5: Gibbs Phenomenon

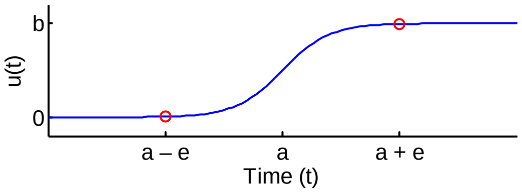
Discontinuities

A function, $v(t)$, has a discontinuity of amplitude b at $t = a$ if

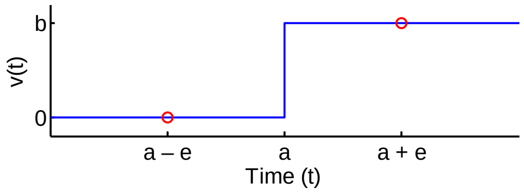
$$\lim_{e \to 0} \left(v(a+e) - v(a-e) \right) = b \neq 0$$

Conversely, $v(t)$, is continuous at $t = a$ if the limit, b, equals zero.

Examples:



Continuous



Discontinuous

We will see that if a periodic function, $v(t)$, is discontinuous, then its Fourier series behaves in a strange way.

Discontinuous Waveform

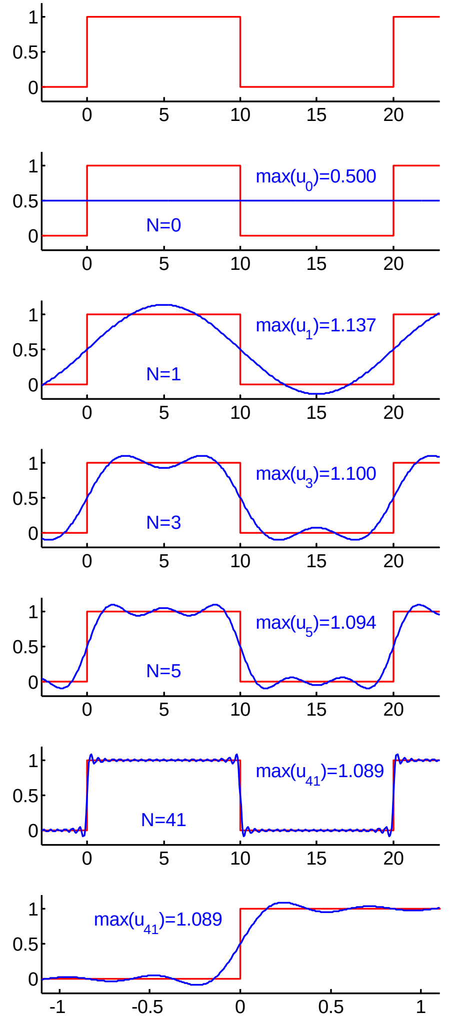
Pulse: $T = \frac{1}{F} = 20$, width$=\frac{1}{2}T$, height $A = 1$

$$U_m = \frac{1}{T}\int_0^{0.5T} Ae^{-i2\pi mFt}dt$$
$$= \frac{i}{2\pi mFT}\left[e^{-i2\pi mFt}\right]_0^{0.5T}$$
$$= \frac{i}{2\pi m}\left(e^{-i\pi m} - 1\right) = \frac{((-1)^m - 1)i}{2\pi m}$$
$$= \begin{cases} 0 & m \neq 0, \text{ even} \\ 0.5 & m = 0 \\ \frac{-i}{m\pi} & m \text{ odd} \end{cases}$$

So, $u(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin 2\pi Ft + \frac{1}{3}\sin 6\pi Ft + \frac{1}{5}\sin 10\pi Ft + \ldots\right)$

Define: $u_N(t) = \sum_{m=-N}^{N} U_m e^{i2\pi mFt}$

$$u_N(0) = 0.5\ \forall N$$
$$\max_t u_N(t) \underset{N\to\infty}{\longrightarrow} \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt \approx 1.0895$$



[Enlarged View: $u_{41}(t)$]

[Powers of -1 and i]

Expressions involving $(-1)^m$ or, less commonly, i^m arise quite frequently and it is worth becoming familiar with them. They can arise in several guises:

$$e^{-i\pi m} = e^{i\pi m} = \left(e^{i\pi}\right)^m = \cos(\pi m) = (-1)^m$$
$$e^{i\frac{1}{2}\pi m} = \left(e^{i\frac{1}{2}\pi}\right)^m = i^m$$
$$e^{-i\frac{1}{2}\pi m} = \left(e^{-i\frac{1}{2}\pi}\right)^m = (-i)^m$$

As m increases these expressions repeat with periods of 2 or 4. Simple expressions involving these quantities make useful sequences.

m	-4	-3	-2	-1	0	1	2	3	4
$(-1)^m = \cos \pi m = e^{i\pi m}$	1	-1	1	-1	1	-1	1	-1	1
$i^m = e^{i0.5\pi m}$	1	i	-1	$-i$	1	i	-1	$-i$	1
$(-i)^m = e^{-i0.5\pi m}$	1	$-i$	-1	i	1	$-i$	-1	i	1
$\frac{1}{2}\left(1+(-1)^m\right)$	1	0	1	0	1	0	1	0	1
$\frac{1}{2}\left(1-(-1)^m\right)$	0	1	0	1	0	1	0	1	0
$\frac{1}{2}\left(i^m+(-i)^m\right) = \cos 0.5\pi m$	1	0	-1	0	1	0	-1	0	1
$\frac{1}{4}\left(1+(-1)^m+i^m+(-i)^m\right)$	1	0	0	0	1	0	0	0	1

Gibbs Phenomenon

Truncated Fourier Series: $u_N(t) = \sum_{m=-N}^{N} U_m e^{i2\pi mFt}$

If $u(t)$ has a discontinuity of height b at $t = a$ then:

(1) $u_N(a) \underset{N\to\infty}{\longrightarrow} \lim_{e\to 0} \frac{u(a-e)+u(a+e)}{2}$

(2) $u_N(t)$ has an overshoot of about 9% of b at the discontinuity. For large N the overshoot moves closer to the discontinuity but does not get smaller (Gibbs phenomenon). In the limit the overshoot equals $\left(-\frac{1}{2} + \frac{1}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt\right) b \approx 0.0895b$.

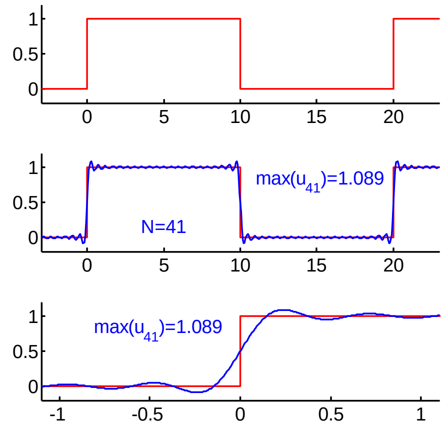
(3) For large m, the coefficients, U_m decrease no faster than $|m|^{-1}$.

Example:

$$u_N(0) \underset{N\to\infty}{\longrightarrow} 0.5$$

$$\max_t u_N(t) \underset{N\to\infty}{\longrightarrow} 1.0895\ldots$$

$$U_m = \begin{cases} 0 & m \neq 0, \text{ even} \\ 0.5 & m = 0 \\ \frac{-i}{m\pi} & m \text{ odd} \end{cases}$$



[Origin of Gibbs Phenomenon]

This topic is included for interest but is not examinable.

Our first goal is to express the partial Fourier series, $u_N(t)$, in terms of the original signal, $u(t)$. We begin by substituting the integral expression for U_n in the partial Fourier series

$$u_N(t) = \sum_{n=-N}^{+N} U_n e^{i2\pi nFt} = \sum_{n=-N}^{+N} \left(\frac{1}{T}\int_0^T u(\tau)e^{-i2\pi nF\tau}d\tau\right)e^{i2\pi nFt}$$

Now we swap the order of the integration and the finite summation (OK if the integral converges $\forall n$)

$$u_N(t) = \frac{1}{T}\int_0^T u(\tau)\left(\sum_{n=-N}^{+N} e^{i2\pi nF(t-\tau)}\right)d\tau$$

Now apply the formula for the sum of a geometric progression with $z = e^{i2\pi F(t-\tau)}$:

$$\sum_{n=-N}^{+N} z^n = \frac{z^{-N}-z^{N+1}}{1-z} = \frac{z^{-(N+0.5)}-z^{N+0.5}}{z^{-0.5}-z^{0.5}}$$

$$u_N(t) = \frac{1}{T}\int_0^T u(\tau)\frac{e^{i2\pi(N+0.5)F(\tau-t)}-e^{-i2\pi(N+0.5)F(\tau-t)}}{e^{i2\pi 0.5F(\tau-t)}-e^{-i2\pi 0.5F(\tau-t)}}d\tau$$

$$= \frac{1}{T}\int_0^T u(\tau)\frac{\sin \pi(2N+1)F(\tau-t)}{\sin \pi F(\tau-t)}d\tau$$

So if we define the Dirichlet Kernel to be $D_N(x) = \frac{\sin((N+0.5)x)}{\sin 0.5x}$, and set $x = 2\pi F(\tau - t)$, we obtain

$$u_N(t) = \frac{1}{T}\int_0^T u(\tau)D_N\left(2\pi F(\tau-t)\right)d\tau$$

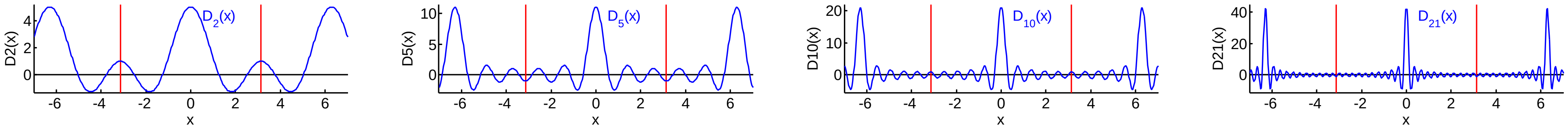
So what we have shown is that $u_N(t)$ can be obtained by multiplying $u(\tau)$ by a time-shifted Dirichlet Kernel and then integrating over one period. Next we will look at the properties of the Dirichlet Kernel.

[Dirichlet Kernel]

This topic is included for interest but is not examinable.

Dirichlet Kernel definition: $D_N(x) = \sum_{n=-N}^{+N} e^{inx} = 1 + 2\sum_{n=1}^{N} \cos nx = \frac{\sin((N+0.5)x)}{\sin 0.5x}$

$D_N(x)$ is plotted below for $N = \{2,\ 5,\ 10,\ 21\}$. The vertical red lines at $\pm\pi$ mark one period.



- Periodic: with period 2π
- Average value: $\langle D_N(x)\rangle = \frac{1}{2\pi}\int_{-\pi}^{+\pi} D_N(x)dx = 1$
- First Zeros: $D_N(x) = 0$ at $x = \pm\frac{\pi}{N+0.5}$ define the main lobe as $-\frac{\pi}{N+0.5} < x < \frac{\pi}{N+0.5}$
- Peak value: $2N+1$ at $x = 0$. The main lobe gets narrower but higher as N increases.
- Main Lobe semi-integral:

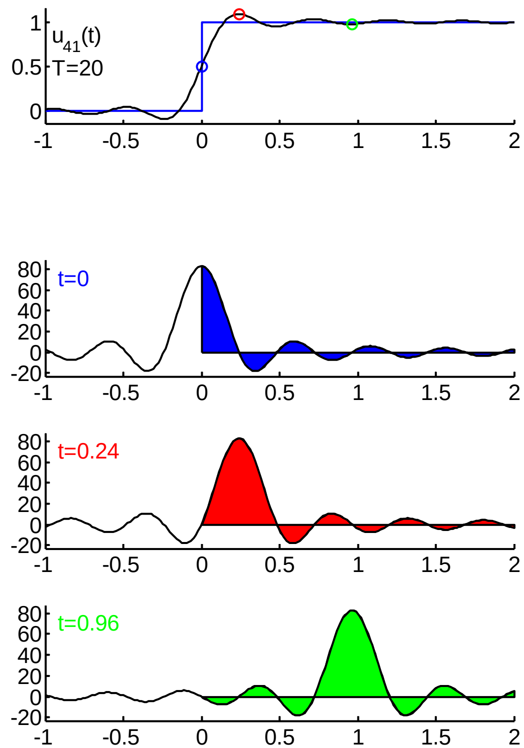
$$\int_{x=0}^{\frac{\pi}{N+0.5}} D_N(x)dx = \int_{x=0}^{\frac{\pi}{N+0.5}} \frac{\sin((N+0.5)x)}{\sin 0.5x}dx = \frac{1}{N+0.5}\int_{y=0}^{\pi} \frac{\sin y}{\sin\frac{y}{2N+1}}dy[y = (N+0.5)x]$$

where we substituted $y = (N+0.5)x$. Now, for large N, we can approximate $\sin\frac{y}{2N+1} \approx \frac{y}{2N+1}$:

$$\int_{x=0}^{\frac{\pi}{N+0.5}} D_N(x)dx \approx \frac{1}{N+0.5}\int_{y=0}^{\pi} \frac{\sin y}{\frac{y}{2N+1}}dy = 2\int_{y=0}^{\pi} \frac{\sin y}{y}dy \approx 3.7038741 \approx 2\pi \times 0.58949$$

We see that, for large enough N, the main lobe semi-integral is independent of N.

[In MATLAB $D_N(x) = (2N+1) \times \text{diric}(x, 2N+1)$]



This topic is included for interest but is not examinable.

The partial Fourier Series, $u_N(t)$, can be obtained by multiplying $u(t)$ by a shifted Dirichlet Kernel and integrating over one period:

$$u_N(t) = \frac{1}{T}\int_0^T u(\tau) D_N\left(2\pi F(\tau - t)\right) d\tau$$

For the special case when $u(t)$ is a pulse of height 1 and width $0.5T$:

$$u_N(t) = \frac{1}{T}\int_0^{0.5T} D_N\left(2\pi F(\tau - t)\right) d\tau$$

Substitute $x = 2\pi F(\tau - t)$

$$u_N(t) = \frac{1}{2\pi F T}\int_{-2\pi F t}^{\pi F T - 2\pi F t} D_N(x)\, dx = \frac{1}{2\pi}\int_{-2\pi F t}^{\pi - 2\pi F t} D_N(x)\, dx$$

- For $t = 0$ (the blue integral and the blue circle on the upper graph):
  $u_N(0) = \frac{1}{2\pi}\int_0^{\pi} D_N(x)\, dx = 0.5$
- For $t = \frac{T}{2N+1}$ (the red integral and the red circle on the upper graph):
  $$u_N\left(\frac{T}{2N+1}\right) = \frac{1}{2\pi}\int_{-\frac{\pi}{N+0.5}}^{\pi - \frac{\pi}{N+0.5}} D_N(x)\, dx = \frac{1}{2\pi}\int_{-\frac{\pi}{N+0.5}}^{0} D_N(x)\, dx + \frac{1}{2\pi}\int_0^{\pi - \frac{\pi}{N+0.5}} D_N(x)\, dx$$
  For large N, we replace the first term by a constant (since it is the semi-integral of the main lobe) and replace the upper limit of the second term by π:
  $$\approx 0.58949 + \frac{1}{2\pi}\int_0^{\pi} D_N(x)\, dx = 1.08949$$
- For $0 \ll t \ll 0.5T$, $u_N(t) \approx 1$ (the green integral and the green circle on the upper graph).

Suppose $u(t) = \sum_{m=-\infty}^{\infty} U_m e^{i2\pi mFt}$

Define $v(t)$ to be the integral of $u(t)$ [boundedness requires $U_0 = 0$]

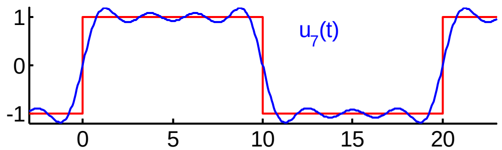
$$v(t) = \int^t u(\tau)d\tau = \int^t \sum_{m=-\infty}^{\infty} U_m e^{i2\pi mF\tau} d\tau$$
$$= \sum_{m=-\infty}^{\infty} U_m \int^t e^{i2\pi mF\tau} d\tau \quad \text{[assume OK to swap } \int \text{ and } \sum\text{]}$$
$$= c + \sum_{m=-\infty}^{\infty} U_m \frac{1}{i2\pi mF} e^{i2\pi mFt}$$
$$= c + \sum_{m=-\infty}^{\infty} V_m e^{i2\pi mFt} \text{ where } c \text{ is an integration constant}$$

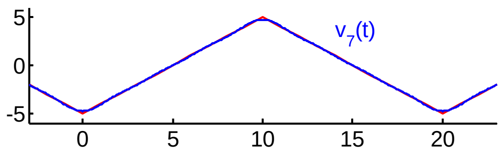
Hence $V_m = \frac{-i}{2\pi mF} U_m$ except for $V_0 = c$ (arbitrary constant)

Example:

Square wave: $U_m = \frac{-2i}{m\pi}$ for odd m (0 for even m)

Triangle wave: $V_m = \frac{-i}{2\pi mF} \times \frac{-2i}{m\pi} = \frac{-1}{\pi^2 m^2 F}$ for odd m (0 for even m)

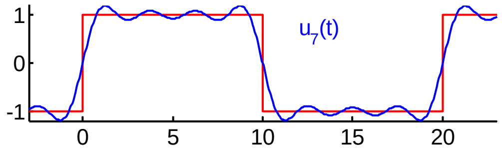


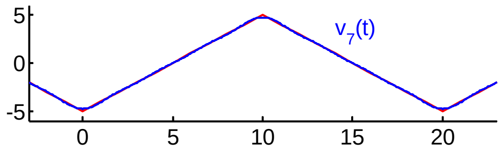


Convergence: $v(t)$ always converges if $u(t)$ does since $V_m \propto \frac{1}{m} U_m$

$v_N(t)$ is a good approximation even for small N

Square wave: $U_m = \frac{-2i}{\pi} m^{-1}$ for odd m (0 for even m)
Triangle wave: $V_m = \frac{-1}{\pi^2 F} m^{-2}$ for odd m (0 for even m)





Integrating $u(t)$ multiplies the U_m by $\frac{-i}{2\pi F} \times m^{-1} \Rightarrow$ they decrease faster.

The rate at which the coefficients, U_m, decrease with m depends on the lowest derivative that has a discontinuity:

- Discontinuity in $u(t)$ itself (e.g. square wave)
  For large $|m|$, U_m decreases as $|m|^{-1}$
- Discontinuity in $u'(t)$ (e.g. triangle wave)
  For large $|m|$, U_m decreases as $|m|^{-2}$
- Discontinuity in $u^{(n)}(t)$
  For large $|m|$, U_m decreases as $|m|^{-(n+1)}$
- No discontinuous derivatives
  For large $|m|$, U_m decreases faster than any power (e.g. $e^{-|m|}$)

Differentiation

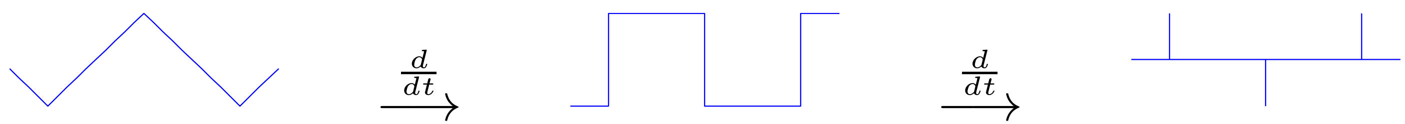
Integration multiplies U_m by $\frac{-i}{2\pi mF}$.

Hence differentiation multiplies U_m by $\frac{2\pi mF}{-i} = i2\pi mF$

If $u(t)$ is a continuous differentiable function and $w(t) = \frac{du(t)}{dt}$ then, provided that $w(t)$ satisfies the Dirichlet conditions, its Fourier coefficients are:

$$W_m = \begin{cases} 0 & m = 0 \\ i2\pi mFU_m & m \neq 0 \end{cases}.$$

Since we are multiplying U_m by m the coefficients W_m decrease more slowly with m and so the Fourier series for $w(t)$ may not converge (i.e. $w(t)$ may not satisfy the Dirichlet conditions).



$U_m \propto |m|^{-2}$ $\xrightarrow{\frac{d}{dt}}$ $U_m \propto |m|^{-1}$ $\xrightarrow{\frac{d}{dt}}$ $U_m \propto |m|^{-0}$

Differentiation makes waveforms spikier and less smooth.

Periodic Extension

Suppose $y(t)$ is only defined over a finite interval (a, b).

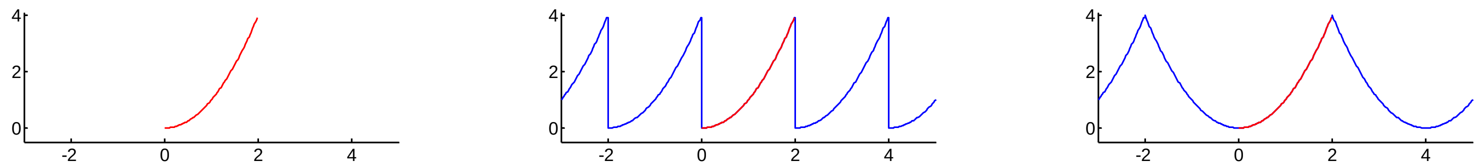
You have two reasonable choices to make a periodic version:

(a) $T = b - a, \qquad u(t) = y(t) \text{ for } a \le t < b$

(b) $T = 2(b-a),\; u(t) = \begin{cases} y(t) & a \le t \le b \\ y(2b - t) & b \le t \le 2b - a \end{cases}$

Example:

$y(t) = t^2$ for $0 \le t < 2$



$y(t)$ (a) $T = 2$ (b) $T = 4$

Option (b) has twice the period, no discontinuities, no Gibbs phenomenon overshoots and if $y(t)$ is continuous the coefficients decrease at least as fast as $|m|^{-2}$.

$y(t) = t^2$ for $0 \le t < 2$

Method (a): $T = \frac{1}{F} = 2$

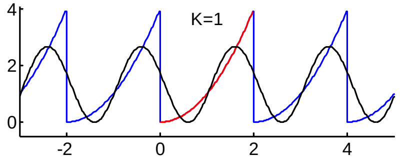
$$U_m = \frac{1}{T}\int_0^T t^2 e^{-i2\pi mFt}dt \qquad U_0 = \frac{1}{T}\int_0^T t^2 dt = \frac{4}{3}$$

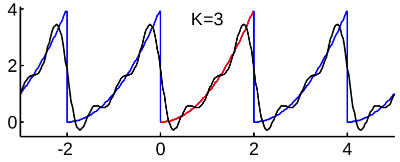
$$= \frac{1}{T}\left[\frac{t^2 e^{-i2\pi mFt}}{-i2\pi mF} - \frac{2te^{-i2\pi mFt}}{(-i2\pi mF)^2} + \frac{2e^{-i2\pi mFt}}{(-i2\pi mF)^3}\right]_0^T$$

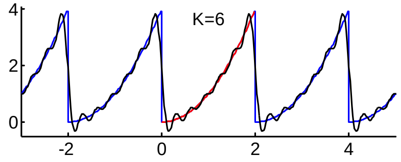
Substitute $e^{-i2\pi mF0} = e^{-i2\pi mFT} = 1$ [for integer m]

$$= \frac{1}{T}\left[\frac{T^2}{-i2\pi mF} - \frac{2T}{(-i2\pi mF)^2}\right]$$

$$= \frac{2i}{\pi m} + \frac{2}{\pi^2 m^2}$$







$$U_{0:3} = [1.333,\ 0.203 + 0.637i,\ 0.051 + 0.318i,\ 0.023 + 0.212i]$$

t^2 Periodic Extension: Method (b)

$y(t) = t^2$ for $0 \le t < 2$

Method (b): $T = \frac{1}{F} = 4$

$$U_m = \frac{1}{T} \int_{-0.5T}^{0.5T} t^2 e^{-i2\pi mFt} dt \qquad U_0 = \frac{1}{T} \int_{-0.5T}^{0.5T} t^2 dt = \frac{4}{3}$$

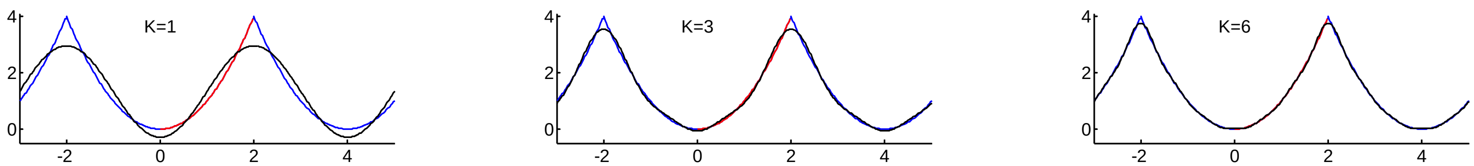
$$= \frac{1}{T} \left[\frac{t^2 e^{-i2\pi mFt}}{-i2\pi mF} - \frac{2te^{-i2\pi mFt}}{(-i2\pi mF)^2} + \frac{2e^{-i2\pi mFt}}{(-i2\pi mF)^3} \right]_{-0.5T}^{0.5T}$$

Substitute $e^{\pm i\pi mFT} = e^{\pm i\pi m} = (-1)^m$ [for integer m]

$$= \frac{(-1)^m}{T} \left[\frac{-2T}{(-i2\pi mF)^2} \right]$$

[all even powers of t cancel out]

$$= \frac{(-1)^m T^2}{2\pi^2 m^2} = \frac{(-1)^m 8}{\pi^2 m^2}$$



$U_{0:3} = [1.333, -0.811, 0.203, -0.090]$ [$u(t)$ real+even $\Rightarrow U_m$ real]

Convergence is noticeably faster than for method (a)

Summary

- Discontinuity at $t = a$
  - Gibbs phenomenon: $u_N(t)$ overshoots by 9% of iump
  - $u_N(a) \to$ mid point of iump
- Integration: If $v(t) = \int^t u(\tau)d\tau$, then $V_m = \frac{-i}{2\pi m F} U_m$ and $V_0 = c$, an arbitrary constant. U_0 must be zero.
- Differentiation: If $w(t) = \frac{du(t)}{dt}$, then $W_m = i2\pi m F U_m$ provided $w(t)$ satisfies Dirichlet conditions (it might not)
- Rate of decay:
  - For large n, U_n decreases at a rate $|n|^{-(k+1)}$ where $\frac{d^k u(t)}{dt^k}$ is the first discontinuous derivative
  - Error power: $\left\langle (u(t) - u_N(t))^2 \right\rangle = \sum_{|n|>N} |U_n|^2$
- Periodic Extension of finite domain signal of length L
  - (a) Repeat indefinitely with period $T = L$
  - (b) Reflect alternate repetitions for period $T = 2L$
    no discontinuities or Gibbs phenomenon

For further details see RHB Chapter 12.4, 12.5, 12.6