3: Complex Fourier Series

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Examples where using $e^{i\theta}$ makes things simpler:

Using $e^{i\theta}$	Using $\cos\theta$ and $\sin\theta$
$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$	

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$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$	$\cos\theta\cos\phi = \frac{1}{2}\cos(\theta+\phi) + \frac{1}{2}\cos(\theta-\phi)$
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The Complex Fourier Series is the Fourier Series but written using $e^{i\theta}$

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Complex Fourier Series

Fourier Series: $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$

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$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n\left(\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}\right) + b_n\left(-\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}\right)\right)$$

$[\theta = 2\pi nFt]$

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$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}a_n - \frac{1}{2}ib_n\right) e^{i2\pi nFt}\right) \quad [\theta = 2\pi nFt]$$

$$+ \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}a_n + \frac{1}{2}ib_n\right) e^{-i2\pi nFt}\right)$$

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$$
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u(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \left(\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}\right) + b_n \left(-\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}\right)\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}a_n - \frac{1}{2}ib_n\right) e^{i2\pi nFt}\right) \qquad [\theta = 2\pi nFt] \\
&\qquad + \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}a_n + \frac{1}{2}ib_n\right) e^{-i2\pi nFt}\right) \\
&= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}
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where

$$U_n = \begin{cases} \frac{1}{2}a_n - \frac{1}{2}ib_n & n \geq 1 \\ \frac{1}{2}a_0 & n = 0 \\ \frac{1}{2}a_{|n|} + \frac{1}{2}ib_{|n|} & n \leq -1 \end{cases}$$

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where $[b_0 \triangleq 0]$

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Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$ [simpler ☺]

Averaging Complex Exponentials

If $x(t)$ has period $\frac{T}{n}$ for some integer n (i.e. frequency $\frac{n}{T} = nF$):

$$\langle x(t) \rangle \triangleq \frac{1}{T} \int_{t=0}^{T} x(t)dt$$

This is the average over an integer number of cycles.

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Hence:

$$\left\langle e^{i2\pi nFt} \right\rangle = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

[☺]

Complex Fourier Analysis

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To find the coefficient, U_n, we multiply by something that makes all the terms involving the other coefficients average to zero.

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To find the coefficient, U_n, we multiply by something that makes all the terms involving the other coefficients average to zero.

$$\left\langle u(t)e^{-i2\pi nFt} \right\rangle = \left\langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi rFt} e^{-i2\pi nFt} \right\rangle$$

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All terms in the sum are zero, except for the one when $n = r$ which equals U_n:

$$U_n = \left\langle u(t)e^{-i2\pi nFt}\right\rangle$$

[☺]

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To find the coefficient, U_n, we multiply by something that makes all the terms involving the other coefficients average to zero.

$$\left\langle u(t)e^{-i2\pi nFt}\right\rangle = \left\langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi rFt} e^{-i2\pi nFt}\right\rangle$$

$$= \left\langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi (r-n)Ft}\right\rangle$$

$$= \sum_{r=-\infty}^{\infty} U_r \left\langle e^{i2\pi (r-n)Ft}\right\rangle$$

All terms in the sum are zero, except for the one when $n = r$ which equals U_n:

$$U_n = \left\langle u(t)e^{-i2\pi nFt}\right\rangle$$ [☺]

This shows that the Fourier series coefficients are unique: you cannot have two different sets of coefficients that result in the same function $u(t)$.

Complex Fourier Analysis

Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

To find the coefficient, U_n, we multiply by something that makes all the terms involving the other coefficients average to zero.

$$\begin{aligned}\left\langle u(t)e^{-i2\pi nFt}\right\rangle &= \left\langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi rFt} e^{-i2\pi nFt}\right\rangle \\ &= \left\langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi (r-n)Ft}\right\rangle \\ &= \sum_{r=-\infty}^{\infty} U_r \left\langle e^{i2\pi (r-n)Ft}\right\rangle\end{aligned}$$

All terms in the sum are zero, except for the one when $n = r$ which equals U_n:

$$U_n = \left\langle u(t)e^{-i2\pi nFt}\right\rangle$$ [☺]

This shows that the Fourier series coefficients are unique: you cannot have two different sets of coefficients that result in the same function $u(t)$.

Note the sign of the exponent: "$+$" in the Fourier Series but "$-$" for Fourier Analysis (in order to cancel out the "$+$").

Fourier Series $\leftrightarrow$ Complex Fourier Series

$$
\begin{aligned}
u(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right) \\
&= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}
\end{aligned}
$$

Fourier Series $\leftrightarrow$ Complex Fourier Series

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$
$$= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

We can easily convert between the two forms.

Fourier Series $\leftrightarrow$ Complex Fourier Series

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$
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We can easily convert between the two forms.

Fourier Coefficients $\rightarrow$ Complex Fourier Coefficients:

$$U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp i b_{|n|}\right)$$

Fourier Series $\leftrightarrow$ Complex Fourier Series

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$
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Fourier Coefficients $\rightarrow$ Complex Fourier Coefficients:

$$U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp i b_{|n|}\right) \qquad [U_n = U_{-n}^*]$$

Fourier Series $\leftrightarrow$ Complex Fourier Series

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$
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Fourier Coefficients $\rightarrow$ Complex Fourier Coefficients:

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Complex Fourier Coefficients $\rightarrow$ Fourier Coefficients:

$$a_n = U_n + U_{-n} = 2\Re\left(U_n\right) \qquad [\Re = \text{“real part”}]$$

Fourier Series $\leftrightarrow$ Complex Fourier Series

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$$
$$= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

We can easily convert between the two forms.

Fourier Coefficients $\rightarrow$ Complex Fourier Coefficients:

$$U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp i b_{|n|}\right) \qquad [U_n = U_{-n}^*]$$

Complex Fourier Coefficients $\rightarrow$ Fourier Coefficients:

$$a_n = U_n + U_{-n} = 2\Re\left(U_n\right) \qquad [\Re = \text{“real part”}]$$
$$b_n = i\left(U_n - U_{-n}\right) = -2\Im\left(U_n\right) \qquad [\Im = \text{“imaginary part”}]$$

Fourier Series ↔ Complex Fourier Series

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$
$$= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

We can easily convert between the two forms.

Fourier Coefficients → Complex Fourier Coefficients:

$$U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp i b_{|n|}\right) \qquad [U_n = U^*_{-n}]$$

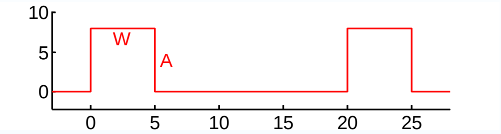
Complex Fourier Coefficients → Fourier Coefficients:

$$a_n = U_n + U_{-n} = 2\Re\left(U_n\right) \qquad [\Re = \text{“real part”}]$$
$$b_n = i\left(U_n - U_{-n}\right) = -2\Im\left(U_n\right) \qquad [\Im = \text{“imaginary part”}]$$

The formula for a_n works even for $n = 0$.

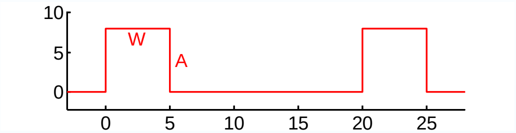
Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Complex Fourier Analysis Example

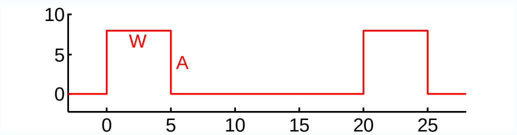
$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



n	a_n	b_n	U_n
-6			
-5			
-4			
-3			
-2			
-1			
0	4		
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	
2	0	$\frac{16}{2\pi}$	
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	
4	0	0	
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	
6	0	$\frac{16}{6\pi}$	

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



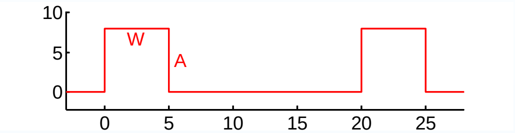
Method 1:

$$U_{\pm n} = \tfrac{1}{2} a_n \mp i \tfrac{1}{2} b_n$$

n	a_n	b_n	U_n
-6			
-5			
-4			
-3			
-2			
-1			
0	4		
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	
2	0	$\frac{16}{2\pi}$	
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	
4	0	0	
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	
6	0	$\frac{16}{6\pi}$	

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



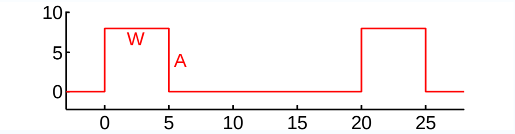
Method 1:

$U_{\pm n} = \frac{1}{2}a_n \mp i\frac{1}{2}b_n$

n	a_n	b_n	U_n
-6			
-5			
-4			
-3			
-2			
-1			
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	
2	0	$\frac{16}{2\pi}$	
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	
4	0	0	
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	
6	0	$\frac{16}{6\pi}$	

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



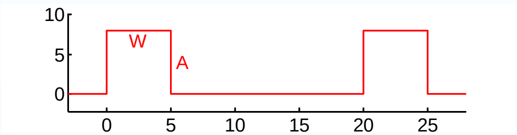
Method 1:

$U_{\pm n} = \frac{1}{2}a_n \mp i\frac{1}{2}b_n$

n	a_n	b_n	U_n
-6			
-5			
-4			
-3			
-2			
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	
4	0	0	
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	
6	0	$\frac{16}{6\pi}$	

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



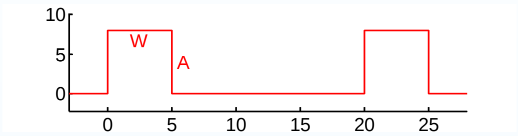
Method 1:

$$U_{\pm n} = \tfrac{1}{2}a_n \mp i\tfrac{1}{2}b_n$$

n	a_n	b_n	U_n
-6			
-5			
-4			
-3			
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	
4	0	0	
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	
6	0	$\frac{16}{6\pi}$	

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



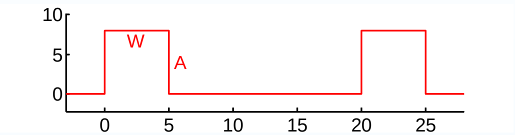
Method 1:

$$U_{\pm n} = \tfrac{1}{2}a_n \mp i\tfrac{1}{2}b_n$$

n	a_n	b_n	U_n
-6			
-5			
-4			
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	
6	0	$\frac{16}{6\pi}$	

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Method 1:

$U_{\pm n} = \frac{1}{2}a_n \mp i\frac{1}{2}b_n$

n	a_n	b_n	U_n
-6			
-5			
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	
6	0	$\frac{16}{6\pi}$	

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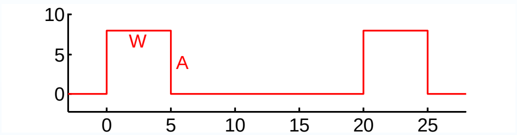


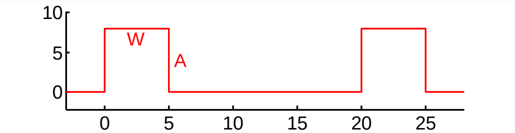
Method 1:

$$U_{\pm n} = \frac{1}{2}a_n \mp i\frac{1}{2}b_n$$

n	a_n	b_n	U_n
-6			
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
6	0	$\frac{16}{6\pi}$	

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



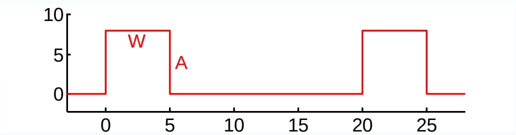
Method 1:

$U_{\pm n} = \frac{1}{2}a_n \mp i\frac{1}{2}b_n$

n	a_n	b_n	U_n
-6			$i\frac{8}{6\pi}$
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
6	0	$\frac{16}{6\pi}$	$i\frac{-8}{6\pi}$

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Method 1:

$$U_{\pm n} = \tfrac{1}{2}a_n \mp i\tfrac{1}{2}b_n$$

Method 2:

$$U_n = \left\langle u(t)e^{-i2\pi nFt} \right\rangle$$

n	a_n	b_n	U_n
-6			$i\frac{8}{6\pi}$
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
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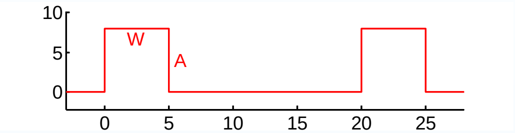
Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Method 1:

$$U_{\pm n} = \tfrac{1}{2}a_n \mp i\tfrac{1}{2}b_n$$

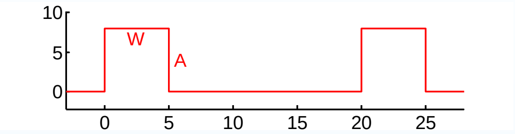
Method 2:

$$U_n = \left\langle u(t)e^{-i2\pi nFt}\right\rangle$$
$$= \tfrac{1}{T}\int_0^T u(t)e^{-i2\pi nFt}dt$$

n	a_n	b_n	U_n
-6			$i\frac{8}{6\pi}$
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
6	0	$\frac{16}{6\pi}$	$i\frac{-8}{6\pi}$

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



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$$U_{\pm n} = \tfrac{1}{2}a_n \mp i\tfrac{1}{2}b_n$$

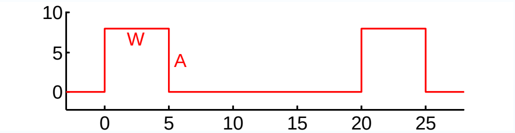
Method 2:

$$U_n = \left\langle u(t)e^{-i2\pi nFt}\right\rangle$$
$$= \tfrac{1}{T}\int_0^T u(t)e^{-i2\pi nFt}dt$$
$$= \tfrac{1}{T}\int_0^W Ae^{-i2\pi nFt}dt$$

n	a_n	b_n	U_n
-6			$i\frac{8}{6\pi}$
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
6	0	$\frac{16}{6\pi}$	$i\frac{-8}{6\pi}$

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Method 1:

$$U_{\pm n} = \frac{1}{2}a_n \mp i\frac{1}{2}b_n$$

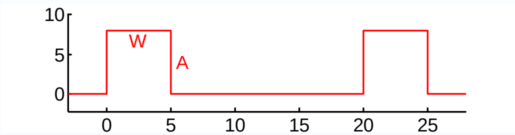
Method 2:

$$U_n = \left\langle u(t)e^{-i2\pi nFt}\right\rangle$$
$$= \frac{1}{T}\int_0^T u(t)e^{-i2\pi nFt}dt$$
$$= \frac{1}{T}\int_0^W Ae^{-i2\pi nFt}dt$$
$$= \frac{A}{-i2\pi nFT}\left[e^{-i2\pi nFt}\right]_0^W$$

n	a_n	b_n	U_n
-6			$i\frac{8}{6\pi}$
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
6	0	$\frac{16}{6\pi}$	$i\frac{-8}{6\pi}$

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Method 1:

$$U_{\pm n} = \tfrac{1}{2} a_n \mp i \tfrac{1}{2} b_n$$

Method 2:

$$U_n = \left\langle u(t) e^{-i2\pi nFt} \right\rangle$$
$$= \frac{1}{T} \int_0^T u(t) e^{-i2\pi nFt} dt$$
$$= \frac{1}{T} \int_0^W A e^{-i2\pi nFt} dt$$
$$= \frac{A}{-i2\pi nFT} \left[e^{-i2\pi nFt} \right]_0^W$$
$$= \frac{A}{i2\pi n} \left(1 - e^{-i2\pi nFW} \right)$$

n	a_n	b_n	U_n
-6			$i\frac{8}{6\pi}$
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
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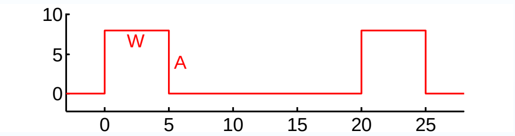
Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Method 1:

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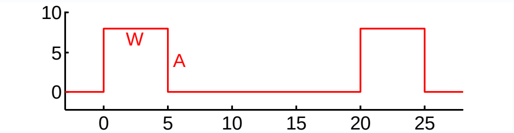

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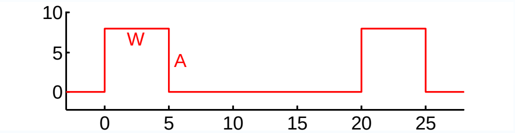

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$$\begin{aligned}
U_n &= \left\langle u(t)e^{-i2\pi nFt}\right\rangle \\
&= \frac{1}{T}\int_0^T u(t)e^{-i2\pi nFt}dt \\
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&= \frac{Ae^{-i\pi nFW}}{i2\pi n}\left(e^{i\pi nFW} - e^{-i\pi nFW}\right) \\
&= \frac{Ae^{-i\pi nFW}}{n\pi}\sin\left(n\pi FW\right) \\
&= \frac{8}{n\pi}\sin\left(\frac{n\pi}{4}\right)e^{-i\frac{n\pi}{4}}
\end{aligned}$$

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Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

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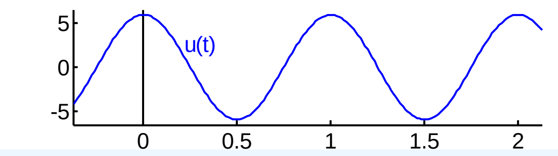
Example:

$u(t) = 6\cos(2\pi Ft)$



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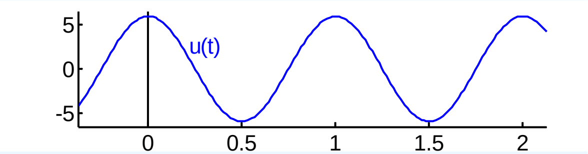

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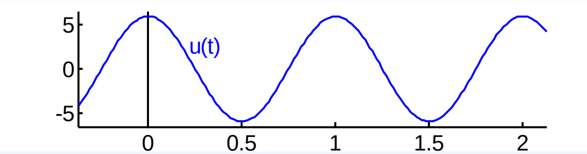

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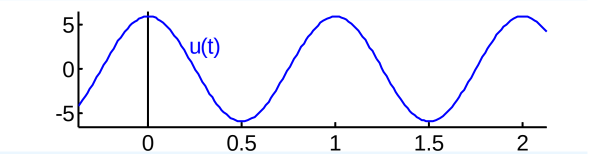

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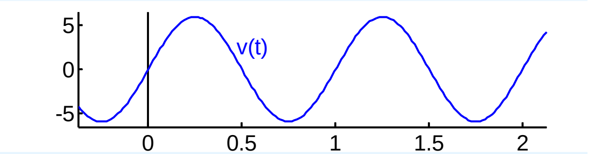

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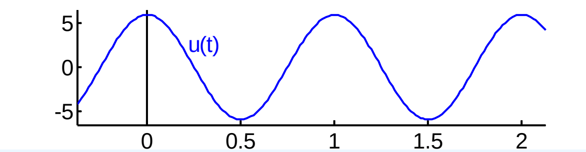

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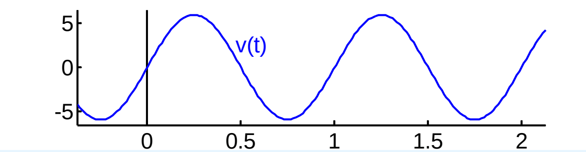

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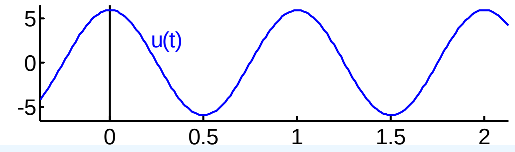

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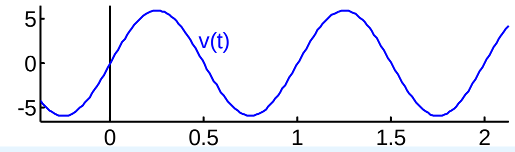

Time delay: $\tau = \frac{T}{4} \Rightarrow F\tau = \frac{1}{4}$

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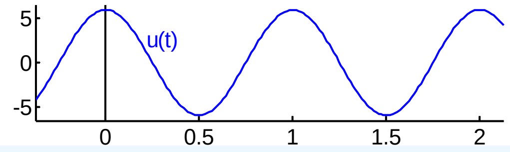

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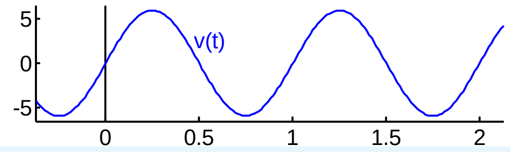

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Note: If $u(t)$ is a sine wave, U_1 equals half the corresponding phasor.

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(1) $u(t)$ real-valued $\Leftrightarrow$ U_n conjugate symmetric $[U_n = U^*_{-n}]$

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Proof of (2): $u(t)$ even $\Rightarrow$ U_n even

$$U_{-n} = \frac{1}{T}\int_0^T u(t)e^{-i2\pi(-n)Ft}dt$$

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$$U_{-n} = \frac{1}{T}\int_0^T u(t)e^{-i2\pi(-n)Ft}dt$$
$$= \frac{1}{T}\int_{x=0}^{-T} u(-x)e^{-i2\pi nFx}(-dx)$$ [substitute $x = -t$]

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$$= \frac{1}{T}\int_{x=-T}^{0} u(-x)e^{-i2\pi nFx}dx \qquad \text{[reverse the limits]}$$

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Proof of (2): $u(t)$ even $\Rightarrow$ U_n even

$$U_{-n} = \frac{1}{T}\int_0^T u(t)e^{-i2\pi(-n)Ft}dt$$

$$= \frac{1}{T}\int_{x=0}^{-T} u(-x)e^{-i2\pi nFx}(-dx) \quad \text{[substitute } x = -t\text{]}$$

$$= \frac{1}{T}\int_{x=-T}^{0} u(-x)e^{-i2\pi nFx}dx \quad \text{[reverse the limits]}$$

$$= \frac{1}{T}\int_{x=-T}^{0} u(x)e^{-i2\pi nFx}dx = U_n \quad \text{[even: } u(-x) = u(x)\text{]}$$

Even/Odd Symmetry

(1) $u(t)$ real-valued $\Leftrightarrow$ U_n conjugate symmetric $[U_n = U_{-n}^*]$

(2) $u(t)$ even $[u(t) = u(-t)]$ $\Leftrightarrow$ U_n even $[U_n = U_{-n}]$

(3) $u(t)$ odd $[u(t) = -u(-t)]$ $\Leftrightarrow$ U_n odd $[U_n = -U_{-n}]$

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Proof of (3): $u(t)$ odd $\Rightarrow$ U_n odd

Same as before, except for the last line:

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Same as before, except for the last line:

$$= \frac{1}{T}\int_{x=-T}^{0} -u(x)e^{-i2\pi nFx}dx = -U_n \qquad \text{[odd: } u(-x) = -u(x)\text{]}$$

Antiperiodic ⇒ Odd Harmonics Only

A waveform, $u(t)$, is anti-periodic if $u(t + \frac{1}{2}T) = -u(t)$.

If $u(t)$ is anti-periodic then $U_n = 0$ for n even.

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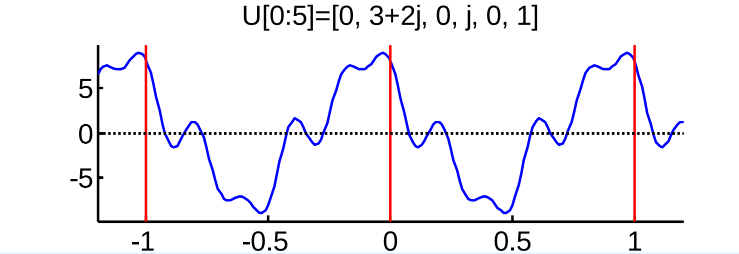
Example:

$U_{0:5} = [0,\ 3+2i,\ 0,\ i,\ 0,\ 1]$

Odd harmonics only ⇔

Second half of each period is the negative of the first half.



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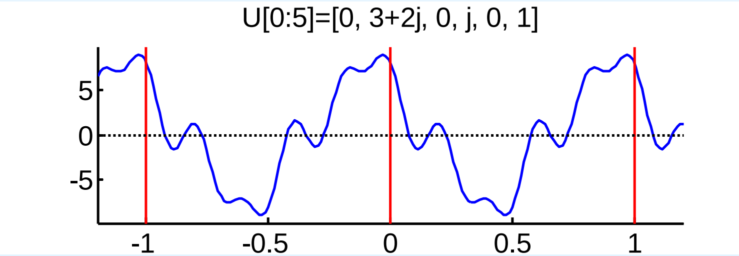

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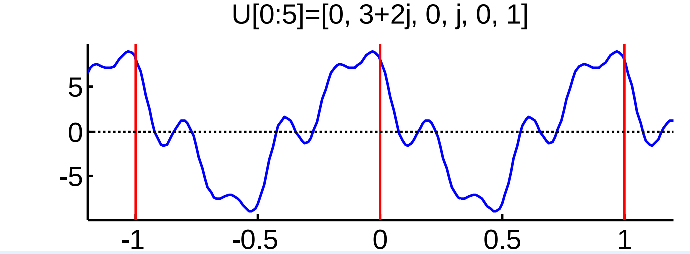

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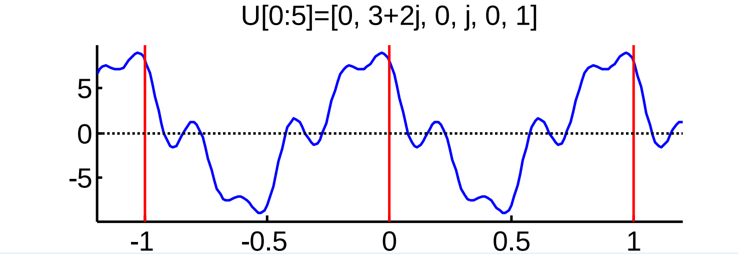

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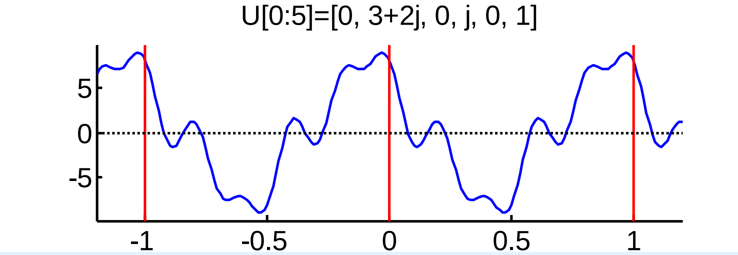

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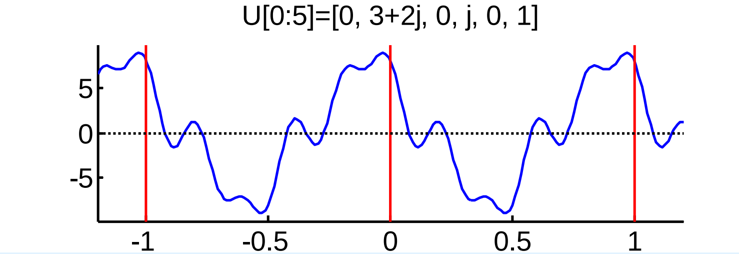

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A waveform, $u(t)$, is anti-periodic if $u(t+\frac{1}{2}T)=-u(t)$.
If $u(t)$ is anti-periodic then $U_n=0$ for n even.

Proof:

Define $v(t)=u(t+\frac{T}{2})$, then

(1) $v(t)=-u(t)\Rightarrow V_n=-U_n$

(2) $v(t)$ equals $u(t)$ but delayed by $-\frac{T}{2}$

$$\Rightarrow V_n=U_n e^{i2\pi nF\frac{T}{2}}=U_n e^{in\pi}$$

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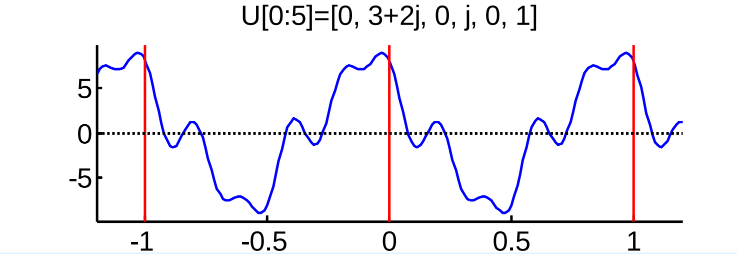

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$$\Rightarrow V_n = U_n e^{i2\pi nF\frac{T}{2}} = U_n e^{in\pi} = \begin{cases} U_n & n \text{ even} \\ -U_n & n \text{ odd} \end{cases}$$

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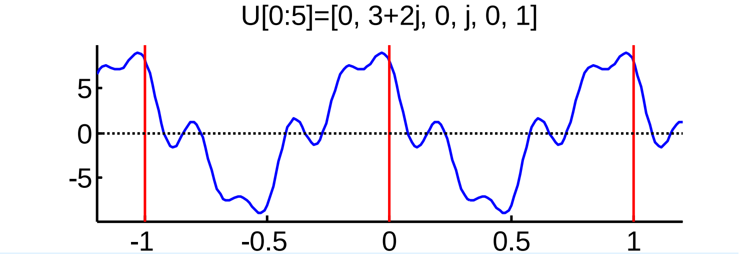

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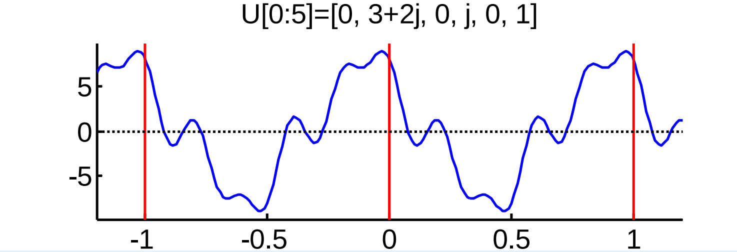

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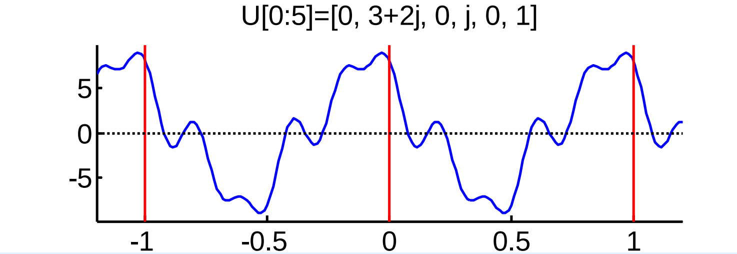

Symmetry Examples

All these examples assume that $u(t)$ is real-valued $\Leftrightarrow U_{-n} = U_{+n}^*$.

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(1) Even $u(t) \Leftrightarrow$ real U_n

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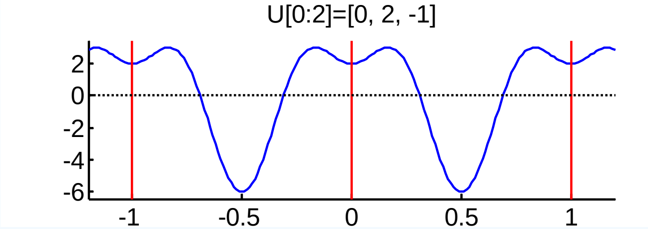

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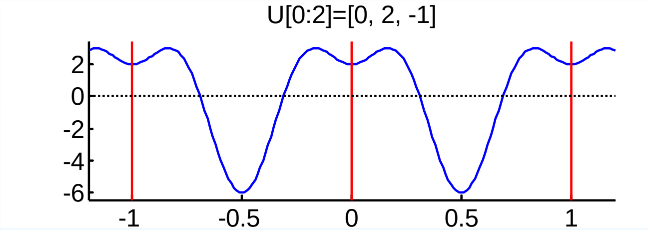
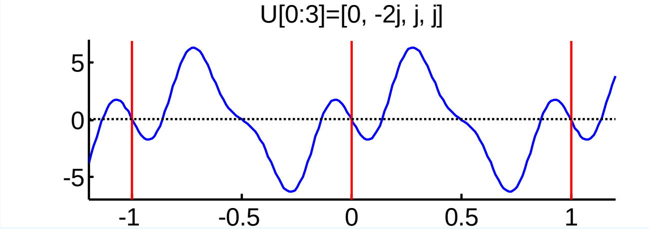


(2) Odd $u(t) \Leftrightarrow$ imaginary U_n

$$U_{0:3} = [0,\ -2i,\ i,\ i]$$



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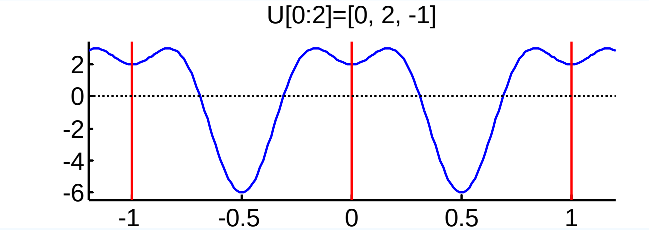

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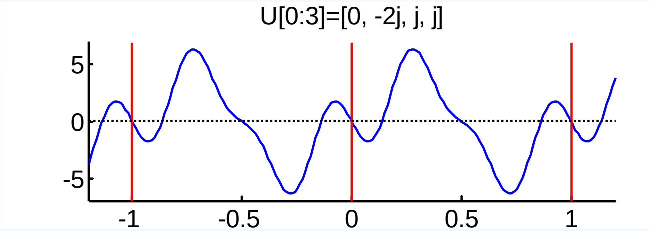
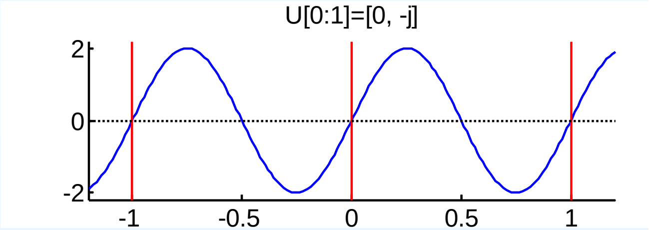


(3) Anti-periodic $u(t)$

$\Leftrightarrow$ odd harmonics only

$$U_{0:1} = [0,\ -i]$$



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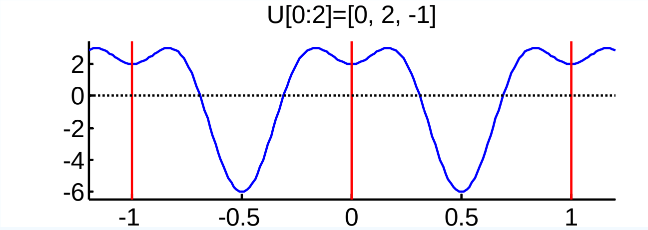

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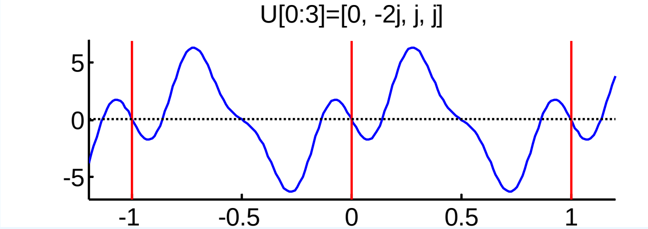

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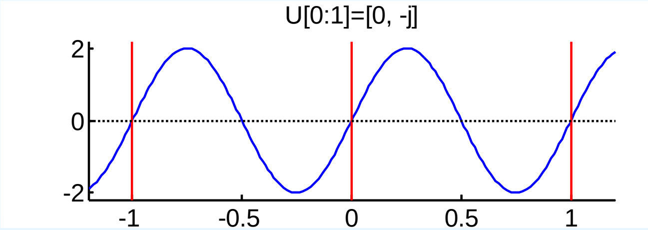
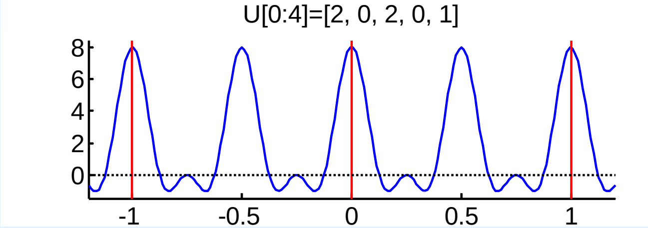


(4) Even harmonics only

$\Leftrightarrow$ period is really $\frac{1}{2}T$

$$U_{0:4} = [2,\ 0,\ 2,\ 0,\ 1]$$



Summary

- Fourier Series:
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  - FS→CFS: $U_{\pm n} = \frac{1}{2}a_{|n|} \mp i\frac{1}{2}b_{|n|}$
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  $\Leftrightarrow U_n$ is purely imaginary and odd $\Leftrightarrow a_n = 0\ \forall n$
- $u(t)$ anti-periodic $\Leftrightarrow u(t + \frac{T}{2}) = -u(t)$
  $\Leftrightarrow$ odd harmonics only $\Leftrightarrow a_{2n} = b_{2n} = U_{2n} = 0\ \forall n$

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  $\Leftrightarrow U_n$ is purely imaginary and odd $\Leftrightarrow a_n = 0\ \forall n$
- $u(t)$ anti-periodic $\Leftrightarrow u(t + \frac{T}{2}) = -u(t)$
  $\Leftrightarrow$ odd harmonics only $\Leftrightarrow a_{2n} = b_{2n} = U_{2n} = 0\ \forall n$

For further details see RHB 12.3 and 12.7.