frac{(z+1)^2}{(\bar{\kappa}(z-1) + \bar{\alpha}(z+1))^2 + (z+1)^2} \\
&= \frac{z^2 + 2z + 1}{((\bar{\alpha} + \bar{\kappa})z + (\bar{\alpha} - \bar{\kappa}))^2 + (z+1)^2} \\
&= \frac{z^2 + 2z + 1}{((\bar{\alpha} + \bar{\kappa})^2 + 1)z^2 + 2(\bar{\alpha}^2 - \bar{\kappa}^2 + 1)z + ((\bar{\alpha} - \bar{\kappa})^2 + 1)}.
\end{aligned}$$

Substituting in the values gives

$$\begin{aligned}
\bar{\kappa} &= \frac{19582}{5000} = 3.916 \\
\bar{\alpha} &= \frac{800}{5000} = 0.16
\end{aligned}$$

$$\begin{aligned}
F(z) &= \frac{z^2 + 2z + 1}{((0.16 + 3.916)^2 + 1)z^2 + 2(0.16^2 - 3.916^2 + 1)z + ((0.16 - 3.916)^2 + 1)} \\
&= \frac{z^2 + 2z + 1}{(16.616 + 1)z^2 + 2(0.026 - 15.338 + 1)z + (14.110 + 1)} \\
&= \frac{z^2 + 2z + 1}{17.616z^2 - 28.624z + 15.110}.
\end{aligned}$$

Thus the numerator and denominator filter coefficients are $[1, 2, 1]$ *and* $[17.616, -28.624, 15.110]$ *or, after normalizing,* $[0.057, 0.1135, 0.057]$ *and* $[1, -1.625, 0.858]$.

Presumably because they misunderstood this equivalence in the previous part of the question, several people substituted $\Omega_0 = -js$ *to give* $H(s) = \frac{-s^2}{(s+\alpha)^2 - s^2}$ *which is a completely different filter. Some gave unevaluated answers, e.g.* $\frac{25000000}{440425924}$ *instead of 0.057; this loses marks because it is clearly forbidden both in the question and in the rubric at the beginning of the paper. Not everyone normalized the polynomial correctly: the "leading coefficient" of a polynomial is that of the highest power of* z *(in this case* z^0*).*

e) Using the values given in part c)ii), determine the pole and zero positions of $H(s)$, $G(z)$ and $F(z)$ and comment on their relationship to the properties of the three filters. [4]

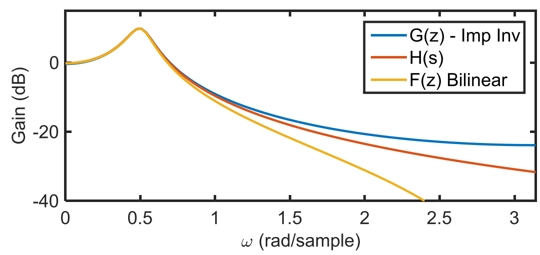
[U] For $H(s) = \frac{\Omega_0^2}{(s+\alpha)^2 + \Omega_0^2}$, *we have a complex conjugate pair of poles at* $-\alpha \pm j\Omega_0 = -800 \pm j5000$.

For $G(z) = \frac{0.221z^{-1}}{1 - 1.620z^{-1} + 0.852z^{-2}} = \frac{0.221z}{z^2 - 1.620z + 0.852}$, *we have a zero at* $z = 0$ *and poles at* $z = \frac{1.62 \pm \sqrt{2.624 - 3.408}}{2} = 0.810 \pm j0.443 = 0.923\angle \pm 0.5$.

For $F(z) = \frac{z^2 + 2z + 1}{17.616(z^2 - 1.625z + 0.858)}$ *we have a double zero at* $z = -1$ *and poles at* $z = \frac{1.625 \pm \sqrt{2.640 - 3.431}}{2} = 0.812 \pm j0.445 = 0.926\angle \pm 0.5$.

Thus we see that $G(z)$ *and* $H(s)$ *both have a single complex pole pair (excluding the pole at* $z = 0$ *which affects only the phase response) which, by construction, gives rise to identical impulse responses consisting of an exponentially decaying sine wave.* $F(z)$ *also has a complex pole pair at almost the same place but, in addition, has a double zero at* $z = -1$ *which means that* $F(e^{j\pi}) = 0$ *unlike* $G(e^{j\pi}) = \frac{-0.221}{1 + 1.62 + 0.852} = -0.0637$.

From the magnitude response plot shown below (not requested) we see that the responses are very similar at low frequencies but differ substantially at high frequencies.



Some people wrongly thought that $H(s)$ *had a zero at* $s = 0$.

3. The FM radio baseband spectrum shown in Figure 3.1 comprises (i) a mono signal (L+R) with a bandwidth of 15 kHz, (ii) a 19 kHz pilot tone and (iii) stereo information (L–R) modulated on a suppressed 38 kHz subcarrier. To demodulate the stereo component it is necessary to regenerate the 38 kHz subcarrier by isolating the 19 kHz pilot tone and multiplying its frequency by 2. The baseband signal is sampled at $f_s = 200$ kHz.

All filters in this question have real coefficients and are lowpass FIR filters with a stopband attenuation of 60 dB whose order may be estimated using the datatsheet formula $M = \frac{60}{3.5\Delta\omega}$ where $\Delta\omega$ is the transition bandwidth in rad/sample.

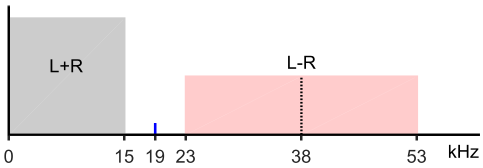


Figure 3.1

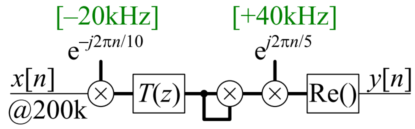


Figure 3.2

a) A block diagram for obtaining the 38 kHz subcarrier, $y[n]$, is shown in Figure 3.2 in which complex-valued signal paths are shown as bold lines. The baseband FM signal, $x[n]$, is translated down in frequency by 20 kHz and lowpass filtered by $T(z)$ to isolated the pilot tone component. The output of $T(z)$ is squared and translated up in frequency by 40 kHz and then the subcarrier, $y[n]$, is obtained by taking the real part of the signal.

i) The pilot tone component of $x[n]$ is given by $x_p[n] = \cos\omega_p n$ and has a frequency of $\omega_p = 2\pi \times \frac{19}{200} = 0.597$ rad/sample.

Give the signed frequencies, in rad/sample, of the complex exponential components of the pilot tone signal at each stage of the processing, i.e. for each horizontal line segment in Figure 3.2. [3]

[U] At successive nodes along the signal path, the true frequency is

$$\{\pm 19, -39\,\&\,-1, -1, -2, 38, \pm 38\} \text{ kHz.}$$

To obtain the normalized frequencies, we multiply these values by $\frac{2\pi}{f_s} = 3.14 \times 10^{-5}$ *to obtain*

$$\{\pm 0.597, -1.225\,\&\,-0.031, -0.031, -0.063, 1.194, \pm 1.194\} \text{ rad/sample.}$$

No one mentioned the frequency component that starts as -19 kHz is removed by $T(z)$ and then re-introduced by $\Re()$.

---

ii) Determine the passband edge frequency and the width of the transition band, $\Delta\omega$, for the lowpass filter, $T(z)$. Hence determine the required FIR filter order using the formula given at the start of the question. [3]

[U] The lowpass filter must pass the pilot tone at -1 kHz but must block the nearby signal components at $\{15, 23\} - 20 = \{-5, 3\}$.

Hence the transition band is $[1, 3]$ kHz *and its width is* $2\,\text{kHz} = 0.063\,\text{rad/sample}$. *Substituting this into the given formula (and rounding up) gives* $M = 273$. *Note that provided* $T(z)$ *has real coefficients, its magnitude frequency response will be symmetrical around* $\omega = 0$ *even though the wanted signal is asymmetric.*

Most got this right.

---

iii) Explain why squaring the output of $T(z)$ doubles the frequency of the pilot tone component. [3]

[T] Following, $T(z)$, *the pilot tone components is* $e^{j\omega_p n}$ *(assuming that the passband gain of* $T(z)$ *is* $1\angle 0$*). Squaring this gives* $\left(e^{j\omega_p n}\right)^2 = e^{j2\omega_p n}$ *which is a complex exponential with twice the frequency.*

Some people gave a correct but more complicated explanation that involved convolving the spectrum of $e^{j\omega_p n}$ *with itself.*

---

iv) Estimate the number of real multiplications per second needed to implement the block diagram of Figure 3.2. [3]

[U] Multiplications per input sample are: 2 for the frequency downshift, $2 \times 274 = 548$ *for* $T(z)$ *(assuming that it has real coefficients), 3 for squaring a complex number and 2 for the frequency upshift (since only the real part is required). The total is therefore 555 per input sample or* 1.11×10^8 *per second. Note that* $(a + jb)^2 = a^2 - b^2 + 2jab$ *and so squaring a complex number only requires three real multiplications rather than the four you might expect.*

Several people forgot that they needed to double the multiplication rate for $T(z)$ *because its signal is complex. Some took the sample frequency to be 200 instead of 200,000. Instead of adding 3 multiplications for the squaring operation, some people multiplied the number of multiplications by 3 instead which gives far too large a number..*

[This question is continued on the next page]

b) An alternative block diagram for generating the 38 kHz subcarrier is shown in Figure 3.3 in which $T(z)$ has been replaced by a lowpass filter, $G(z)$, operating at a sample frequency of 10 kHz.

i) Explain the reason that the lowpass filters $F(z)$ and $H(z)$ are needed. [2]

[B] The antialiasing filter, $F(z)$, eliminates any signal components above the new Nyquist frequency of $0.5K^{-1}f_s$ to prevent aliasing. The reconstruction filter, $H(z)$, with the same cutoff frequency eliminates the image components that are introduced by the upsampling.

Most got this right (although often called the images aliasing).

---

ii) Determine the passband edge, transition band width and filter order for each of the lowpass filters $F(z)$, $G(z)$ and $H(z)$. [6]

[U] The intermediate sample frequency is $10\,\text{kHz}$. The antialiasing filter, $F(z)$, must pass the pilot tone at $-1\,\text{kHz} = -0.597\,\text{rad/s}$ but must block anything that might alias into the passband+transition band of $G(z)$, namely $\pm 3\,\text{kHz}$. Thus anything outside $\mp 3 \pm 10 = \pm 7\,\text{kHz}$ must be blocked. Thus, the passband edge is $1\,\text{kHz} = 0.031\,\text{rad/s}$ and the transition band width is $7-1 = 6\,\text{kHz} = 0.189\,\text{rad/s}$ giving $M_F = 90.9 \approx 91$. Normally, it would be sufficient just to block components that would alias into the passband, but in this case the aliasing component are much larger than the wanted pilot tone signal and, unless the transition band of $G(z)$ is very steep, they might not be attenuated enough if they are passed by $F(z)$ into the transition band of $G(z)$.

For $G(z)$, the passband edge is $1\,\text{kHz} = 0.628\,\text{rad/s}$ and the transition band width is, as before, $2\,\text{kHz} = 1.257\,\text{rad/s}$ giving $M_G = 13.6 \approx 14$.

For $H(z)$, the pilot tone is now at $-2\,\text{kHz}$ so the passband edge is now $2\,\text{kHz} = 0.063\,\text{rad/s}$ and the nearest image frequency is at $-2+10 = 8\,\text{kHz}$ giving a transition band width of $8-2 = 6\,\text{kHz} = 0.189\,\text{rad/s}$ and a filter order of $M_H = 90.9 \approx 91$.

Despite answering the previous part correctly, many used $F(z)$ to filter out the L+R and L-R signal components (which will actually be removed by $G(z)$) rather than just doing what is needed to avoid aliasing. Several people assumed the spacing between images was $20\,\text{kHz}$ (presumably because this was the intermediate sample frequency used in the lecture note example) rather than $10\,\text{kHz}$.

---

iii) Estimate the number of real multiplications per second needed to implement the block diagram assuming that $F(z)$ and $H(z)$ both use a polyphase implementation that incorporates the associated upsampler/downsampler. You may assume without proof that a polyphase filter of order M acting on a complex-valued signal requires $(2M+2)$ multiplications per sample at the lower of the two sample rates. [3]

[U] The frequency downshift requires $2f_s$, the filters F, H, G require

$$(2M_F+2)K^{-1}f_s+(2M_G+2)K^{-1}f_s+(2M_H+2)K^{-1}f_s=354K^{-1}f_s,$$

the squaring requires $3K^{-1}f_s$ and the frequency upshift requires $2f_s$ for a total of $(2+23.2+0.3+2)f_s=27.52f_s=5.5\times10^6$. This is very much less than in part a)iv).

Most people got this right although several omitted the K^{-1} factors (despite the information given in the question) which then gives for more multiplications than are actually needed.

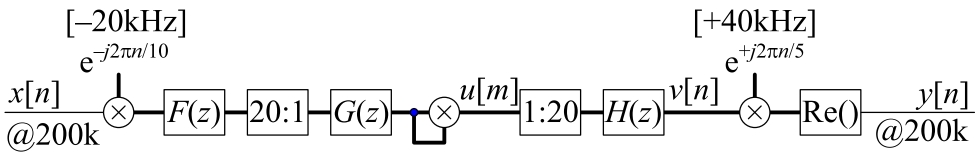


Figure 3.3

c) Suppose now that the upsampling is performed in two stages as illustrated in Figure 3.4 which replaces the blocks "1 : 20" and "$H(z)$" in Figure 3.3.

i) Determine the cutoff frequency, transition bandwidth and filter order for each of the lowpass filters $P(z)$ and $Q(z)$. [4]

[U] Following $P(z)$ the sample rate is $20\,\text{kHz}$ and the transition band of the filter needs to be $[2,8]\,\text{kHz}$ for a transition band width of $6\,\text{kHz}=1.885\,\text{rad/s}$ and $M_P=9.1\approx10$.

Following $Q(z)$ the sample rate is $200\,\text{kHz}$ so the transition band of the filter needs to be $[2,18]\,\text{kHz}$ for a transition band width of $16\,\text{kHz}=0.503\,\text{rad/s}$ and $M_Q=34.1\approx35$.

Note that in each case, the centre of the transition band is the old Nyquist frequency: 5 and $10\,\text{kHz}$ respectively.

Some people use transition bandwidths of 10 and $20\,\text{kHz}$ instead of 6 and $16\,\text{kHz}$ because they measured it to -2 instead of $+2\,\text{kHz}$.

ii) Estimate the number of real multiplications per second needed to implement the block diagram of Figure 3.4 assuming that a polyphase implementation is used for $P(z)$ and $Q(z)$. Compare this with the number of multiplications needed for the corresponding part of Figure 3.3. [3]

[U] The number of multiplications needed is

$$\frac{2M_P+2}{20}f_s+\frac{2M_Q+2}{10}f_s=\left(\frac{22}{20}+\frac{72}{10}\right)f_s=(1.1+7.2)f_s=8.3f_s=1.66\times10^6.$$

We can compare this to the multiplication rate required for $H(z)$

which is

$$\frac{(2M_H+2)}{20}f_s = \frac{186}{20}f_s = 9.3f_s = 1.86\times 10^6.$$

So performing the decimation in stages reduces the computation a little and halves the number of coefficients to store..

Most got the calculations right for $P(z)$ and $Q(z)$ although few people compared with $H(z)$.

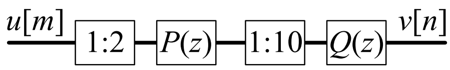


Figure 3.4

4. a) Explain briefly the advantages of processing signals in subbands. [2]

[B] After splitting a signal into subbands, the subband signals are bandlimited and so it is possible to reduce the sample rate within each subband. This generally results in lower computational requirements than for full-band processing. A second advantage is that adaptive filtering applications converge more rapidly because the signal power spectrum is flatter in any subband than in the full band. Finally, processing the signal independently in subbands allows parallelism.

Surprisingly few people stated any of these advantages precisely although most siad that the computational complexity would be reduced.

b) Figure 4.1 shows the analysis and synthesis stages of a 2-subband system. Show that $Y(z) = T(z)X(z)$ where $T(z) = \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z))$. [4]

For $p \in [0, 1]$ you may assume without proof that $W_p(z) = U_p(z^2)$ and that $U_p(z) = \frac{1}{2}\left\{V_p\left(z^{\frac{1}{2}}\right) + V_p\left(-z^{\frac{1}{2}}\right)\right\}$.

[B] Working backwards from the output to the input, we can write

$$
\begin{aligned}
Y(z) &= H(z)W_0(z) - H(-z)W_1(z) \\
&= H(z)U_0\left(z^2\right) - H(-z)U_1\left(z^2\right) \\
&= \frac{1}{2}\left(H(z)\left(V_0(z) + V_0(-z)\right) - H(-z)\left(V_1(z) + V_1(-z)\right)\right) \\
&= \frac{1}{2}\left(H(z)\left(H(z)X(z) + H(-z)X(-z)\right) - H(-z)\left(H(-z)X(z) + H(z)X(-z)\right)\right) \\
&= \frac{1}{2}\left(H^2(z) - H^2(-z)\right)X(z) \\
&= \frac{1}{2}\left(H(z) - H(-z)\right)\left(H(z) + H(-z)\right)X(z)
\end{aligned}
$$

x

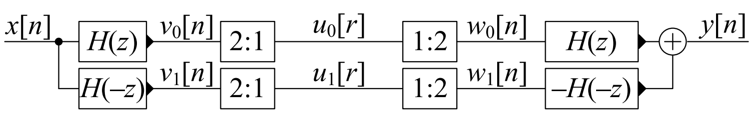


Figure 4.1

c) Given that the impulse response, $h[n]$, is causal and of odd order M, we define

$$t[n] = \frac{1}{2}\left(h[n] + (-1)^n h[n]\right) * \left(h[n] - (-1)^n h[n]\right)$$

where * denotes convolution.

i) Show that the z-transform of $t[n]$ is [3]

$$T(z) = \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z)).$$

[U] If we define $s[n] = (-1)^n h[n]$, we can write

$$S(z) = \sum s[n]z^{-n} = \sum h[n](-1)^n z^{-n}$$
$$= \sum h[n]\left(\frac{z}{-1}\right)^{-n} = \sum h[n](-z)^{-n} = H(-z).$$

Hence

$$T(z) = \frac{1}{2}(H(z) - S(z))(H(z) + S(z)) = \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z))$$

where convolution in the time domain corresponds to multiplication in the z-transform domain.

Quite a few people just assumed the z-transform of $(-1)^n h[n]$ without proving it. A more general theorem is that the z-transform of $a^n h[n]$ is $H\left(\frac{z}{a}\right)$. Because of the ordering of the factors in the question, quite a few people wrongly assumed that the z-transform of $(-1)^n h[n]$ was $-H(-z)$.

ii) Show that, if $h[n]$ satisfies the symmetry condition $h[M-n] = h[n]$, then $t[n]$ satisfies the condition $t[2M-n] = t[n]$. [3]

[U] The easiest method is to use the relationship that the z-transform of $h[M-n]$ is $\sum_n h[M-n]z^{-n} = \sum_n h[n]z^{n-M} = z^{-M}\sum_n h[n]z^n = z^{-M}H(z^{-1})$. So then we can write

$$\begin{aligned}
T(z) &= \frac{1}{2}(H(z) - H(-z))(H(z) + H(-z)) \\
&= \frac{1}{2}\left(z^{-M}H(z^{-1}) - z^{-M}H(-z^{-1})\right)\left(z^{-M}H(z^{-1}) + z^{-M}H(-z^{-1})\right) \\
&= z^{-2M}\frac{1}{2}\left(H(z^{-1}) - H(-z^{-1})\right)\left(H(z^{-1}) + H(-z^{-1})\right) \\
&= z^{-2M}T(z^{-1})
\end{aligned}$$

which is the z-transform of $t[2M-n]$.

Alternatively, doing it all in the time domain, we can write out the convolution explicitly as

$$2t[n] = \sum_r (h[r] - (-1)^r h[r])\left(h[n-r] + (-1)^{n-r}h[n-r]\right).$$

Now we can write was $-H(-z$

$$\begin{aligned}
2t[2M-n] &= \sum_r (h[r]+(-1)^r h[r])\left(h[2M-n-r]-(-1)^{2M-n-r}h[2M-n-r]\right)\\
&= \sum_r (h[M-r]+(-1)^r h[M-r])\left(h[n+r-M]-(-1)^{2M-n-r}h[n+r-M]\right)\\
&= \sum_s \left(h[s]+(-1)^{M-s}h[s]\right)\left(h[n-s]-(-1)^{M-n+s}h[n-s]\right)\\
&= \sum_s \left(h[s]+(-1)^M(-1)^s h[s]\right)\left(h[n-s]-(-1)^M(-1)^{n-s}h[n-s]\right)\\
&= \sum_s (h[s]-(-1)^s h[s])\left(h[n-s]+(-1)^{n-s}h[n-s]\right) = 2t[n]
\end{aligned}$$

where the second line follows from $h[M-n]=h[n]$, the third line uses the substitution $s=M-r$ and the fifth line uses $(-1)^M=-1$ since M is odd.

If M is even, the result is still true but the proof requires the factors in the summand to be interchanged.

There was some confusion between $H(z^{-1})$ and $H(-z)$; $H(z^{-1})$ is the z-transform of $h[-n]$ which reverses the order of the coefficients while $H(-z)$ is the z-transform of $(-1)^n h[n]$ which keeps the coefficients in the same order but negates those with odd n. Many people thought that $h[M-n]=h[n]$ must imply that $h[2M-n]=h[n]$ as well but this is not true.

---

iii) Deduce the group delay function, $\tau_T\left(e^{j\omega}\right)$, of the filter $T(z)$ from the symmetry condition of part ii). [2]

[U] The group delay of a symmetric filter satisfying $t[2M-n]=t[n]$ is independent of ω and equals M samples. Another way of looking at this is that $s[n]=t[n+M]$ is a symmetric filter (easily seen since $s[-n]=t[-n+M]=t[+n+M]=s[n]$. The symmetric fiter, $s[n]$, has 0 group delay, so $t[n]$ must have a group delay of M samples.

The proof (not required) is

$$\begin{aligned}
T(e^{j\omega}) &= \frac{1}{2}\left(\sum_n t[n]e^{-j\omega n}+\sum_n t[2M-n]e^{-j\omega n}\right)\\
&= \frac{1}{2}\left(\sum_n t[n]e^{-j\omega n}+\sum_n t[n]e^{-j\omega(2M-n)}\right)\\
&= \frac{1}{2}\sum_n t[n]\left(e^{-j\omega n}+e^{-j\omega(2M-n)}\right)\\
&= \frac{1}{2}e^{-j\omega M}\sum_n t[n]\left(e^{j\omega(M-n)}+e^{-j\omega(M-n)}\right)\\
&= \frac{1}{2}e^{-j\omega M}\sum_n t[n]\cos(\omega(M-n))\\
\Rightarrow \angle T(e^{j\omega}) &= -\omega M+\frac{\pi}{2}\left(1-\operatorname{sgn}\left(\sum_n t[n]\cos(\omega(M-n))\right)\right)
\end{aligned}$$

$$\Rightarrow \frac{-d\angle T(e^{j\omega})}{d\omega} = M$$

Surprisingly many people did not get this right.

[This question is continued on the next page]

d) i) By using the inverse DTFT, show that the impulse response of an ideal lowpass filter whose frequency response is

$$G\left(e^{j\omega}\right)=\begin{cases}e^{-j0.5M\omega} & |\omega|\leq\frac{\pi}{2}\\ 0 & \frac{\pi}{2}<|\omega|\leq\pi\end{cases}$$

is given by [3]

$$g[n]=\frac{\sin\left(0.5\pi\left(n-0.5M\right)\right)}{\pi\left(n-0.5M\right)}.$$

[U] From the inverse DTFT given in the datasheet,

$$\begin{aligned}g[n]&=\frac{1}{2\pi}\int_{-\pi}^{\pi}G\left(e^{j\omega}\right)e^{j\omega n}d\omega\\ &=\frac{1}{2\pi}\int_{-0.5\pi}^{0.5\pi}e^{-j0.5M\omega}e^{j\omega n}d\omega\\ &=\frac{1}{2\pi}\left[\frac{e^{j\omega(n-0.5M)}}{j(n-0.5M)}\right]_{-0.5\pi}^{0.5\pi}\\ &=\frac{2j\sin\left(0.5\pi(n-0.5M)\right)}{2\pi j(n-0.5M)}\\ &=\frac{\sin\left(0.5\pi(n-0.5M)\right)}{\pi(n-0.5M)}\end{aligned}$$

Most people got this right although a few were unable to do the integration (perhaps no realizing that the two exponentials could be combined).

---

ii) A causal Hamming window of length $M+1$ is given by

$$w[n]=0.54-0.46\cos\left(\frac{2n\pi}{M}\right)$$

for $n\in[0,M]$. Using the window design method with $g[n]$ and $w[n]$, design a causal FIR filter, $H(z)$, of order $M=7$ with a cutoff frequency of $\omega=\frac{\pi}{2}$. Determine the numerical values of the filter coefficients, $h[n]$, to 3 decimal places. [4]

[U] Substituting $n\in[0,M]$ into the given formula gives

$$w[n]=[0.080,0.253,0.642,0.954,0.954,0.642,0.253,0.080].$$

Likewise, from part i), we obtain

$$g[n]=[-0.064,-0.090,0.150,0.450,0.450,0.150,-0.090,-0.064].$$

Multiplying these together gives

$h[n] = [-0.005, -0.023, 0.096, 0.429, 0.429, 0.096, -0.023, -0.005]$.

Quite a lot of people calculated $h[n] = g[n-0.5M]w[n-0.5M]$ without realizing that $g[]$ and $w[$ were already causal due to the terms $0.5M$ in their expressions.

---

iii) The filter, $H(z)$, from part ii) is used in the block diagram shown in Figure 4.1. If $T(z) = \frac{Y(z)}{X(z)}$, determine the magnitude gain, $\left|T(e^{j\omega})\right|$ for $\omega = 0$, $\frac{\pi}{2}$ and π. [4]

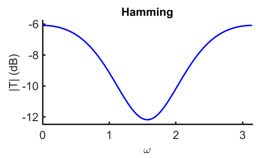
[U] The gain is given by $\left|T(e^{j\omega})\right| = 0.5\left|H^2(e^{j\omega}) - H^2(-e^{j\omega})\right|$. Taking $\mathbf{h}$ to be the column vector of filter coefficients,

$$
\begin{aligned}
H(e^{j0}) &= H(1) = [1\,1\,1\,1\,1\,1\,1\,1]\mathbf{h} = 0.994\\
H(e^{j\frac{\pi}{2}}) &= H(j) = [1\; j\; -1\; -j\; 1\; j\; -1\; -j]\mathbf{h} = 0.351 - 0.351j\\
&\Rightarrow H(-j) = 0.351 + 0.351j\\
H(e^{j\pi}) &= H(-1) = [1\; -1\; 1\; -1\; 1\; -1\; 1\; -1]\mathbf{h} = 0
\end{aligned}
$$

from which

$$
\begin{aligned}
\left|T(e^{j0})\right| &= 0.5\left|H^2(1) - H^2(-1)\right| = 0.5 \times 0.994^2 = 0.5 \times 0.988 = 0.494\\
\left|T(e^{j\frac{\pi}{2}})\right| &= 0.5\left|H^2(j) - H^2(-j)\right| = 0.5 \times |-0.246j - 0.246j| = 0.5 \times 0.492 = 0.246\\
\left|T(e^{j\pi})\right| &= 0.5\left|H^2(-1) - H^2(1)\right| = 0.5 \times 0.994^2 = 0.5 \times 0.988 = 0.494.
\end{aligned}
$$

The response, $T(e^{j\omega})$ has a 6 dB dip at $\omega = \frac{\pi}{2}$. The complete response (although not requested) is



x

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e) A "Johnston half-band filter" selects the coefficients, $h[n]$, to minimize the cost function

$$
\alpha \int_{\frac{\pi}{2}+\Delta}^{\pi} \left|H(e^{j\omega})\right|^2 d\omega + (1-\alpha) \int_0^{\pi} \left(\left|H^2(e^{j\omega}) - H^2(-e^{j\omega})\right| - 1\right)^2 d\omega
$$

for suitable choices of α and Δ.

i) Explain the significance of the two integrals in the cost function and hence explain the effect of reducing the value of α. [2]

[B] The filter $H(z)$ has a pass band of $(0, \frac{\pi}{2})$ and a stop band $(\frac{\pi}{2}, \pi)$. The first term of the cost function integrates the squared response over the stopband but allows a transition region $(\frac{\pi}{2}, \frac{\pi}{2}+\Delta)$; it is therefore a measure of the stop band attenuation. The overall magnitude gain $\left|T(e^{j\omega})\right|$ should ideally equal 1 at all frequencies. The second term in the cost function integrates the squared error in the overall magnitude gain over the entire band and is therefore a measure of the flatness of the overall response. Decreasing α will therefore make the overall response, $\left|T(e^{j\omega})\right|$, flatter but at the expense of reducing the stopband attenuation of $H(z)$.

Surprisingly few people understood the meaning of the two terms in the cost function.

---

ii) For $M = 7$, $\alpha = 0.5$ and $\Delta = 0.07$, the $h[n]$ are given by

$$h[0] = h[7] = 0.009, \qquad h[1] = h[6] = -0.071$$
$$h[2] = h[5] = 0.069, \qquad h[3] = h[4] = 0.490$$

Determine the magnitude gain, $\left|T(e^{j\omega})\right|$ for $\omega = 0, \frac{\pi}{2}$ and π. [3]

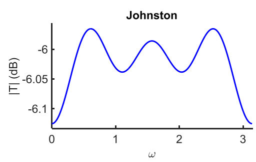
[U] The gain is given by $\left|T(e^{j\omega})\right| = H^2(e^{j\omega}) - H^2(-e^{j\omega})$. Taking $\mathbf{h}$ to be the column vector of filter coefficients,

$$H(e^{j0}) = H(1) = [1\,1\,1\,1\,1\,1\,1\,1]\mathbf{h} = 0.994$$
$$H(e^{j\frac{\pi}{2}}) = H(j) = [1\;j\;-1\;-j\;1\;j\;-1\;-j]\mathbf{h} = 0.501 - 0.501j$$
$$\Rightarrow H(-j) = 0.501 + 0.501j$$
$$H(e^{j\pi}) = H(-1) = [1\;-1\;1\;-1\;1\;-1\;1\;-1]\mathbf{h} = 0$$

from which

$$\left|T(e^{j0})\right| = 0.5\left|H^2(1) - H^2(-1)\right| = 0.5 \times 0.994^2 = 0.5 \times 0.988 = 0.494$$
$$\left|T(e^{j\frac{\pi}{2}})\right| = 0.5\left|H^2(j) - H^2(-j)\right| = 0.5 \times |-0.502j - 0.502j| = 0.5 \times 1.004 = 0.502$$
$$\left|T(e^{j\pi})\right| = 0.5\left|H^2(-1) - H^2(1)\right| = 0.5 \times 0.994^2 = 0.5 \times 0.988 = 0.494.$$

The complete response (not requested) is shown below and varies by only a small fraction of a dB; it is much flatter than the Hamming window design in part d).



Many people omitted the last two or three sub-parts of this question, presumably through lack of time.

Datasheet:

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1}z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$	$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$	$x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N} \sum_{n=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r) \bmod N] \quad \Leftrightarrow \quad V[k] = X[k] Y[k]$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N} \sum_{r=0}^{N-1} X[r] Y[(k-r) \bmod N]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}\left(\right)$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

where a =stop band attenuation in dB, $b =$ peak-to-peak passband ripple in dB and $\Delta\omega =$ width of smallest transition band in radians per sample.

z-plane Transformations

A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}$, $\rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

Noble Identities

$$-\boxed{Q{:}1}-\boxed{H(z)}- \quad = \quad -\boxed{H(z^Q)}-\boxed{Q{:}1}-$$

$$-\boxed{H(z)}-\boxed{1{:}Q}- \quad = \quad -\boxed{1{:}Q}-\boxed{H(z^Q)}-$$

Multirate Spectra

Upsample: $v[n] - \boxed{1{:}Q} - x[r]$ $\quad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample: $v[n] - \boxed{Q{:}1} - y[m]$ $\quad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators

Input Commutator	Output Commutator