FORMULA SHEET AVAILABLE IN EXAM

The following formulae will be available in the exam:

Where a question requires a numerical answer, it must be given as a fully evaluated decimal number and not as an unevaluated arithmetic expression.

Notation

- All signals and filter coefficients are real-valued unless explicitly noted otherwise.
- Unless otherwise specified, upper and lower case letters are used for sequences and their z-transforms respectively. The signal at a block diagram node V is $v[n]$ and its z-transform is $V(z)$.
- $x[n] = [a, b, c, d, e, f]$ means that $x[0] = a, \ldots x[5] = f$ and that $x[n] = 0$ outside this range.
- $\Re(z)$, $\Im(z)$, z^*, $|z|$ and $\angle z$ denote respectively the real part, imaginary part, complex conjugate, magnitude and argument of a complex number z.
- The expected value of x is denoted $E\{x\}$.
- In block diagrams: solid arrows denote the direction of signal flow; an open triangle denotes a gain element with the gain indicated adjacently; a "$+$" in a circle denotes an adder/subtractor whose inputs may be labelled "$+$" or "$-$" according to their sign; the sample rate, f, of a signal in Hz may be indicated in the form "@f".

Abbreviations

BIBO	Bounded Input, Bounded Output	IIR	Infinite Impulse Response
CTFT	Continuous-Time Fourier Transform	LTI	Linear Time-Invariant
DCT	Discrete Cosine Transform	MDCT	Modified Discrete Cosine Transform
DFT	Discrete Fourier Transform	PSD	Power Spectral Density
DTFT	Discrete-Time Fourier Transform	SNR	Signal-to-Noise Ratio
FIR	Finite Impulse Response		

Standard Sequences

- $\delta[n] = 1$ for $n = 0$ and 0 otherwise.
- $\delta_{\text{condition}}[n] = 1$ whenever "condition" is true and 0 otherwise.
- $u[n] = 1$ for $n \geq 0$ and 0 otherwise.

Geometric Progression

- $\sum_{n=0}^{r} \alpha^n z^{-n} = \frac{1-\alpha^{r+1} z^{-r-1}}{1-\alpha z^{-1}}$ provided that $\alpha z^{-1} \neq 1$.
- $\sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}$ provided that $\left|\alpha z^{-1}\right| < 1$.

Forward and Inverse Transforms

z:	$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$	$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$
CTFT:	$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
DTFT:	$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$	$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
DFT:	$X[k] = \sum_{0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$	$x[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}$
DCT:	$X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$	$x[n] = \frac{X[0]}{N} + \frac{2}{N} \sum_{n=1}^{N-1} X[k] \cos \frac{2\pi(2n+1)k}{4N}$
MDCT:	$X[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$	$y[n] = \frac{1}{N} \sum_{0}^{N-1} X[k] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$

Convolution

DTFT: $v[n] = x[n] * y[n] \triangleq \sum_{r=-\infty}^{\infty} x[r] y[n-r] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \circledast Y\left(e^{j\omega}\right) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Y\left(e^{j(\omega-\theta)}\right) d\theta$

DFT: $v[n] = x[n] \circledast_N y[n] \triangleq \sum_{r=0}^{N-1} x[r] y[(n-r) \mod N] \quad \Leftrightarrow \quad V[k] = X[k] Y[k]$

$v[n] = x[n] y[n] \quad \Leftrightarrow \quad V[k] = \frac{1}{N} X[k] \circledast_N Y[k] \triangleq \frac{1}{N} \sum_{r=0}^{N-1} X[r] Y[(k-r) \mod N]$

Group Delay

The group delay of a filter, $H(z)$, is $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \Re\left(\frac{-z}{H(z)} \frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \Re\left(\frac{\mathscr{F}(nh[n])}{\mathscr{F}(h[n])}\right)$ where $\mathscr{F}\,()$ denotes the DTFT.

Order Estimation for FIR Filters

Three increasingly sophisticated formulae for estimating the minimum order of an FIR filter with unity gain passbands:

1. $M \approx \frac{a}{3.5\Delta\omega}$
2. $M \approx \frac{a-8}{2.2\Delta\omega}$
3. $M \approx \frac{a-1.2-20\log_{10} b}{4.6\Delta\omega}$

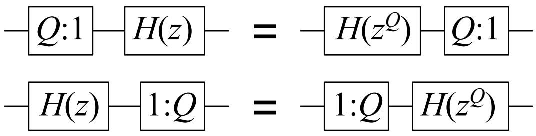
where a =stop band attenuation in dB, b = peak-to-peak passband ripple in dB and $\Delta\omega$ = width of smallest transition band in radians per sample.

z-plane Transformations

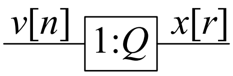
A lowpass filter, $H(z)$, with cutoff frequency ω_0 may be transformed into the filter $H(\hat{z})$ as follows:

Target $H(\hat{z})$	Substitute	Parameters
Lowpass $\hat{\omega} < \hat{\omega}_1$	$z^{-1} = \frac{\hat{z}^{-1}-\lambda}{1-\lambda\hat{z}^{-1}}$	$\lambda = \frac{\sin\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}{\sin\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}$
Highpass $\hat{\omega} > \hat{\omega}_1$	$z^{-1} = -\frac{\hat{z}^{-1}+\lambda}{1+\lambda\hat{z}^{-1}}$	$\lambda = \frac{\cos\left(\frac{\omega_0+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\omega_0-\hat{\omega}_1}{2}\right)}$
Bandpass $\hat{\omega}_1 < \hat{\omega} < \hat{\omega}_2$	$z^{-1} = -\frac{(\rho-1)-2\lambda\rho\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\rho\hat{z}^{-1}+(\rho-1)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \cot\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$
Bandstop $\hat{\omega}_1 \not< \hat{\omega} \not< \hat{\omega}_2$	$z^{-1} = \frac{(1-\rho)-2\lambda\hat{z}^{-1}+(\rho+1)\hat{z}^{-2}}{(\rho+1)-2\lambda\hat{z}^{-1}+(1-\rho)\hat{z}^{-2}}$	$\lambda = \frac{\cos\left(\frac{\hat{\omega}_2+\hat{\omega}_1}{2}\right)}{\cos\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)}, \rho = \tan\left(\frac{\hat{\omega}_2-\hat{\omega}_1}{2}\right)\tan\left(\frac{\omega_0}{2}\right)$

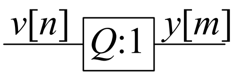
Noble Identities



Multirate Spectra

Upsample: 

$\Rightarrow \quad x[r] = \begin{cases} v\left[\frac{r}{Q}\right] & \text{if } Q \mid r \\ 0 & \text{if } Q \nmid r \end{cases} \quad \Rightarrow \quad X(z) = V(z^Q)$

Downsample: 

$\Rightarrow \quad y[m] = v[Qm] \quad \Rightarrow \quad Y(z) = \frac{1}{Q}\sum_{k=0}^{Q-1} V\left(e^{\frac{-j2\pi k}{Q}} z^{\frac{1}{Q}}\right)$

Multirate Commutators

Input Commutator	Output Commutator