2: Fourier Series

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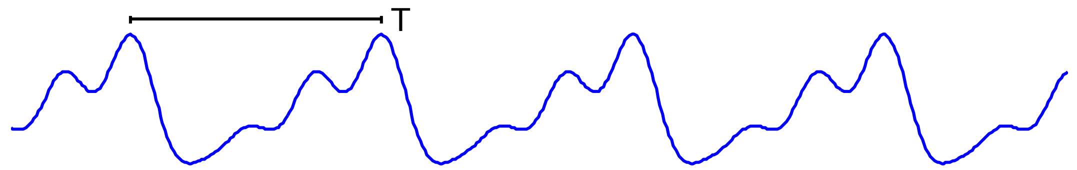

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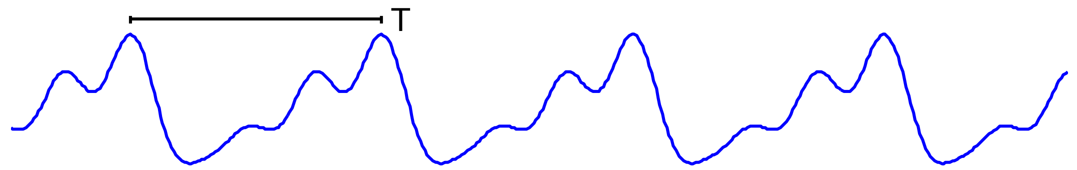

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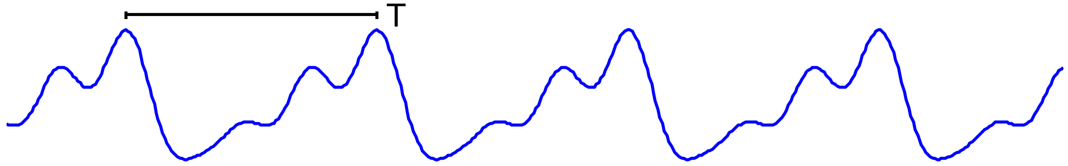

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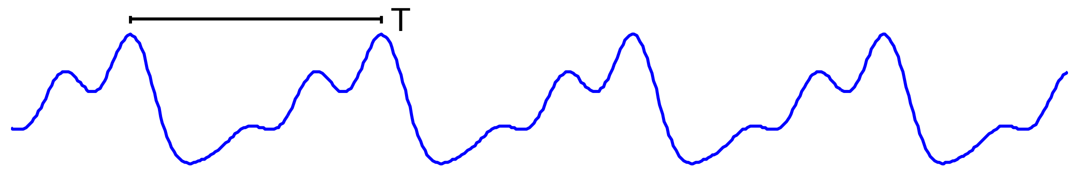

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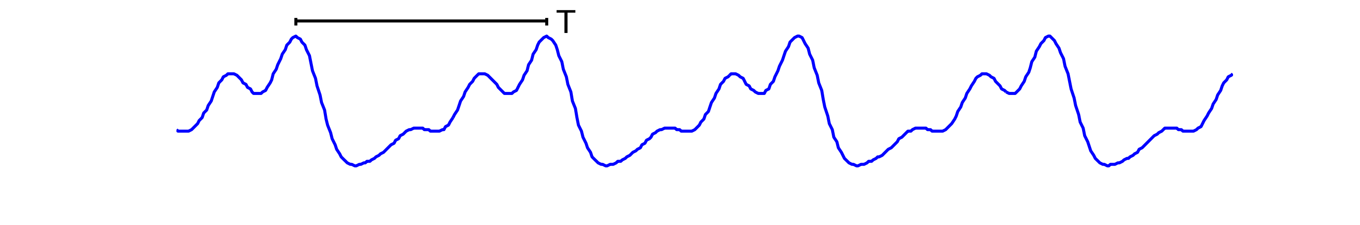

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- $u(t) = \cos 4\pi t \quad \Rightarrow \quad T_u = \frac{2\pi}{4\pi} = 0.5$
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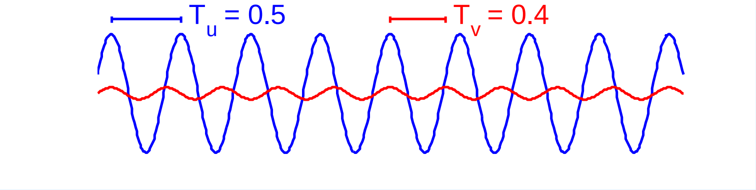

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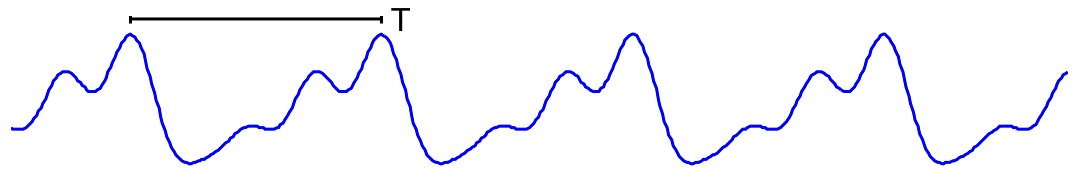

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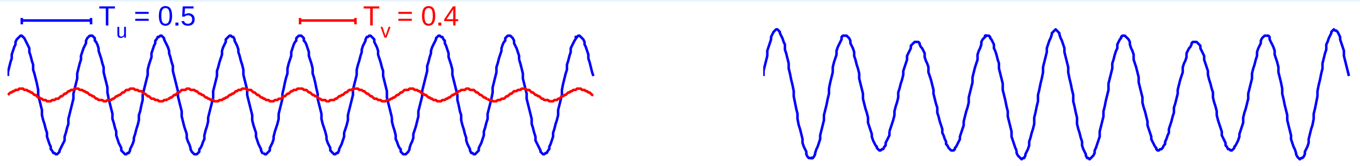

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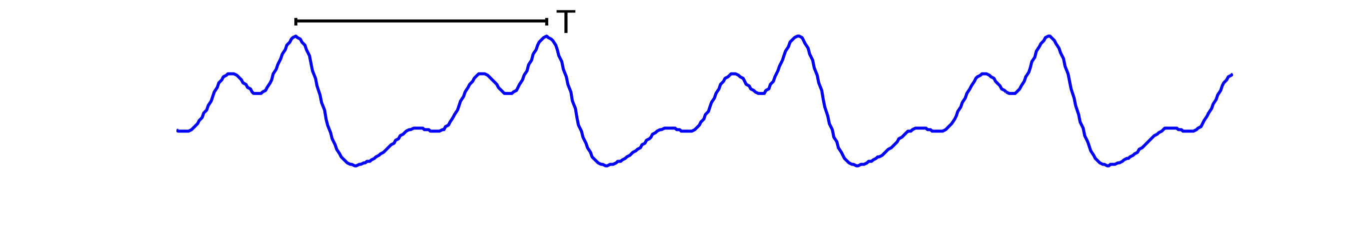

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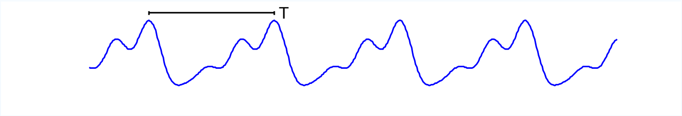
Fourier Series

If $u(t)$ has fundamental period T and fundamental frequency $F = \frac{1}{T}$ then, in most cases, we can express it as a (possibly infinite) sum of sine and cosine waves with frequencies $0, F, 2F, 3F, \cdots$.



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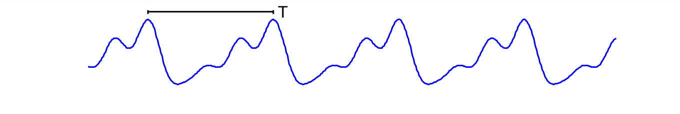

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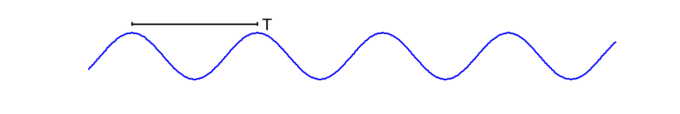

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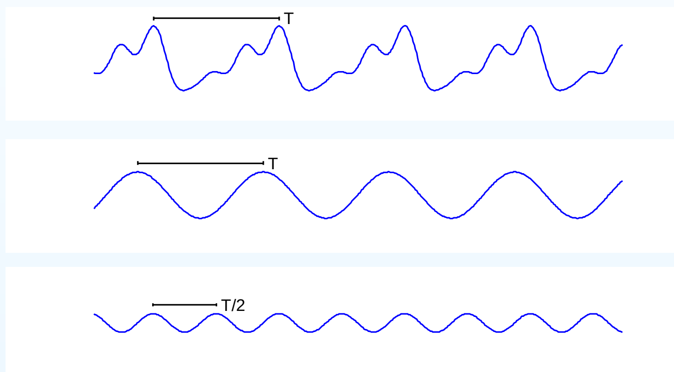

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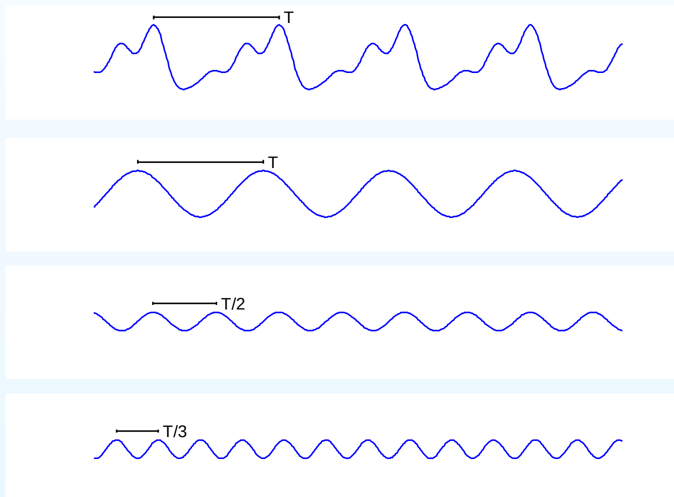

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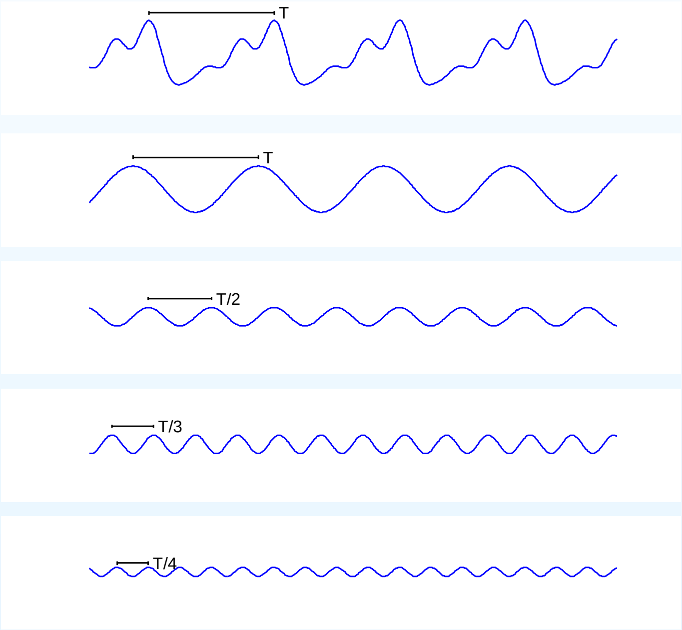

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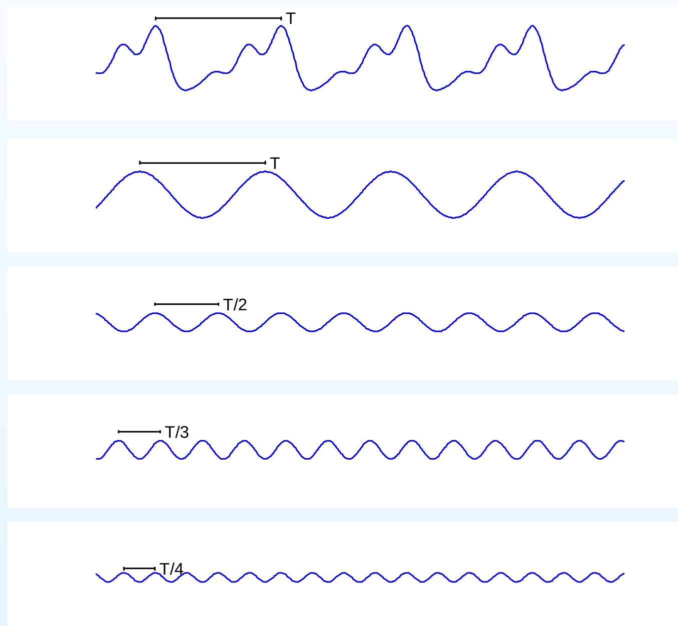

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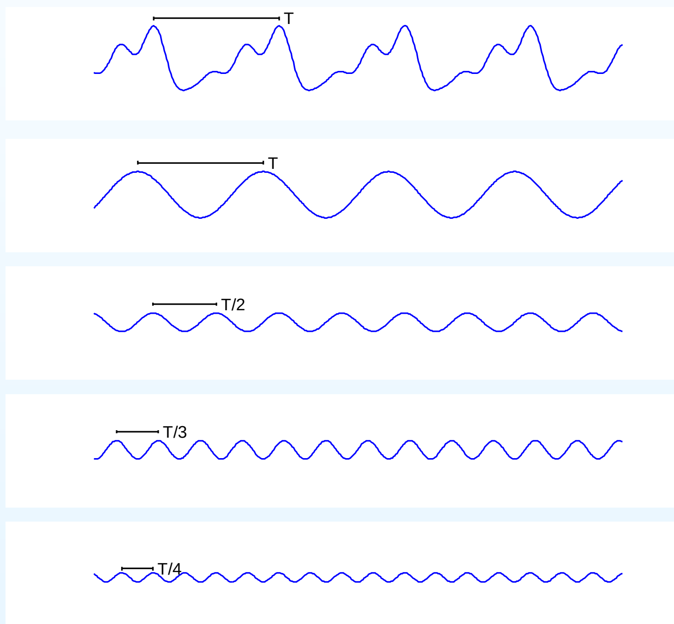

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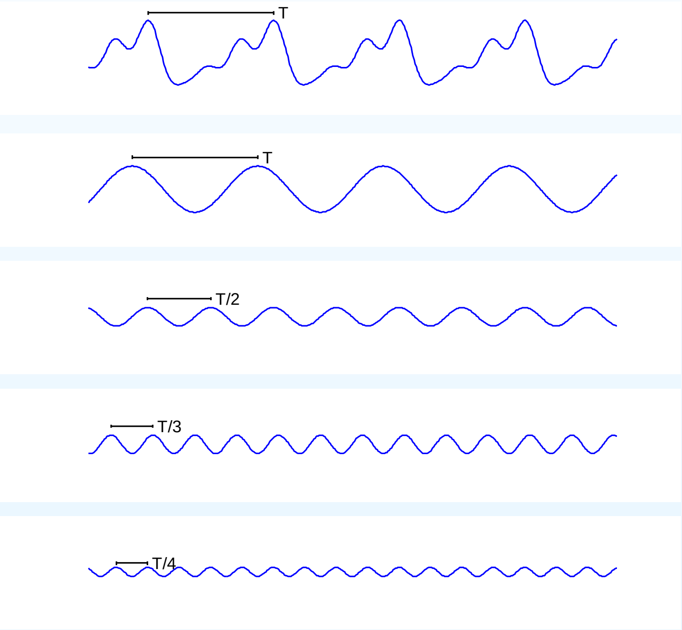

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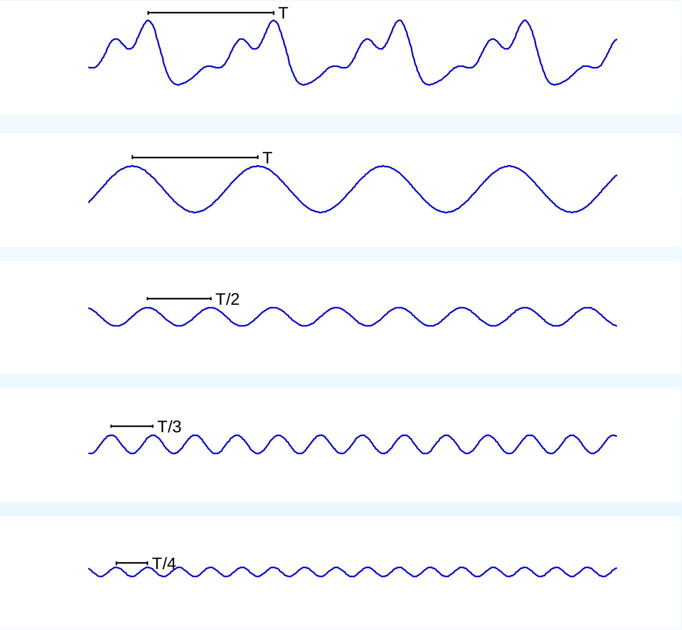

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The n^{th} harmonic of the fundamental is the component at a frequency nF.

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Answer: Because ...

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In these lectures we will use T for the fundamental period and $F = \frac{1}{T}$ for the fundamental frequency.

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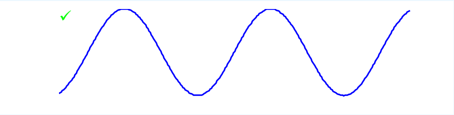

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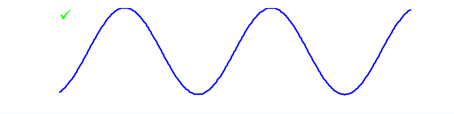

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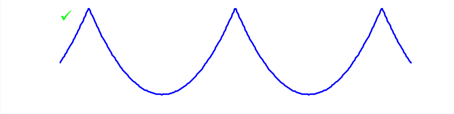

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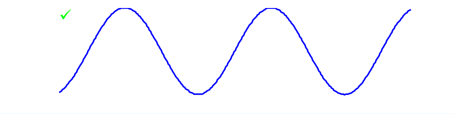

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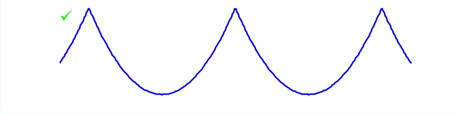

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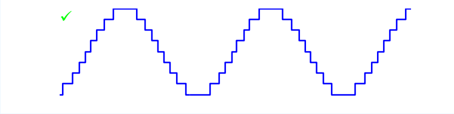

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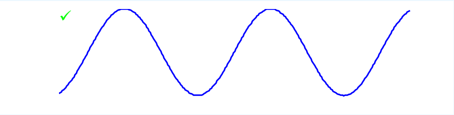

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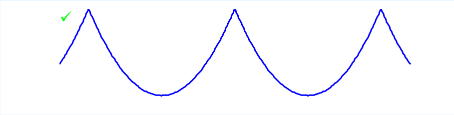

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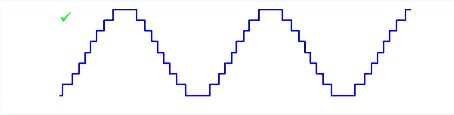

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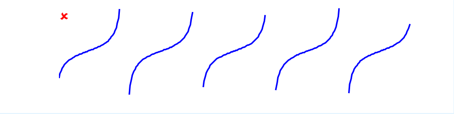

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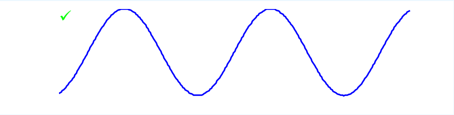

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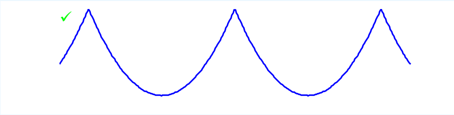

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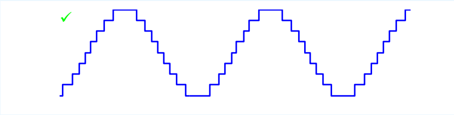

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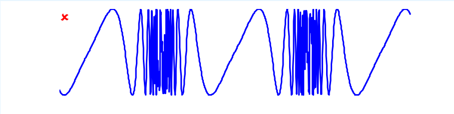

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1805-1859

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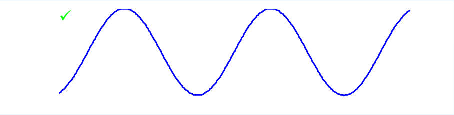
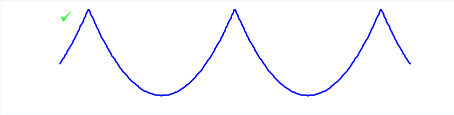
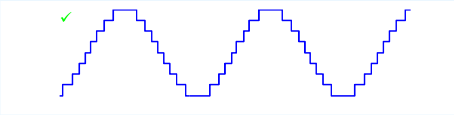


$\sin(t)$

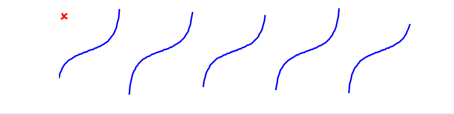


t^2

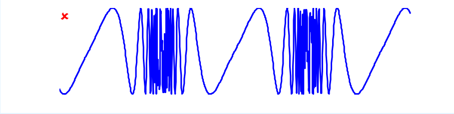


quantized

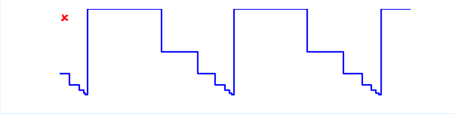
Bad:



$\tan(t)$



$\sin\left(\frac{1}{t}\right)$



∞ halving steps

Fourier Analysis

Suppose that $u(t)$ satisfies the Dirichlet conditions so that

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

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Finding a_n and b_n is called Fourier analysis.

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Proof [a_0]: $2\left\langle u(t)\cos\left(2\pi 0Ft\right)\right\rangle = 2\left\langle u(t)\right\rangle$

Fourier Analysis

Find a_n and b_n in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$

Answer: $a_n = 2 \langle u(t) \cos(2\pi nFt) \rangle \triangleq \frac{2}{T} \int_0^T u(t) \cos(2\pi nFt)\, dt$

$b_n = 2 \langle u(t) \sin(2\pi nFt) \rangle \triangleq \frac{2}{T} \int_0^T u(t) \sin(2\pi nFt)\, dt$

Proof [a_0]: $2 \langle u(t) \cos(2\pi 0Ft) \rangle = 2 \langle u(t) \rangle = 2 \times \frac{a_0}{2} = a_0$

Fourier Analysis

Find a_n and b_n in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$

Answer: $a_n = 2\left\langle u(t)\cos\left(2\pi nFt\right)\right\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos\left(2\pi nFt\right)dt$

$b_n = 2\left\langle u(t)\sin\left(2\pi nFt\right)\right\rangle \triangleq \frac{2}{T}\int_0^T u(t)\sin\left(2\pi nFt\right)dt$

Proof [a_0]: $2\left\langle u(t)\cos\left(2\pi 0Ft\right)\right\rangle = 2\left\langle u(t)\right\rangle = 2\times\frac{a_0}{2} = a_0$

Proof [$a_n,\ n>0$]:

$$2\left\langle u(t)\cos\left(2\pi nFt\right)\right\rangle$$
$$= 2\left\langle \frac{a_0}{2}\cos\left(2\pi nFt\right)\right\rangle + \sum_{r=1}^{\infty} 2\left\langle a_r\cos\left(2\pi rFt\right)\cos\left(2\pi nFt\right)\right\rangle$$
$$+ \sum_{r=1}^{\infty} 2\left\langle b_r\sin\left(2\pi rFt\right)\cos\left(2\pi nFt\right)\right\rangle$$

Fourier Analysis

Find a_n and b_n in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$

Answer: $a_n = 2\left\langle u(t)\cos\left(2\pi nFt\right)\right\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos\left(2\pi nFt\right)dt$

$b_n = 2\left\langle u(t)\sin\left(2\pi nFt\right)\right\rangle \triangleq \frac{2}{T}\int_0^T u(t)\sin\left(2\pi nFt\right)dt$

Proof [a_0]: $2\left\langle u(t)\cos\left(2\pi 0Ft\right)\right\rangle = 2\left\langle u(t)\right\rangle = 2 \times \frac{a_0}{2} = a_0$

Proof [$a_n,\ n > 0$]:

$$2\left\langle u(t)\cos\left(2\pi nFt\right)\right\rangle$$
$$= 2\left\langle \frac{a_0}{2}\cos\left(2\pi nFt\right)\right\rangle + \sum_{r=1}^{\infty} 2\left\langle a_r \cos\left(2\pi rFt\right)\cos\left(2\pi nFt\right)\right\rangle$$
$$+ \sum_{r=1}^{\infty} 2\left\langle b_r \sin\left(2\pi rFt\right)\cos\left(2\pi nFt\right)\right\rangle$$

Term 1: $2\left\langle \frac{a_0}{2}\cos\left(2\pi nFt\right)\right\rangle = 0$

Fourier Analysis

Find a_n and b_n in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$

Answer: $a_n = 2\langle u(t)\cos(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt$

$b_n = 2\langle u(t)\sin(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\sin(2\pi nFt)\,dt$

Proof [a_0]: $2\langle u(t)\cos(2\pi 0Ft)\rangle = 2\langle u(t)\rangle = 2 \times \frac{a_0}{2} = a_0$

Proof [$a_n,\ n > 0$]:

$$\begin{aligned} &2\langle u(t)\cos(2\pi nFt)\rangle \\ &\quad = 2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle + \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle \\ &\qquad + \sum_{r=1}^{\infty} 2\langle b_r \sin(2\pi rFt)\cos(2\pi nFt)\rangle \end{aligned}$$

Term 1: $2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle = 0$

Term 2: $2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = \begin{cases} a_n & r = n \\ 0 & r \neq n \end{cases}$

Fourier Analysis

Find a_n and b_n in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$

Answer: $a_n = 2\langle u(t)\cos(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt$

$b_n = 2\langle u(t)\sin(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\sin(2\pi nFt)\,dt$

Proof [a_0]: $2\langle u(t)\cos(2\pi 0Ft)\rangle = 2\langle u(t)\rangle = 2 \times \frac{a_0}{2} = a_0$

Proof [$a_n,\ n > 0$]:

$$2\langle u(t)\cos(2\pi nFt)\rangle$$
$$= 2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle + \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle$$
$$+ \sum_{r=1}^{\infty} 2\langle b_r \sin(2\pi rFt)\cos(2\pi nFt)\rangle$$

Term 1: $2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle = 0$

Term 2: $2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = \begin{cases} a_n & r = n \\ 0 & r \neq n \end{cases}$

$$\Rightarrow \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = a_n$$

Fourier Analysis

Find a_n and b_n in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$

Answer: $a_n = 2\langle u(t)\cos(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt$

$b_n = 2\langle u(t)\sin(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\sin(2\pi nFt)\,dt$

Proof [a_0]: $2\langle u(t)\cos(2\pi 0Ft)\rangle = 2\langle u(t)\rangle = 2 \times \frac{a_0}{2} = a_0$

Proof [$a_n,\ n > 0$]:

$$\begin{aligned}&2\langle u(t)\cos(2\pi nFt)\rangle\\ &\quad = 2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle + \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle\\ &\qquad + \sum_{r=1}^{\infty} 2\langle b_r \sin(2\pi rFt)\cos(2\pi nFt)\rangle\end{aligned}$$

Term 1: $2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle = 0$

Term 2: $2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = \begin{cases} a_n & r = n \\ 0 & r \neq n \end{cases}$

$\Rightarrow \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = a_n$

Term 3: $2\langle b_r \sin 2\pi rFt \cos(2\pi nFt)\rangle = 0$

Fourier Analysis

Find a_n and b_n in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$

Answer: $a_n = 2\langle u(t)\cos(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt$

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Proof [a_0]: $2\langle u(t)\cos(2\pi 0Ft)\rangle = 2\langle u(t)\rangle = 2 \times \frac{a_0}{2} = a_0$

Proof [$a_n,\ n > 0$]:

$$2\langle u(t)\cos(2\pi nFt)\rangle$$
$$= 2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle + \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle$$
$$+ \sum_{r=1}^{\infty} 2\langle b_r \sin(2\pi rFt)\cos(2\pi nFt)\rangle$$

Term 1: $2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle = 0$

Term 2: $2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = \begin{cases} a_n & r = n \\ 0 & r \neq n \end{cases}$

$$\Rightarrow \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = a_n$$

Term 3: $2\langle b_r \sin 2\pi rFt \cos(2\pi nFt)\rangle = 0$

Proof [$b_n,\ n > 0$]: same method as for a_n

Fourier Analysis Example

Fourier Series:

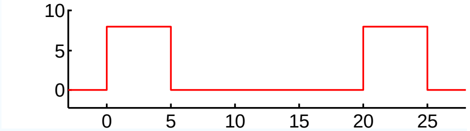
$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

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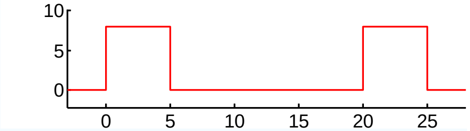


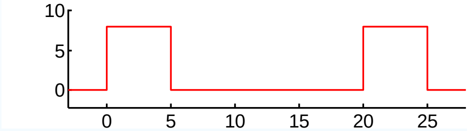
$$a_n = \frac{2}{T} \int_0^T u(t) \cos \frac{2\pi nt}{T} dt$$

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



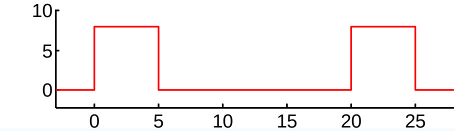
$$a_n = \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt$$
$$= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt$$

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



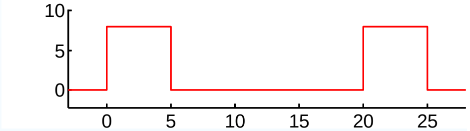
$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \end{aligned}$$

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} \end{aligned}$$

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

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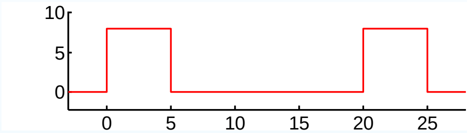


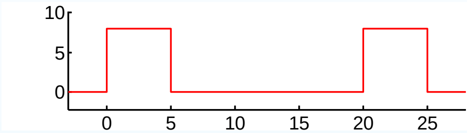
$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T u(t) \cos \frac{2\pi nt}{T} dt \\ &= \frac{2}{T} \int_0^W A \cos \frac{2\pi nt}{T} dt \\ &= \frac{2AT}{2\pi nT} \left[\sin \frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi} \sin \frac{2\pi nW}{T} = \frac{A}{n\pi} \sin \frac{n\pi}{2} \end{aligned}$$

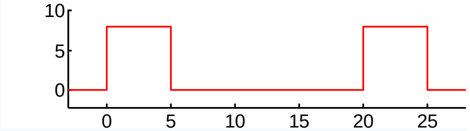
n	0	1	2	3	4	5	6
a_n		$\frac{8}{\pi}$					
b_n							

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

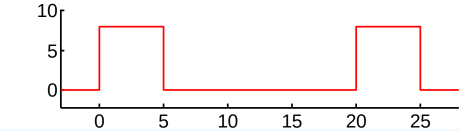
n	0	1	2	3	4	5	6
a_n		$\frac{8}{\pi}$	0				
b_n							

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

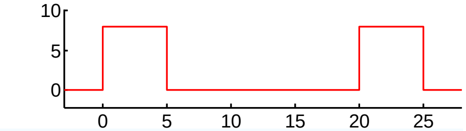
n	0	1	2	3	4	5	6
a_n		$\frac{8}{\pi}$	0	$\frac{-8}{3\pi}$			
b_n							

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

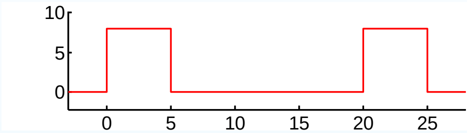
n	0	1	2	3	4	5	6
a_n		$\frac{8}{\pi}$	0	$\frac{-8}{3\pi}$	0		
b_n							

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

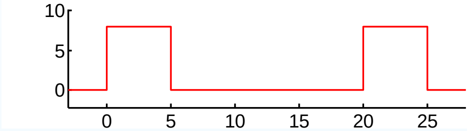
n	0	1	2	3	4	5	6
a_n		$\frac{8}{\pi}$	0	$\frac{-8}{3\pi}$	0	$\frac{8}{5\pi}$	
b_n							

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

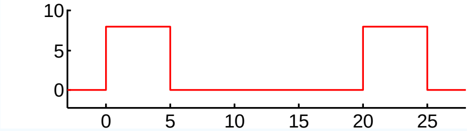
n	0	1	2	3	4	5	6
a_n		$\frac{8}{\pi}$	0	$\frac{-8}{3\pi}$	0	$\frac{8}{5\pi}$	0
b_n							

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

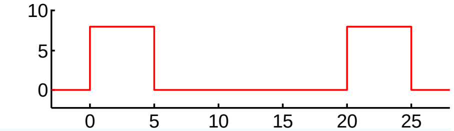
n	0	1	2	3	4	5	6
a_n	4	$\frac{8}{\pi}$	0	$\frac{-8}{3\pi}$	0	$\frac{8}{5\pi}$	0
b_n							

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$



$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

$$b_n = \frac{2}{T}\int_0^T u(t)\sin\frac{2\pi nt}{T}dt$$

n	0	1	2	3	4	5	6
a_n	4	$\frac{8}{\pi}$	0	$\frac{-8}{3\pi}$	0	$\frac{8}{5\pi}$	0
b_n							

Fourier Analysis Example

Fourier Series:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt \right)$$

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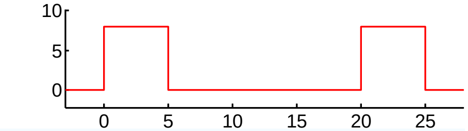

$$a_n = \frac{2}{T} \int_0^T u(t) \cos \frac{2\pi nt}{T} dt$$
$$= \frac{2}{T} \int_0^W A \cos \frac{2\pi nt}{T} dt$$
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$$b_n = \frac{2}{T} \int_0^T u(t) \sin \frac{2\pi nt}{T} dt$$
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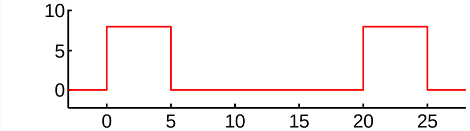

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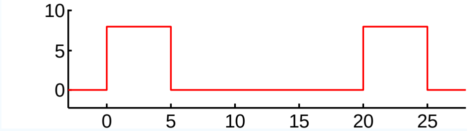

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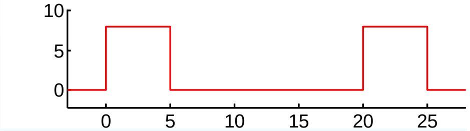

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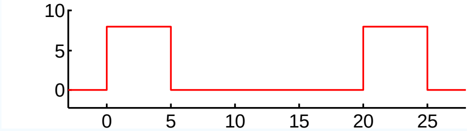

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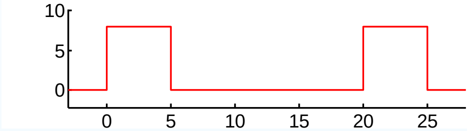

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b_n		$\frac{8}{\pi}$	$\frac{16}{2\pi}$	$\frac{8}{3\pi}$	0		

Fourier Analysis Example

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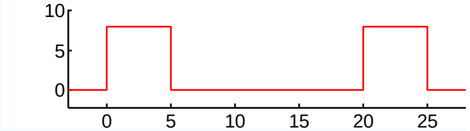

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Fourier Analysis Example

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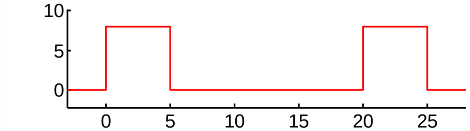

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Fourier Analysis Example

Truncated Series:

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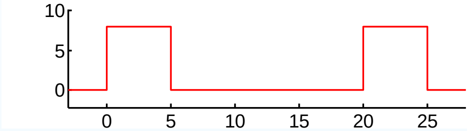

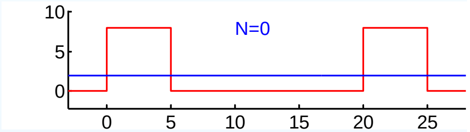

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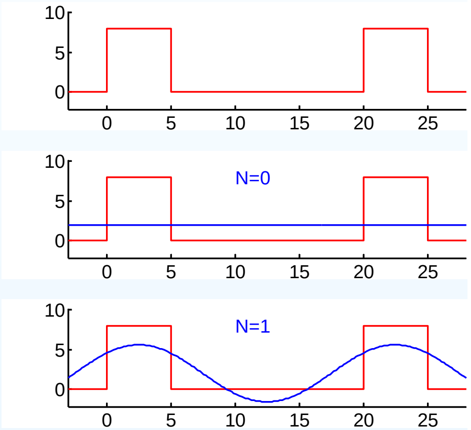

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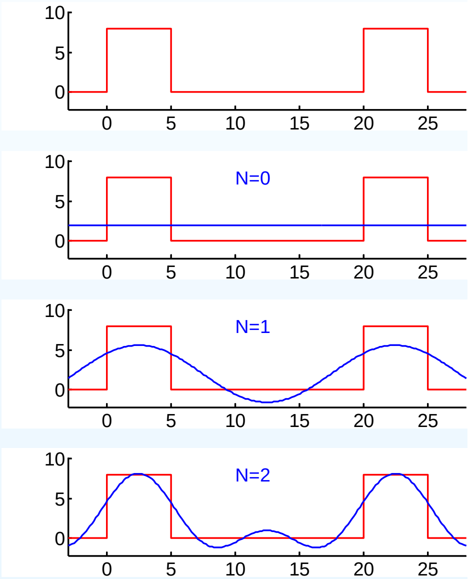

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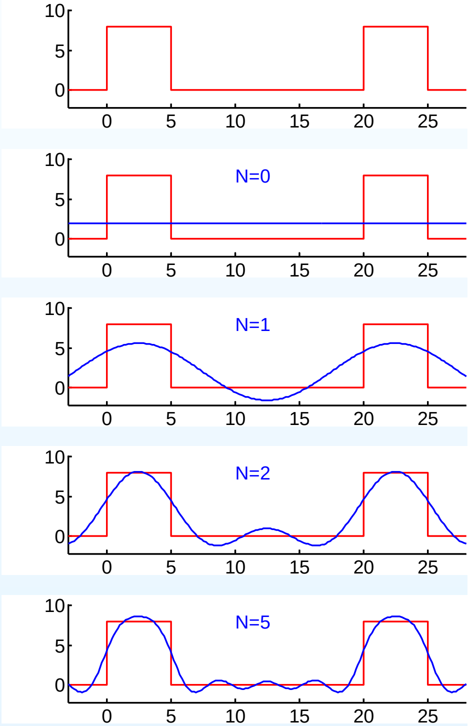

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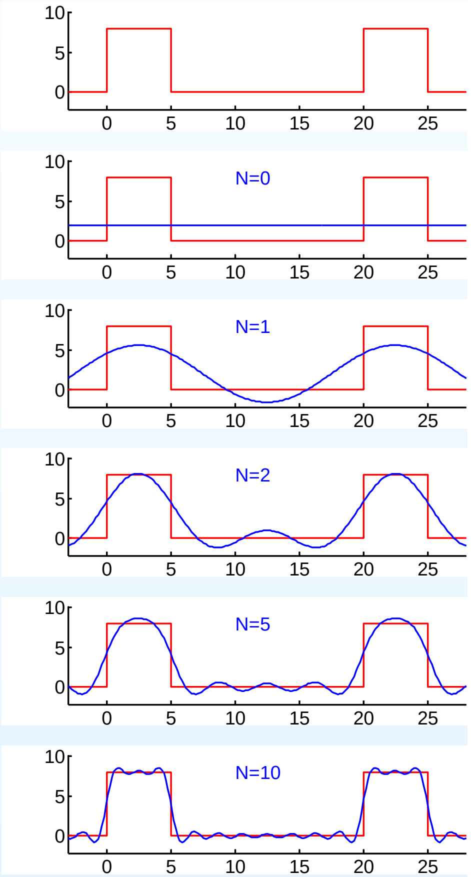

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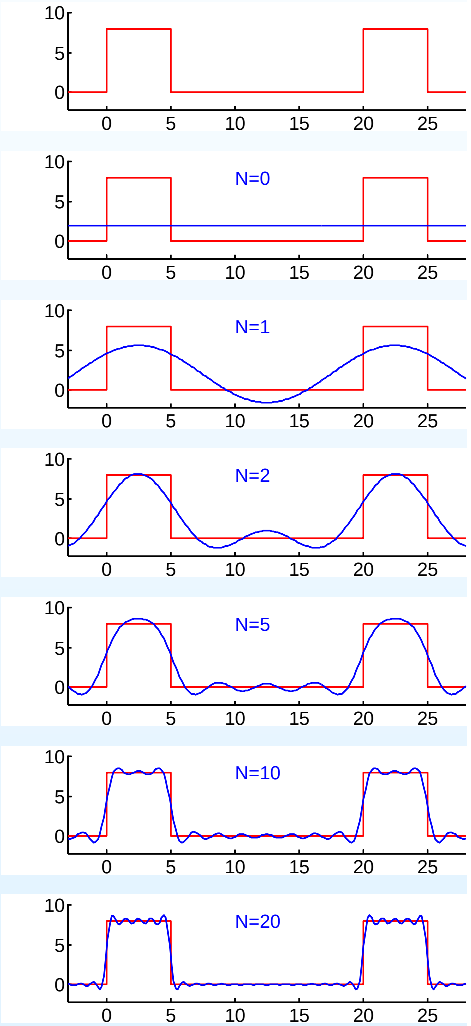

Linearity

Fourier analysis maps a function of time onto a set of Fourier coefficients:

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For further details see RHB 12.1 and 12.2.