E1.10 Fourier Series and Transforms

Mike Brookes

- Syllabus
- Optical Fourier Transform
- Organization

1: Sums and Averages

Syllabus

Main fact: Complicated time waveforms can be expressed as a sum of sine and cosine waves.

Joseph Fourier

1768-1830

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- Fourier transform for aperiodic waveforms (3 lectures)

Optical Fourier Transform

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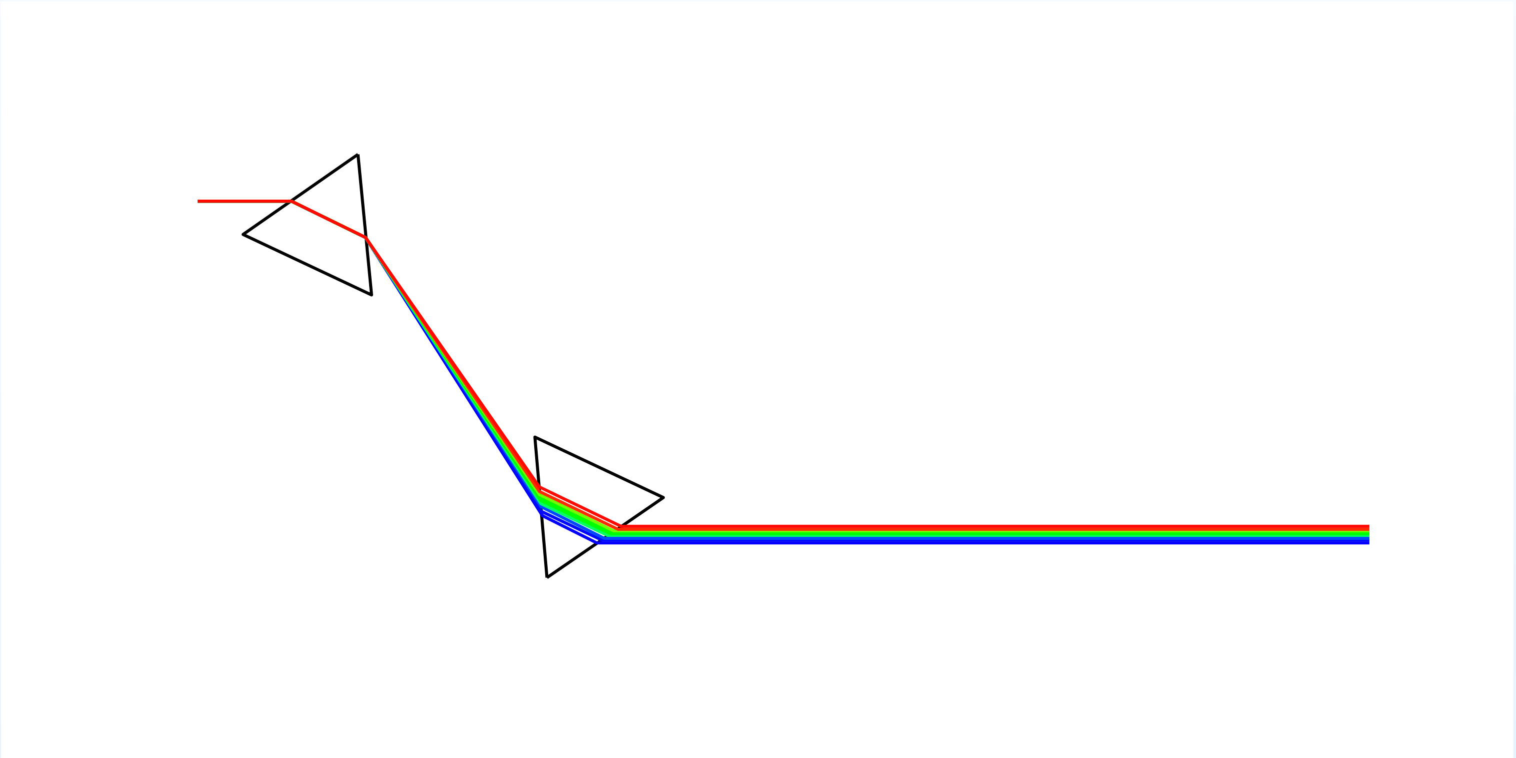

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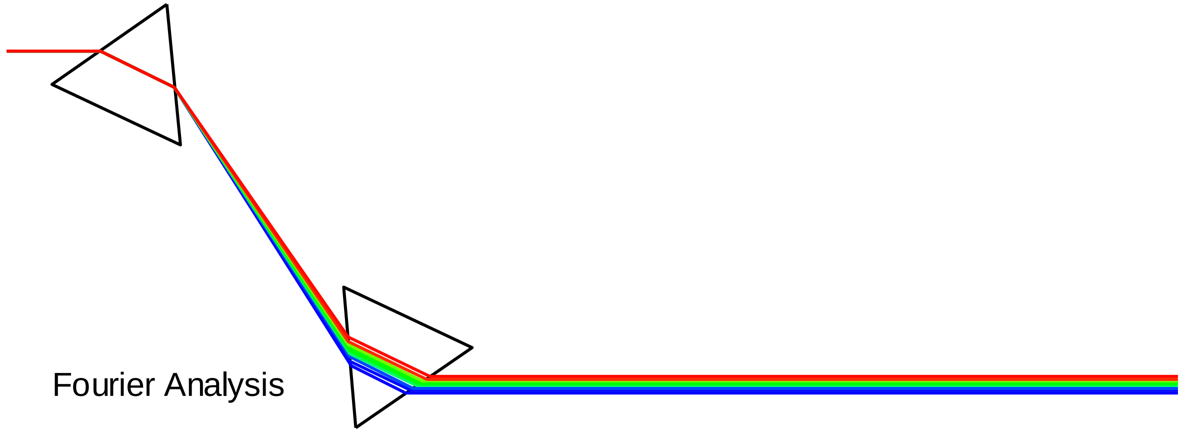

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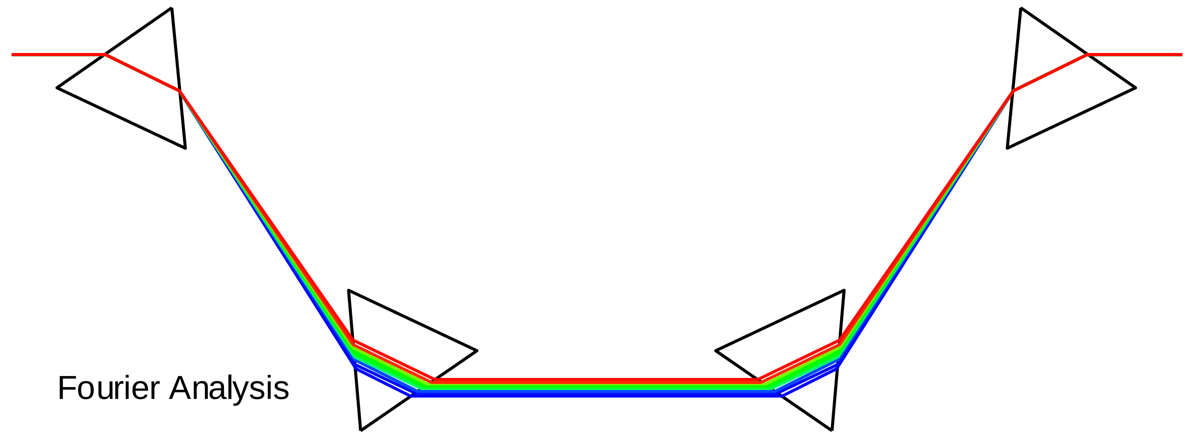

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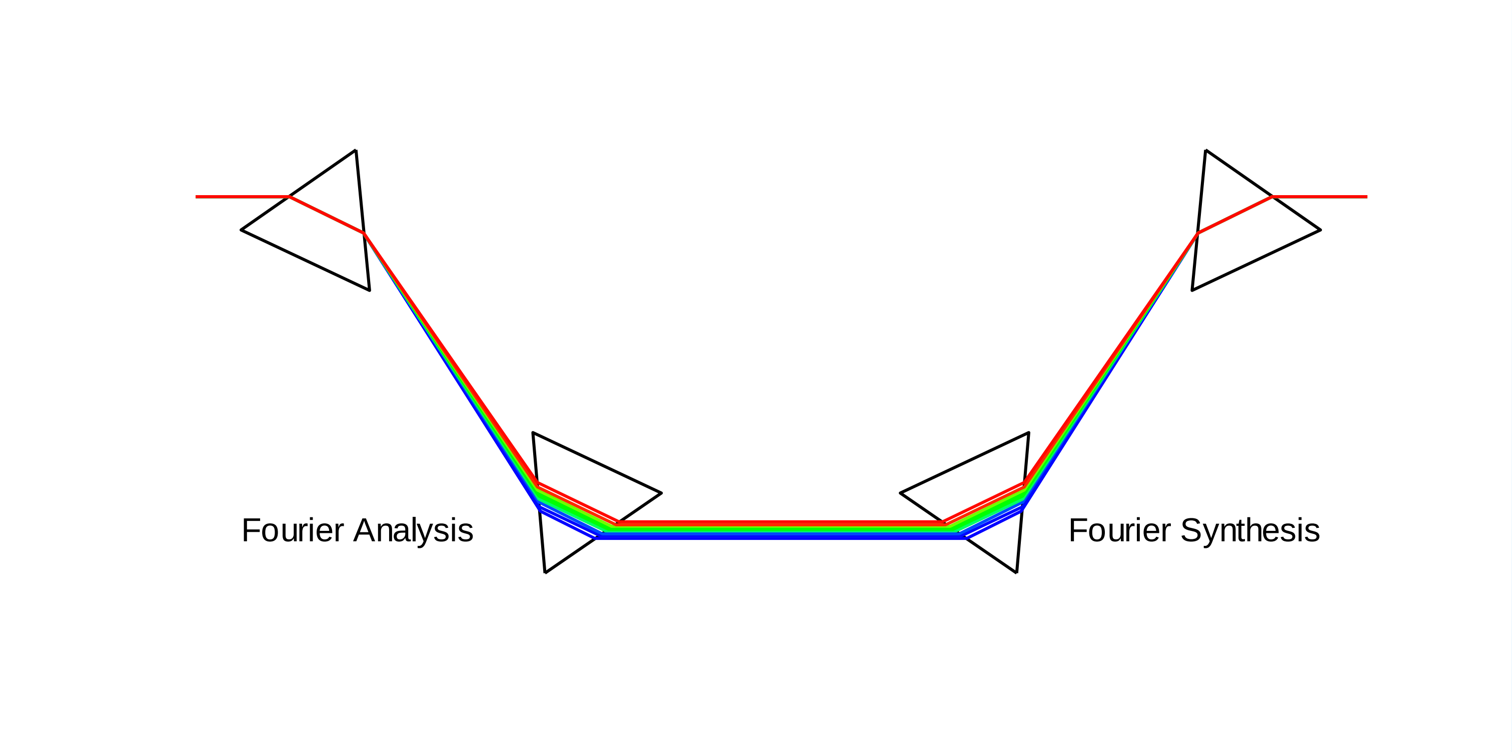

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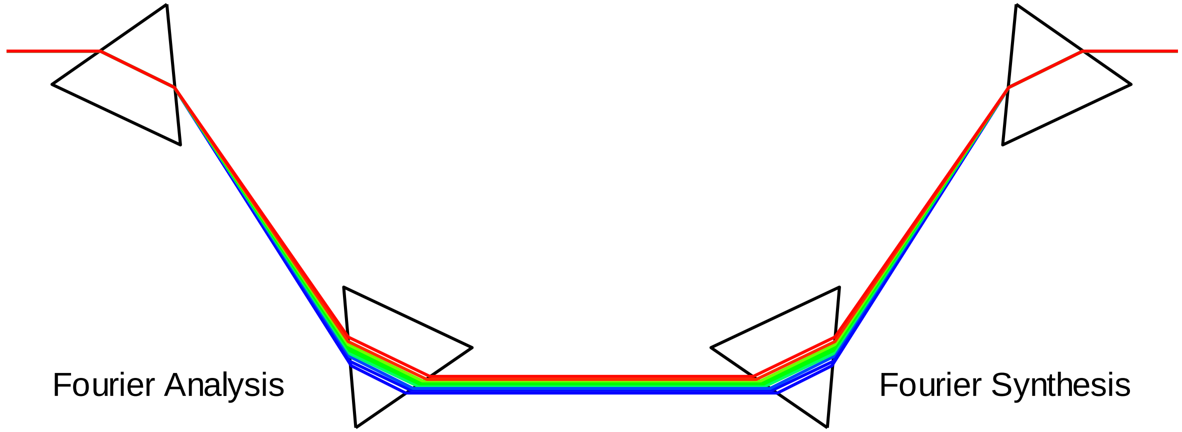

We want to do the same thing with mathematical signals instead of light.

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The formula $S = a + ax + ax^2 + \ldots = \frac{a}{1-x}$ is only valid for $|x| < 1$

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The two dummy variables are both called r but they have no connection with each other at all (or with any other variable called r anywhere else).

Dummy Variable Substitution

We can derive the formula for the geometric series using $\sum$ notation:

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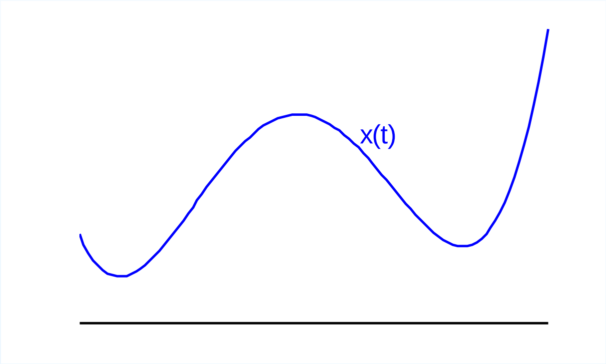
Hence: $S_n = a\frac{1-x^{n+1}}{1-x}$

Averages

If a signal varies with time, we can plot its waveform, $x(t)$.



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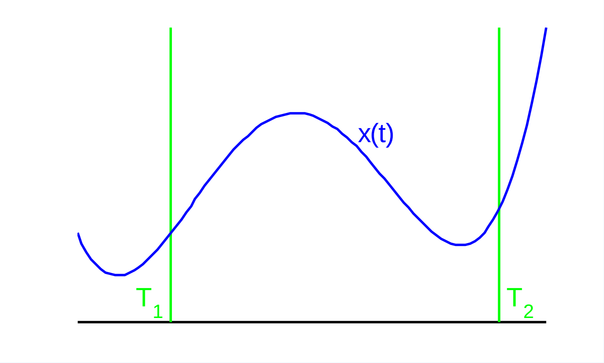

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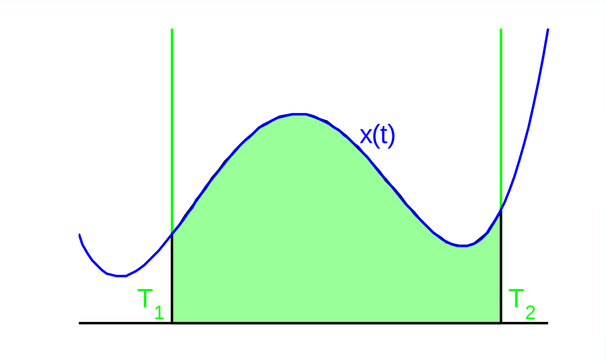

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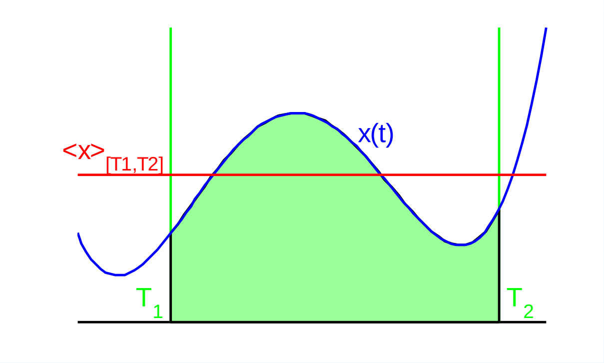

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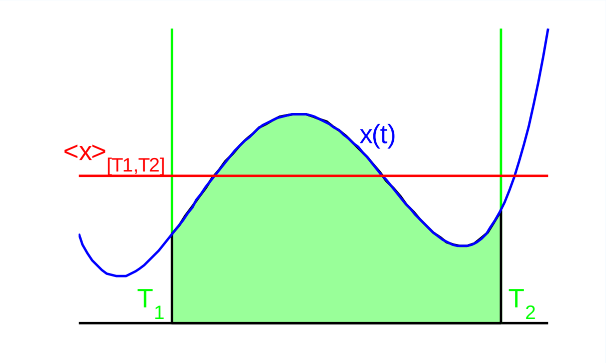

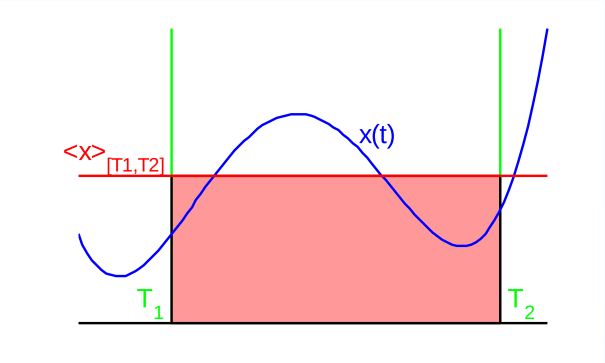

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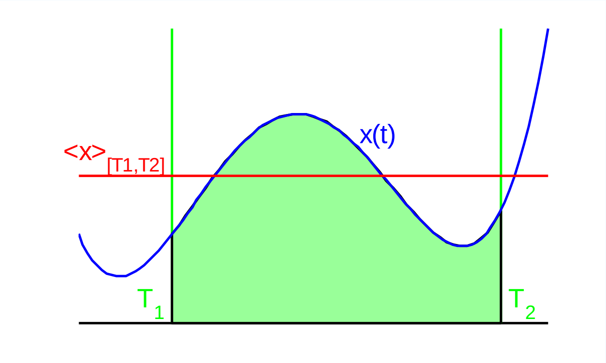

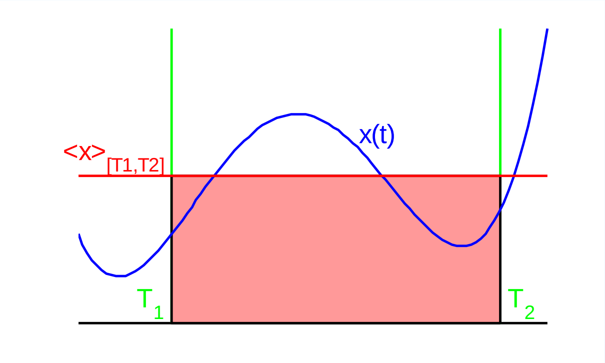

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Angle brackets alone, $\langle x \rangle$, denotes the average value over all time

$$\langle x(t) \rangle = \lim_{A,B \to \infty} \langle x(t) \rangle_{[-A,+B]}$$

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The properties of averages follow from the properties of integrals:

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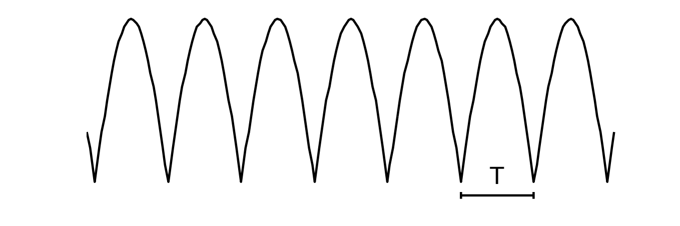
But beware: $\langle x(t) \times y(t) \rangle \neq \langle x(t) \rangle \times \langle y(t) \rangle$.

Periodic Waveforms

A periodic waveform with period T repeats itself at intervals of T:

$$x(t+T) = x(t)$$



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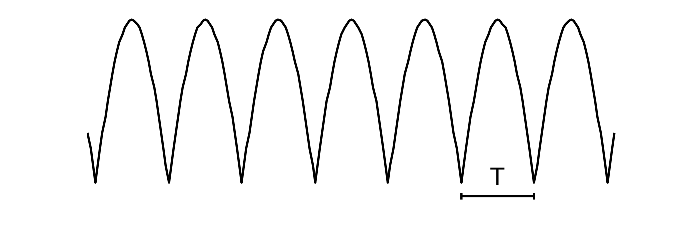

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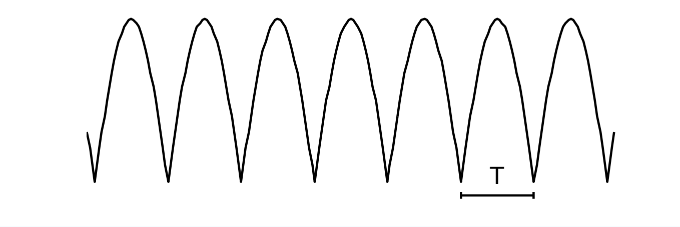

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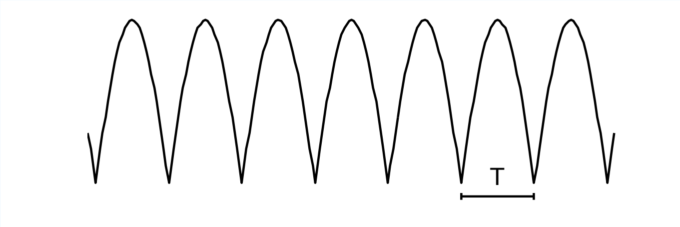

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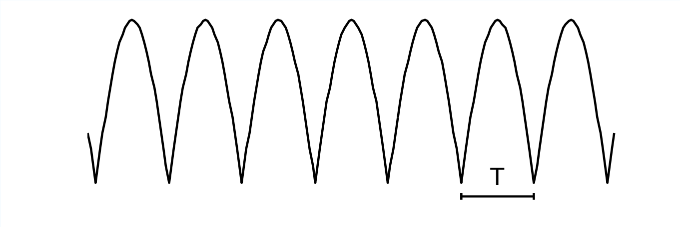

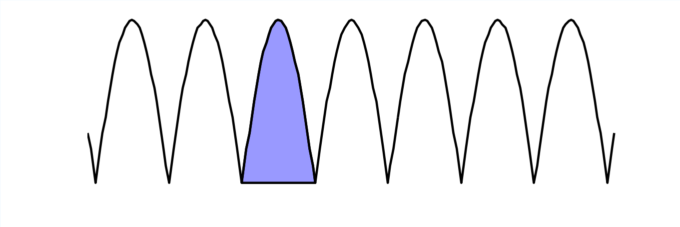

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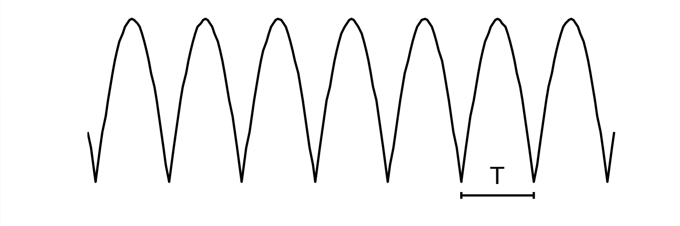

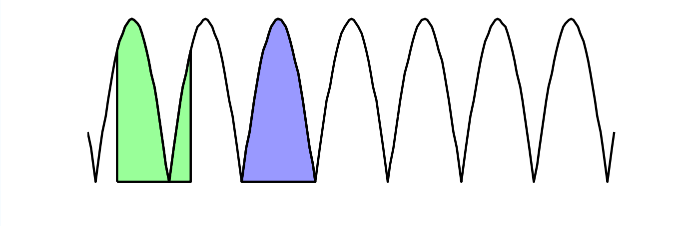

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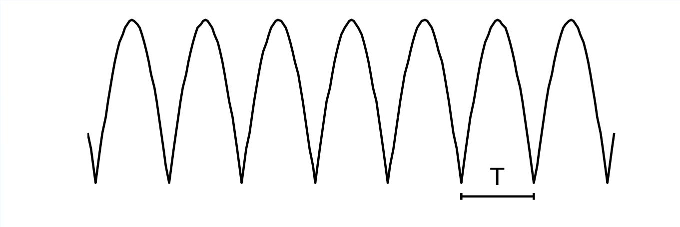

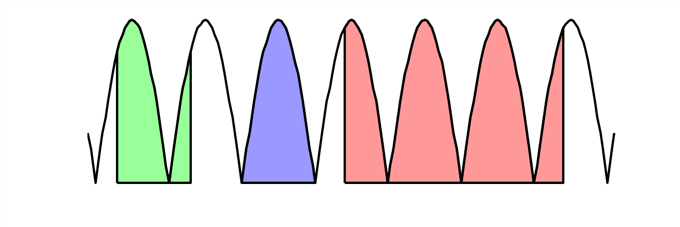

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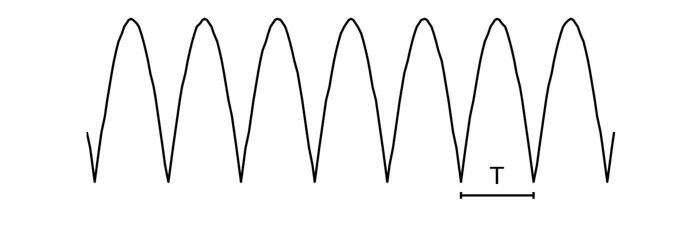

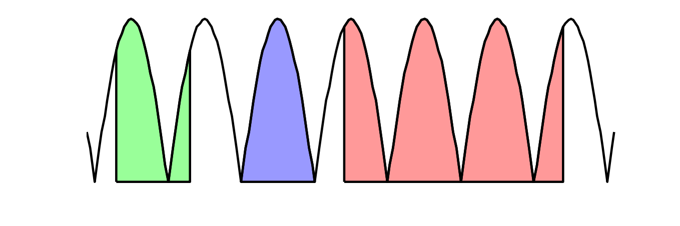

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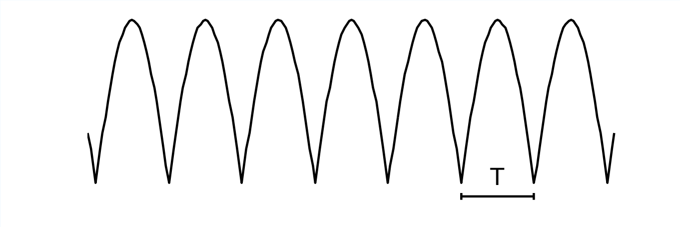

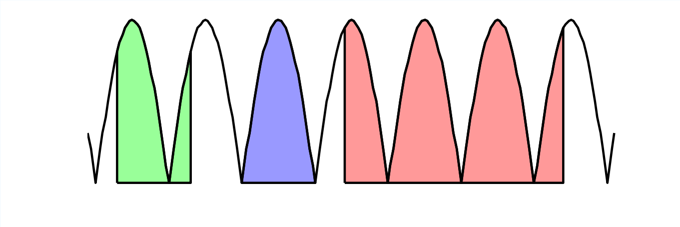

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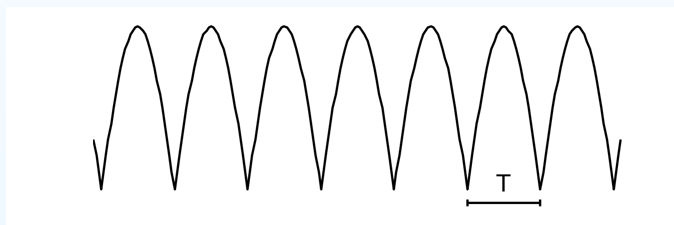

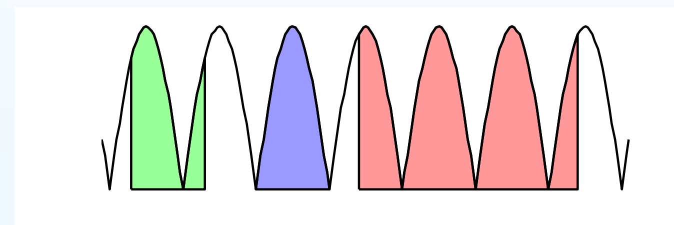

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A periodic waveform with period T repeats itself at intervals of T:

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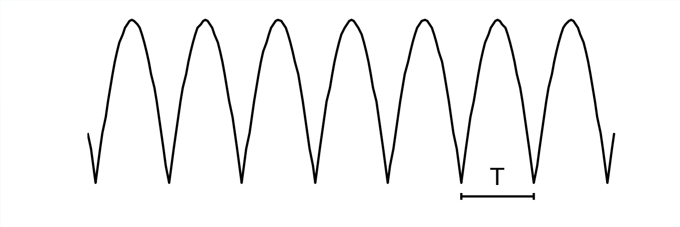

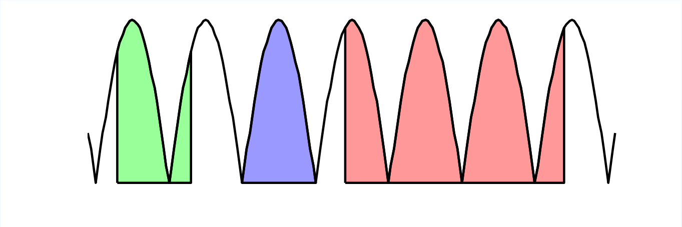

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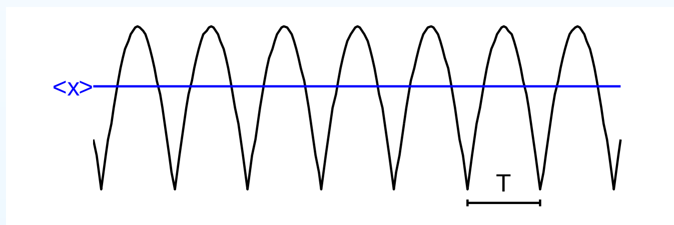

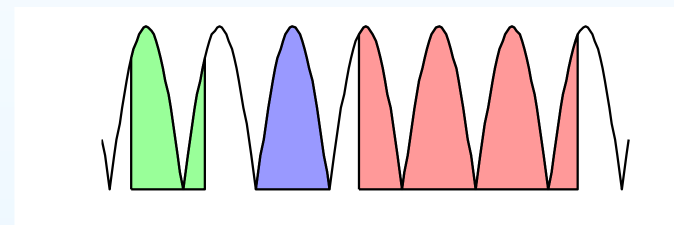

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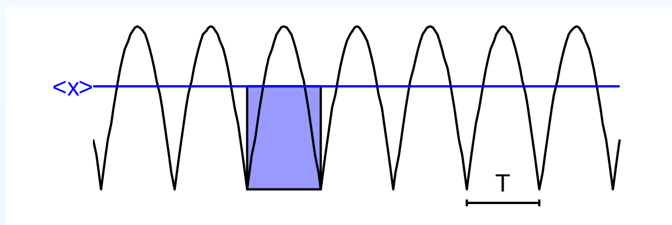

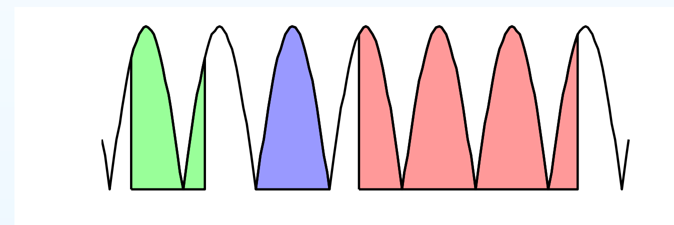

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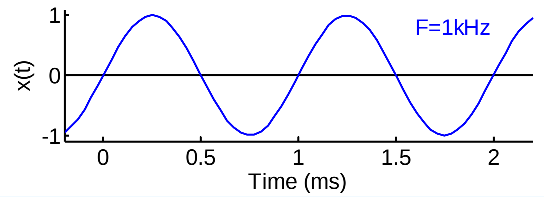

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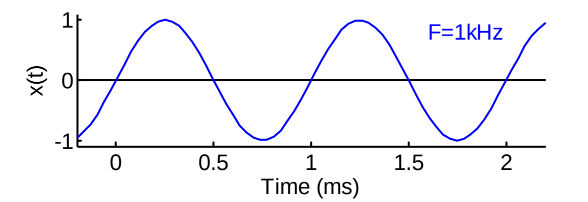

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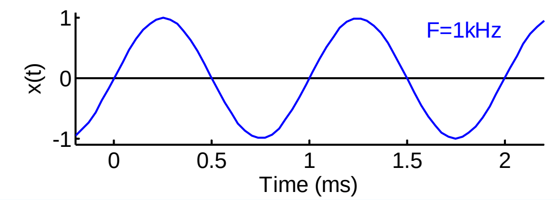

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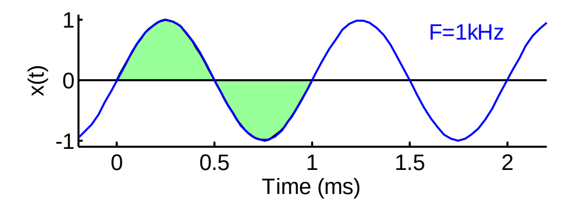

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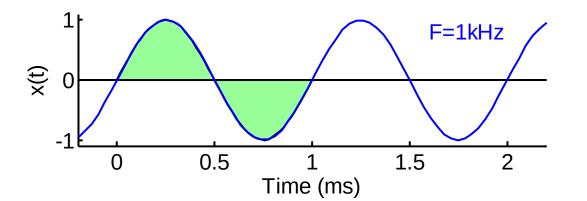

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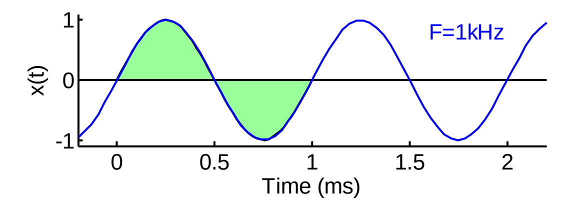

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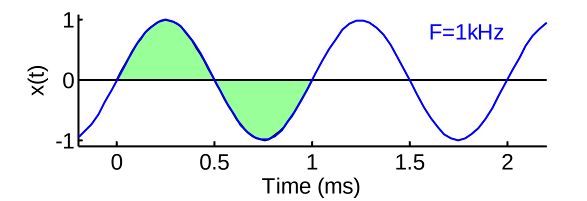

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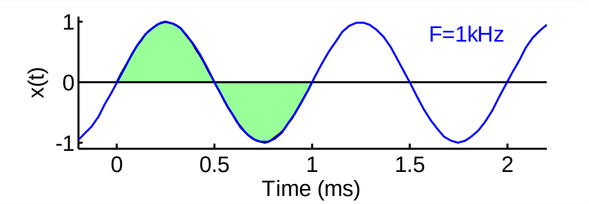

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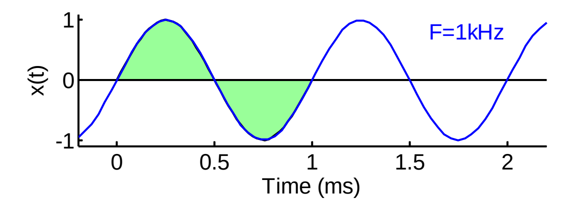

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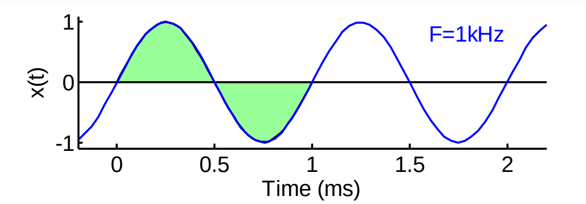

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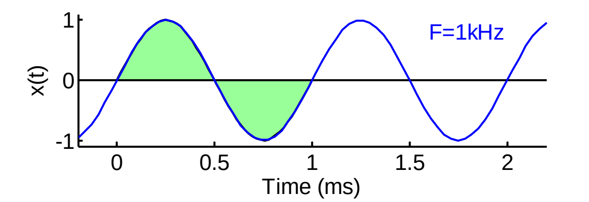

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