6: Fourier Transform

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- Units: if $u(t)$ is in volts, then $U(f)df$ must also be in volts
  $\Rightarrow U(f)$ is in volts/Hz (hence "spectral density").

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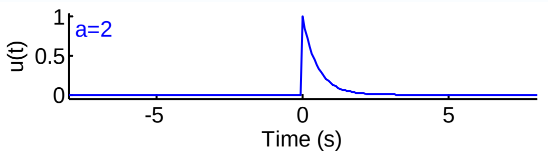

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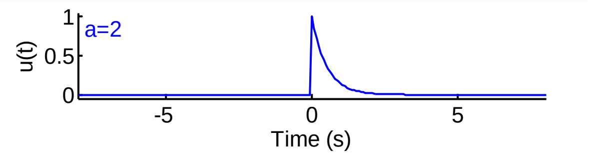

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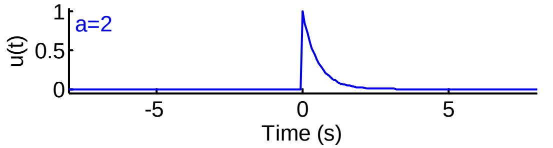

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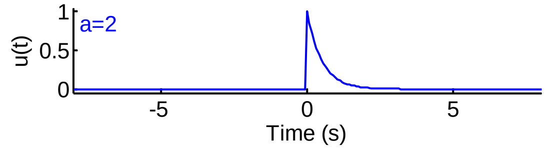

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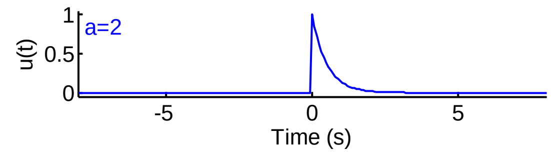

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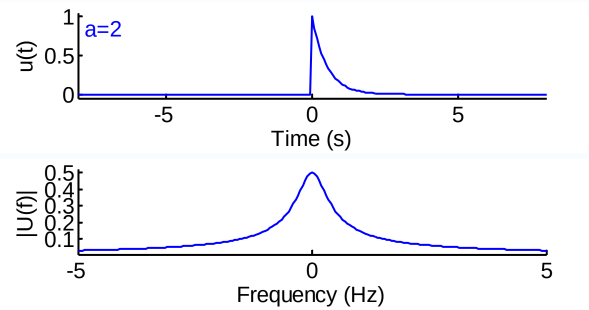

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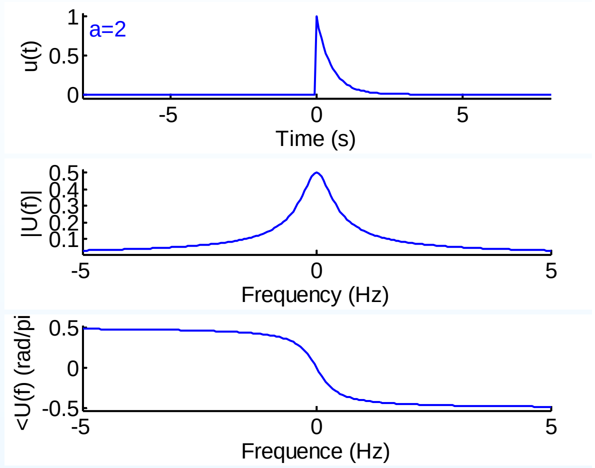

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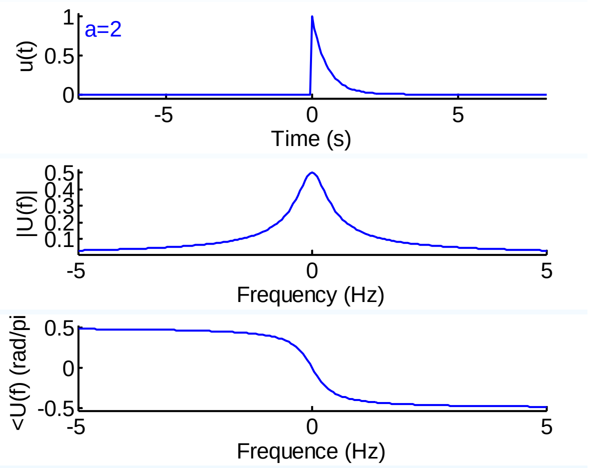

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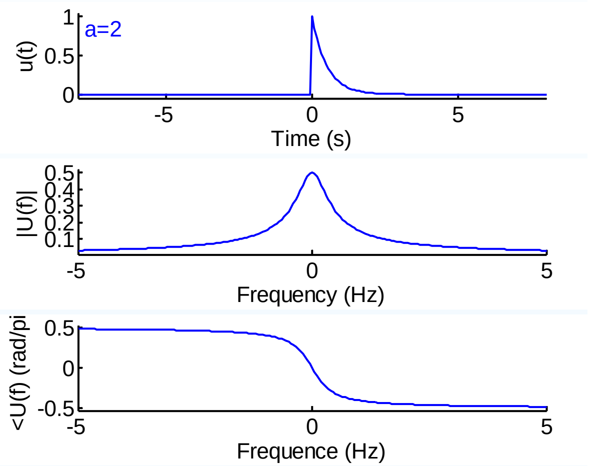

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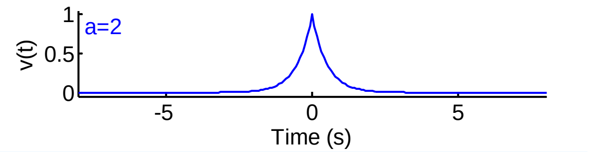

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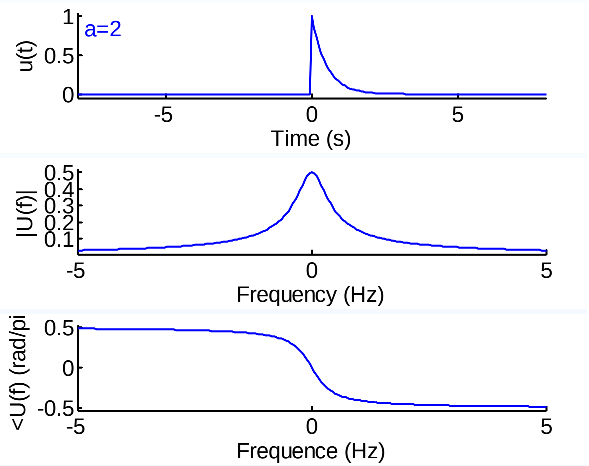

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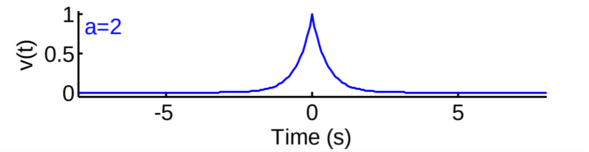

Fourier Transform Examples

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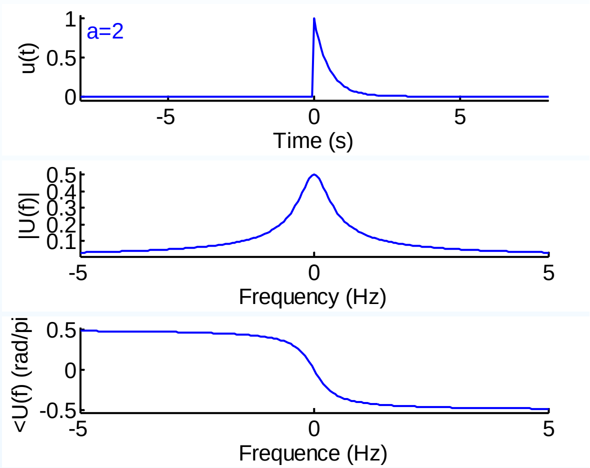

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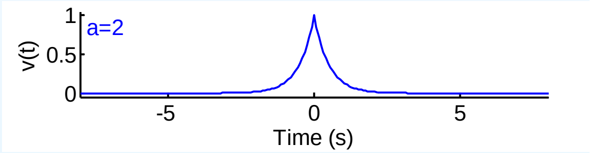

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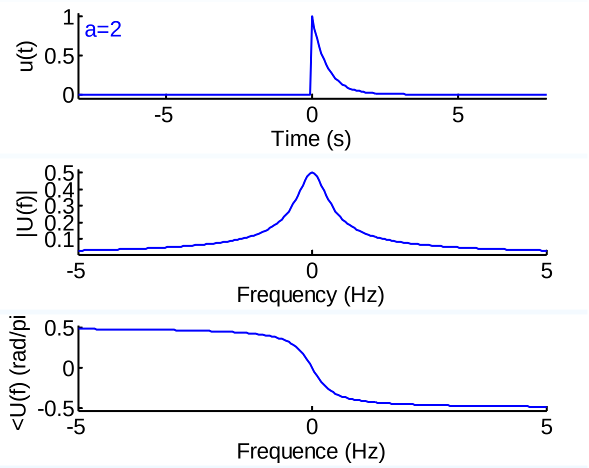

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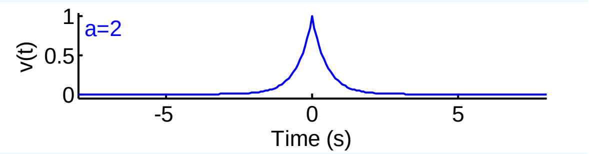

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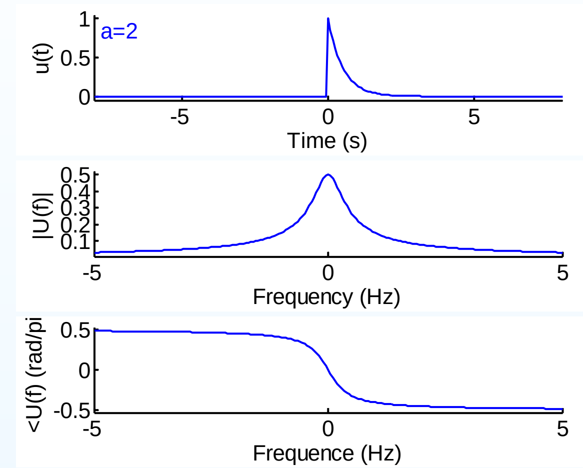

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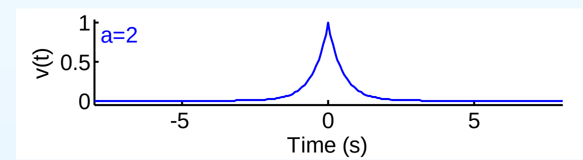

$$\begin{aligned} V(f) &= \int_{-\infty}^{\infty} v(t) e^{-i2\pi f t} dt \\ &= \int_{-\infty}^{0} e^{at} e^{-i2\pi f t} dt + \int_{0}^{\infty} e^{-at} e^{-i2\pi f t} dt \\ &= \frac{1}{a-i2\pi f} \left[e^{(a-i2\pi f)t} \right]_{-\infty}^{0} + \frac{-1}{a+i2\pi f} \left[e^{(-a-i2\pi f)t} \right]_0^{\infty} \\ &= \frac{1}{a-i2\pi f} + \frac{1}{a+i2\pi f} \end{aligned}$$

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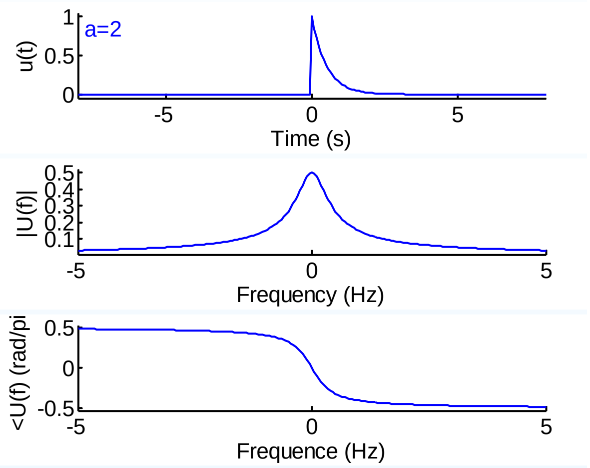

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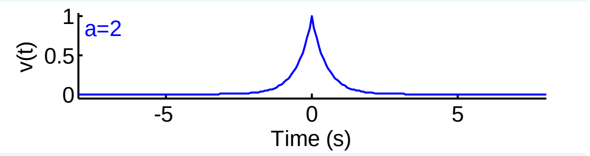

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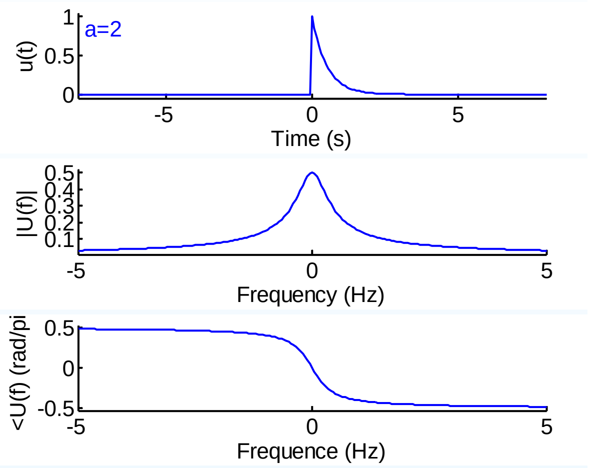

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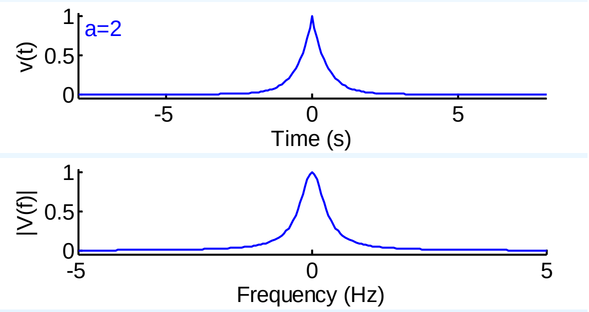

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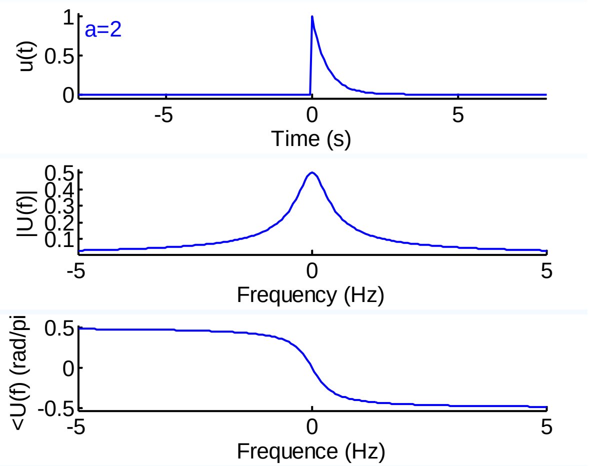

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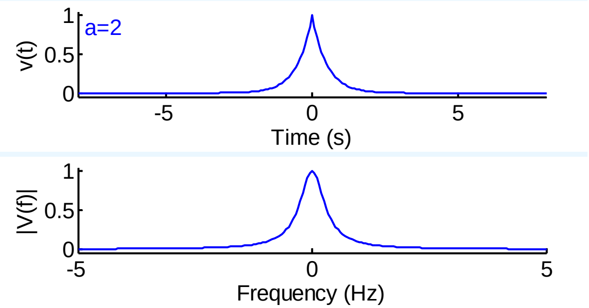

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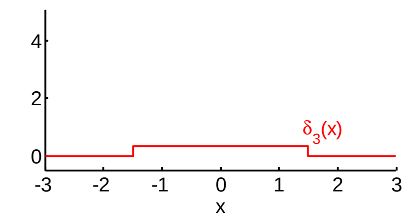
[$v(t)$ real+symmetric
$\Rightarrow V(f)$ real+symmetric]

Dirac Delta Function

We define a unit area pulse of width w as

$$d_w(x) = \begin{cases} \frac{1}{w} & -0.5w \le x \le 0.5w \\ 0 & \text{otherwise} \end{cases}$$



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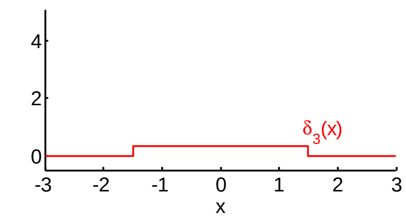

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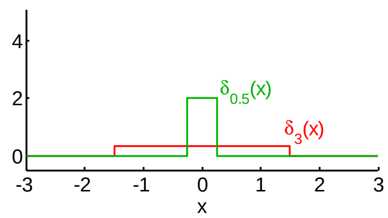

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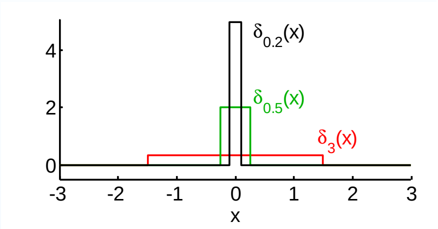

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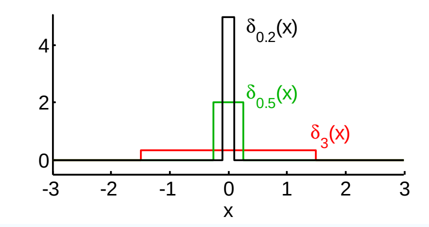

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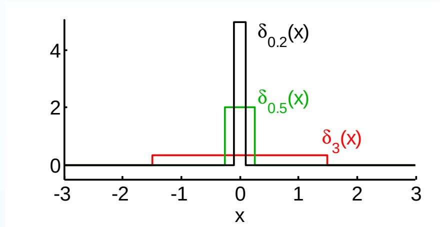

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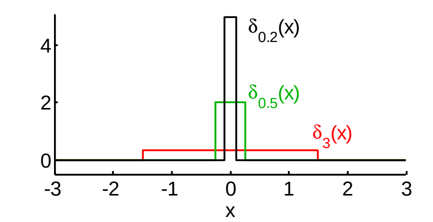

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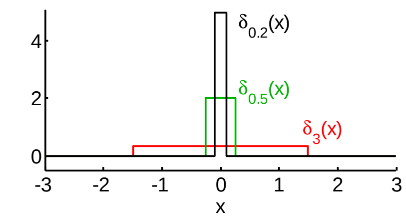

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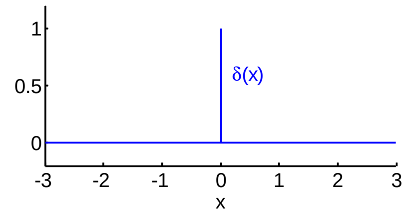

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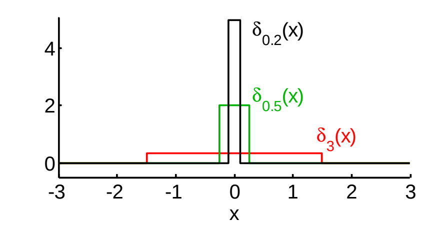

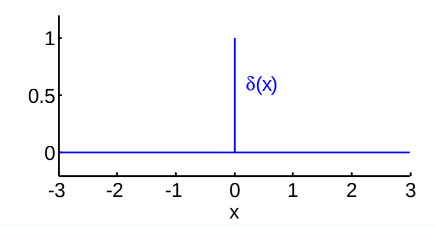

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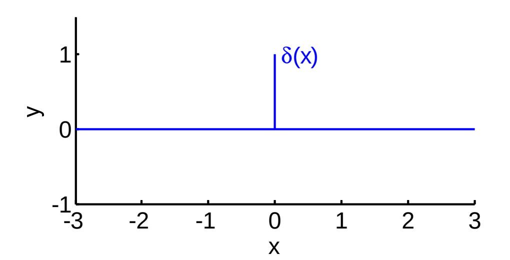
$\delta(x)$ is not quite a proper function: it is called a generalized function.

Dirac Delta Function: Scaling and Translation

Translation: $\delta(x-a)$

$\delta(x)$ is a pulse at $x = 0$



Dirac Delta Function: Scaling and Translation

Translation: $\delta(x-a)$

$\delta(x)$ is a pulse at $x=0$

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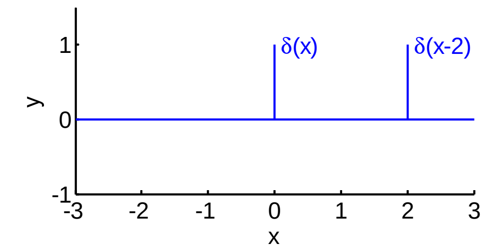

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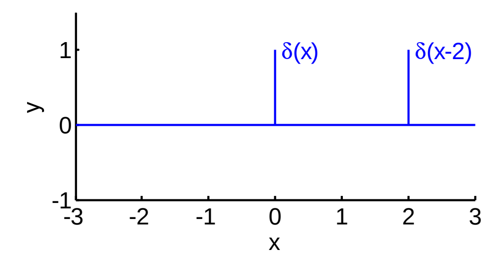

Amplitude Scaling: $b\delta(x)$

$\delta(x)$ has an area of 1

Dirac Delta Function: Scaling and Translation

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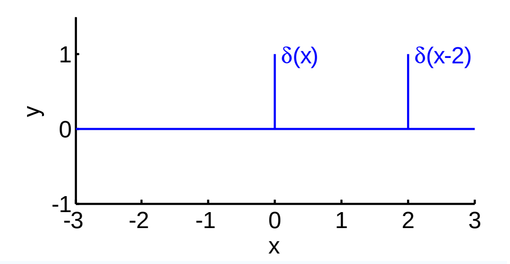

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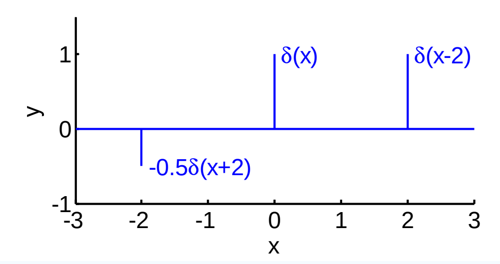

Amplitude Scaling: $b\delta(x)$

$\delta(x)$ has an area of $1 \Leftrightarrow \int_{-\infty}^{\infty} \delta(x)dx = 1$

$b\delta(x)$ has an area of b since

$$\int_{-\infty}^{\infty} (b\delta(x))\, dx$$

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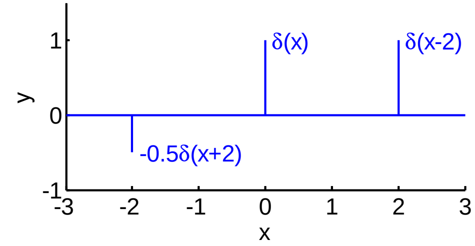

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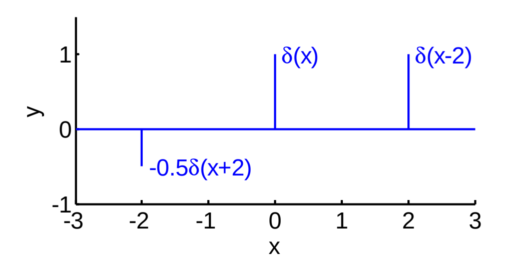

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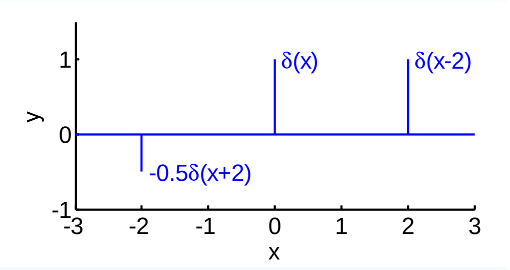

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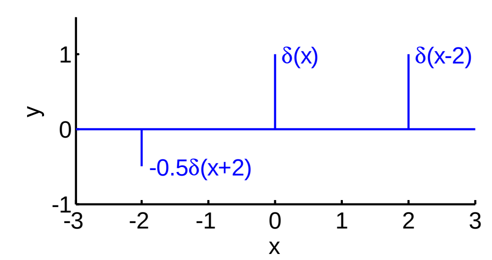
b can be a complex number (on a graph, we then plot only its magnitude)

Dirac Delta Function: Scaling and Translation



Translation: $\delta(x-a)$

$\delta(x)$ is a pulse at $x=0$
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Amplitude Scaling: $b\delta(x)$

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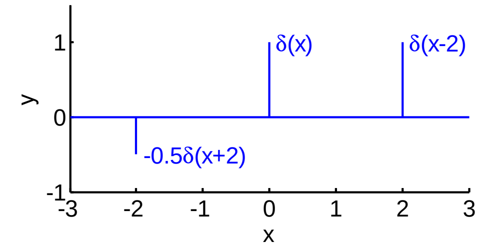

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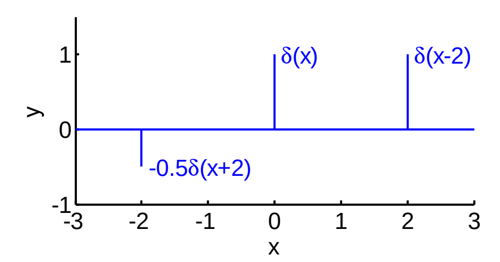

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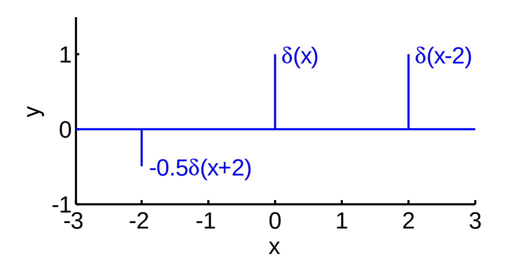

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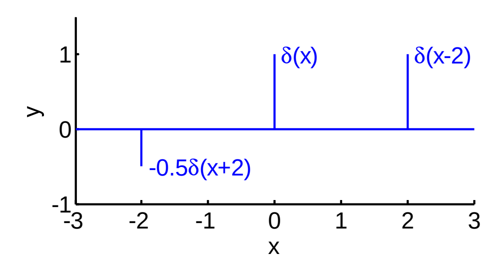

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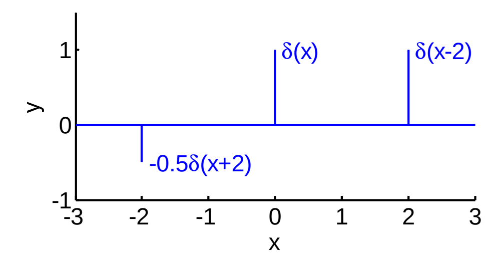

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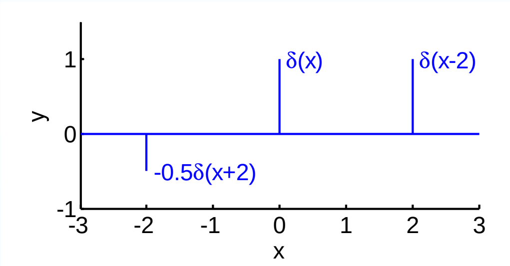

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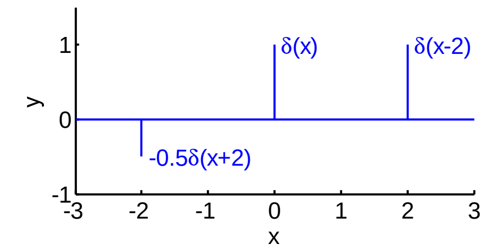

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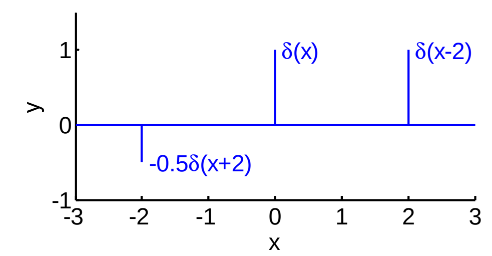

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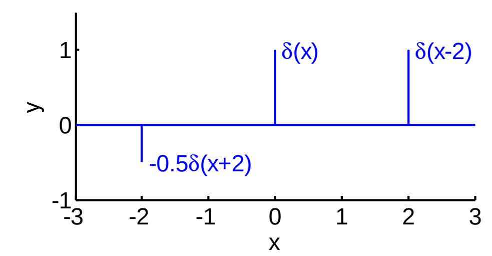

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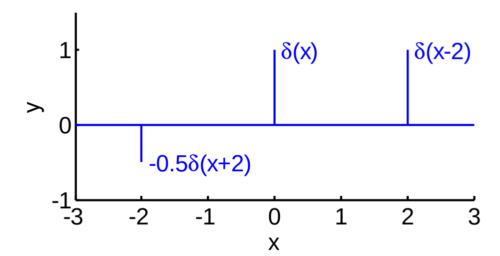

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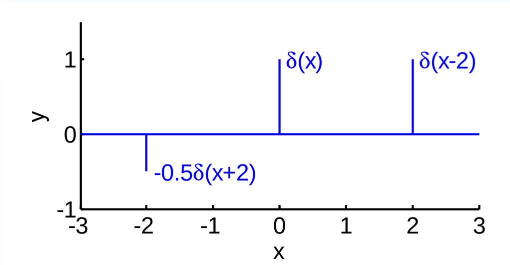

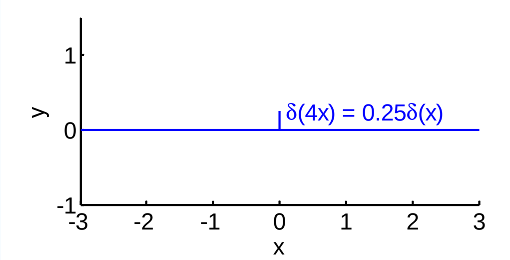

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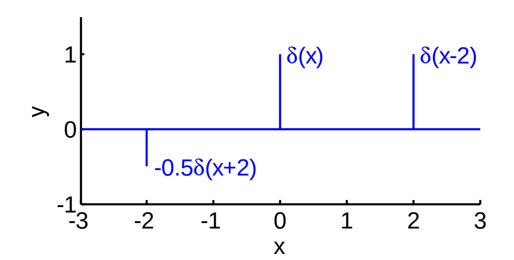

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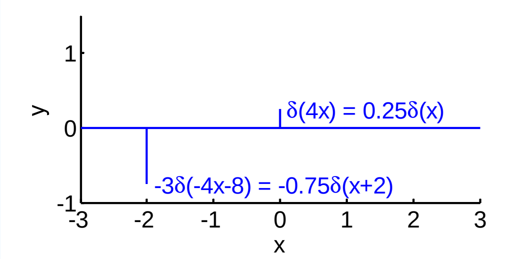

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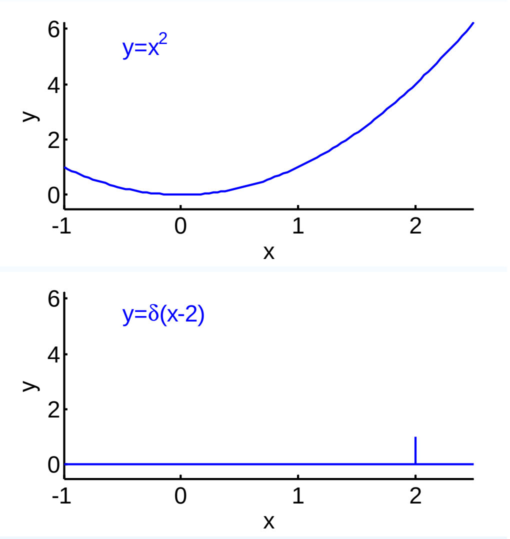
Dirac Delta Function: Products and Integrals

If we multiply $\delta(x-a)$ by a function of x:

$$y = x^2 \times \delta(x-2)$$



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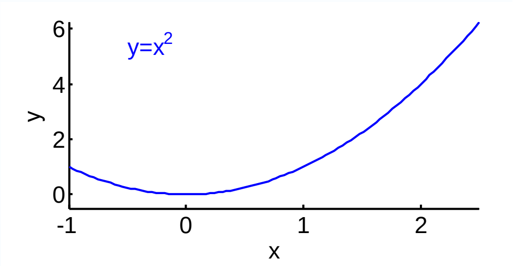

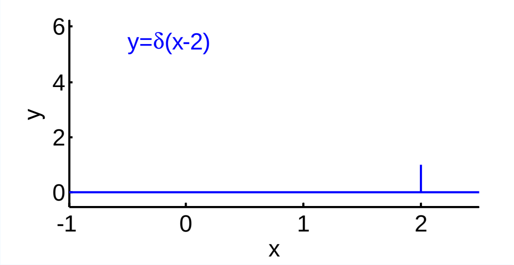

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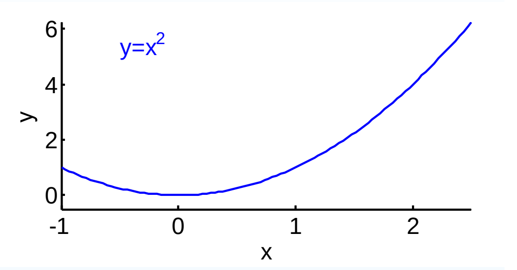

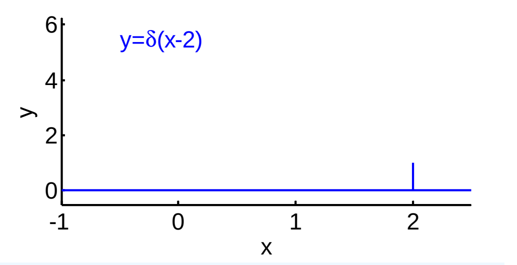

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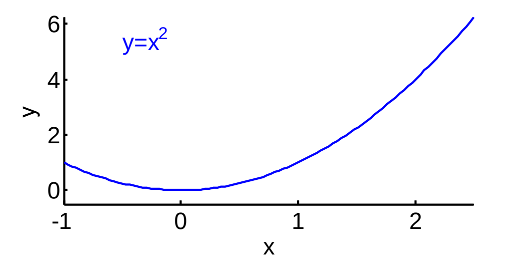

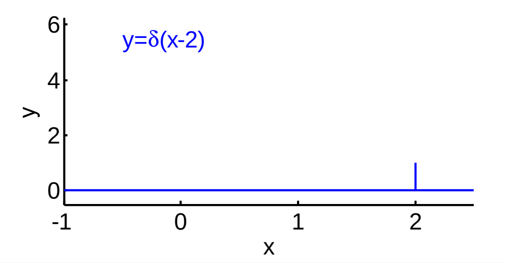

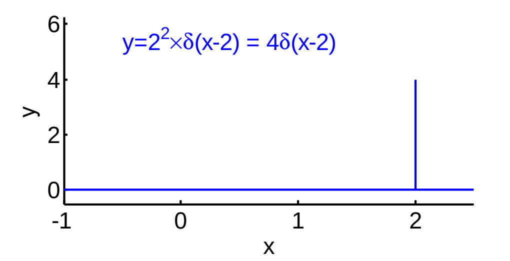

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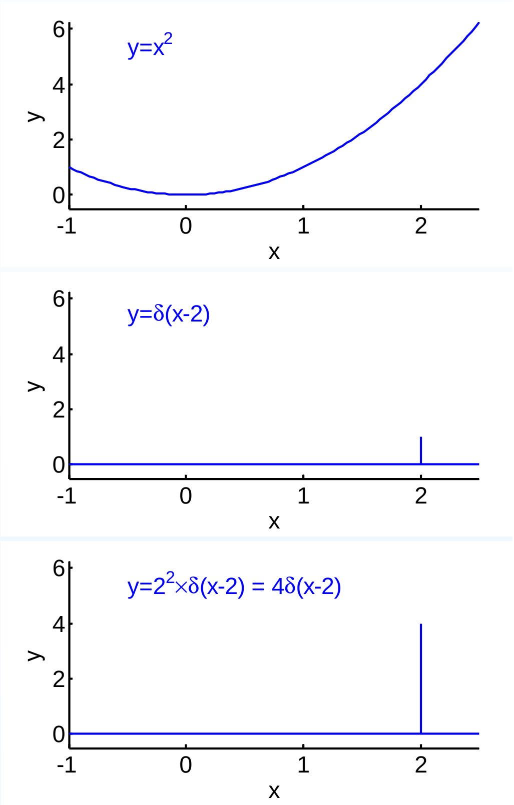

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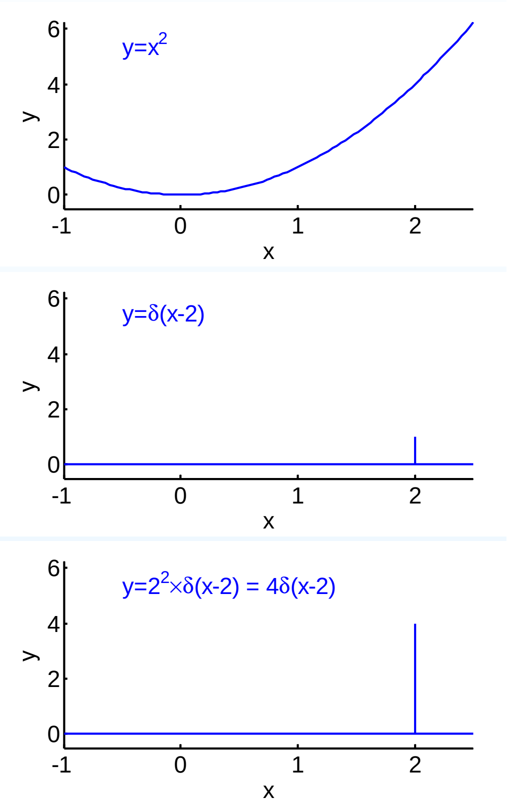

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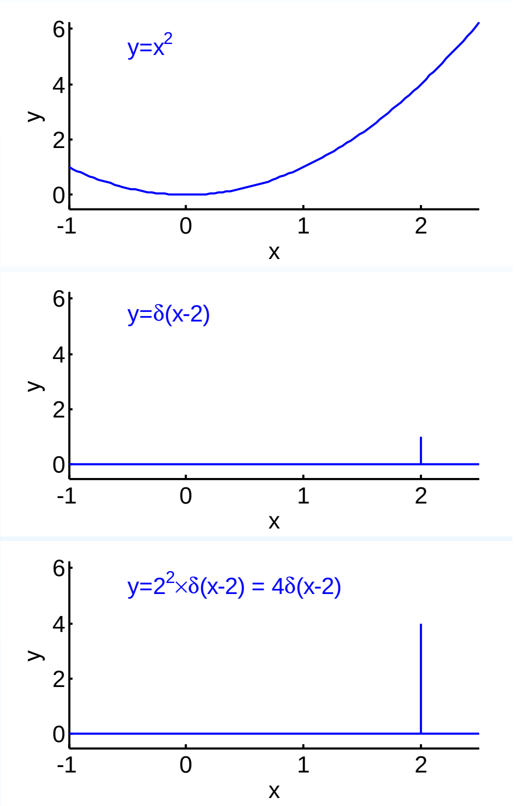

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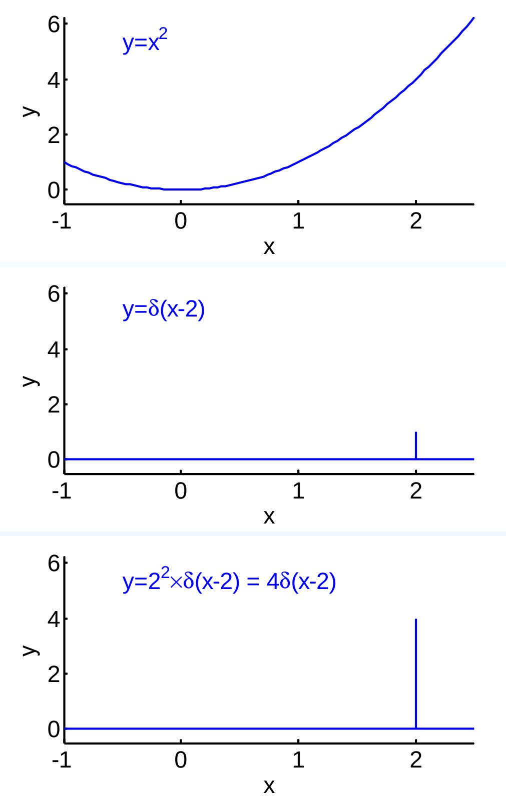

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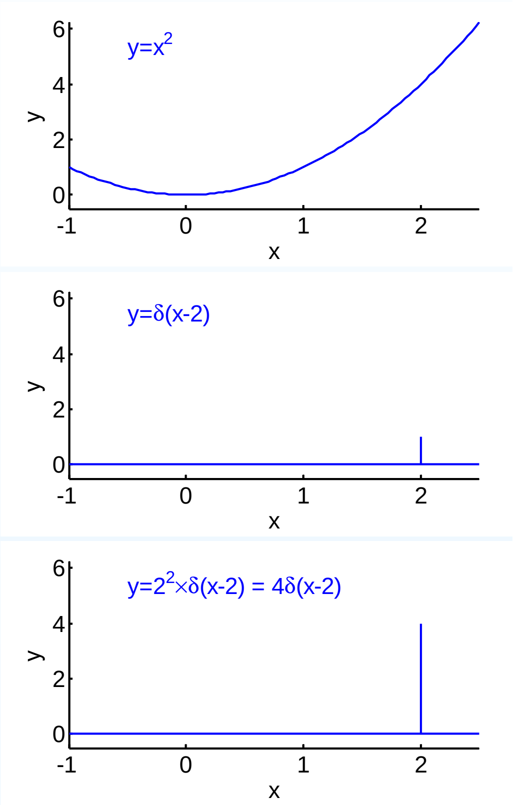
$$y = x^2 \times \delta(x-2)$$

The product is 0 everywhere except at $x = 2$.

So $\delta(x-2)$ is multiplied by the value taken by x^2 at $x = 2$:

$$x^2 \times \delta(x-2) = \left[x^2\right]_{x=2} \times \delta(x-2)$$
$$= 4 \times \delta(x-2)$$



In general for any function, $f(x)$, that is continuous at $x = a$,

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

Integrals:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(a)\delta(x-a)dx$$
$$= f(a)\int_{-\infty}^{\infty} \delta(x-a)dx$$
$$= f(a) \qquad \text{[if } f(x) \text{ continuous at } a\text{]}$$

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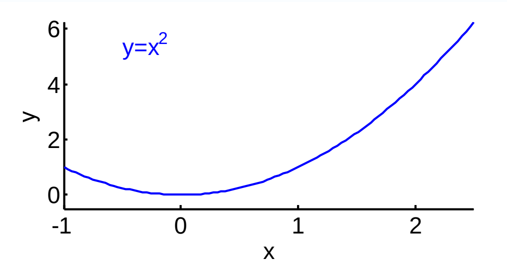

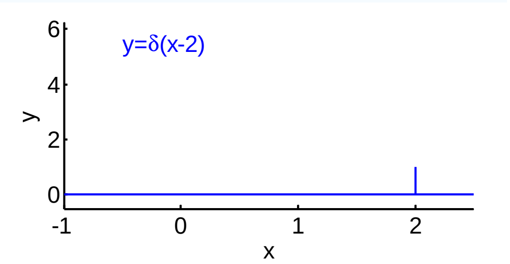

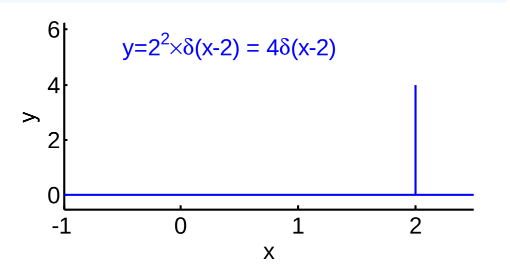

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Example: $\int_{-\infty}^{\infty}\left(3x^2-2x\right)\delta(x-2)dx = \left[3x^2-2x\right]_{x=2} = 8$

Periodic Signals

Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi f t} df$ [Fourier Synthesis]

$U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi f t} dt$ [Fourier Analysis]

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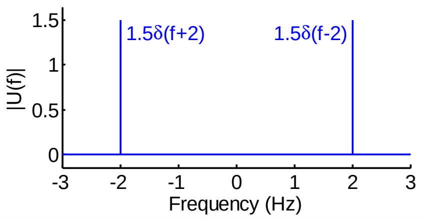
Example: $U(f) = 1.5\delta(f+2) + 1.5\delta(f-2)$



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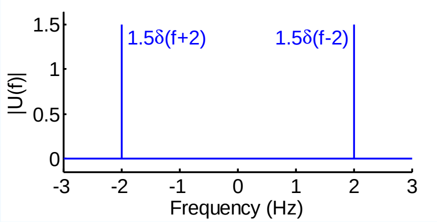

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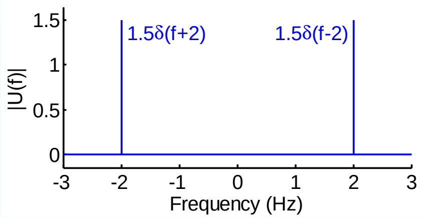

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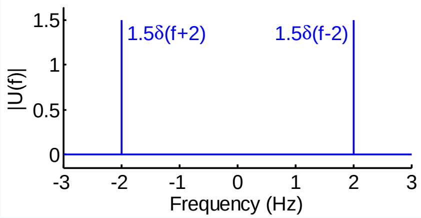

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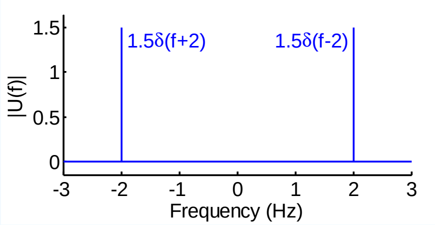

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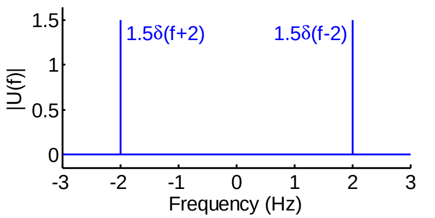

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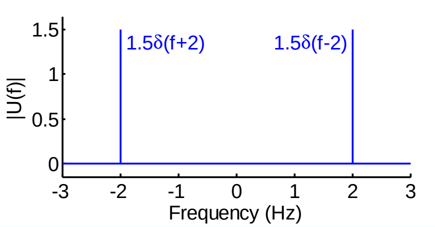

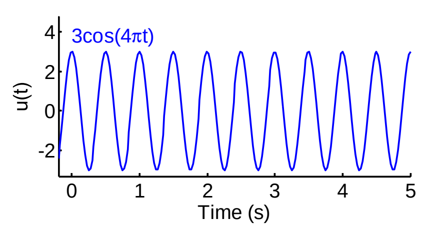

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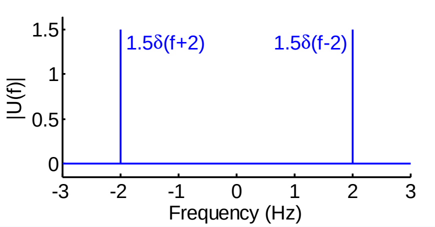

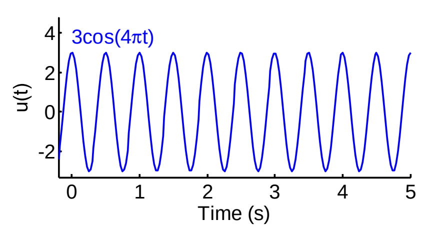

If $u(t)$ is periodic then $U(f)$ is a sum of Dirac delta functions:

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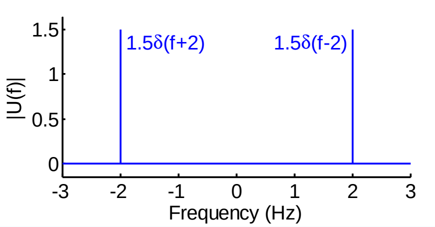

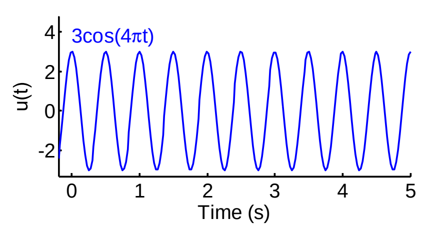

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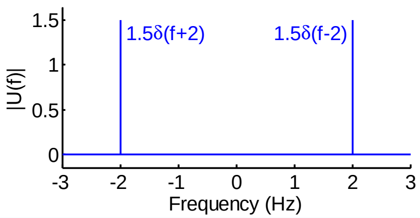

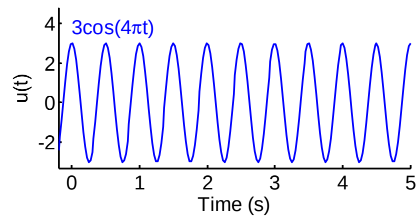

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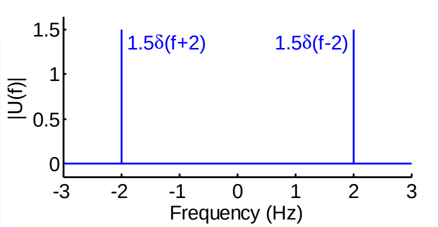

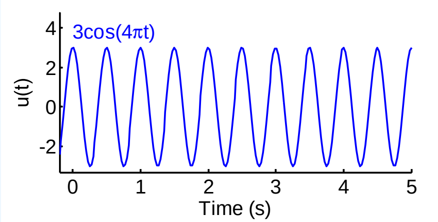

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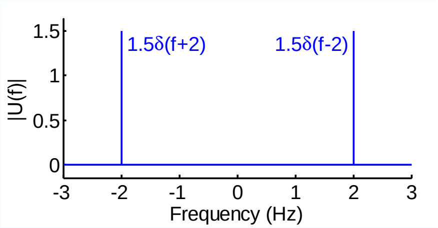

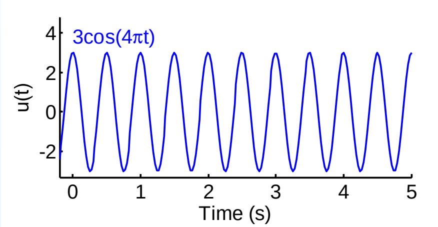

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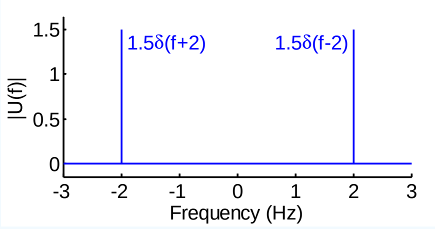

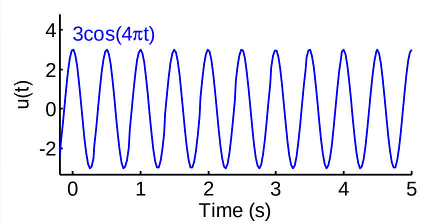

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Duality

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Suppose $v(t) = U(t)$

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Suppose $v(t) = U(t)$, then

$$V(f) = \int_{t=-\infty}^{\infty} v(t)e^{-i2\pi ft}d\tau$$

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$$V(g) = \int_{t=-\infty}^{\infty} U(t) e^{-i2\pi gt} dt$$ [substitute $f = g$, $v(t) = U(t)$]

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$$V(g) = \int_{t=-\infty}^{\infty} U(t)e^{-i2\pi gt}dt$$ [substitute $f = g$, $v(t) = U(t)$]

$$= \int_{f=-\infty}^{\infty} U(f)e^{-i2\pi gf}df$$ [substitute $t = f$]

$$= u(-g)$$

Duality

Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df$ [Fourier Synthesis]

$U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi ft} dt$ [Fourier Analysis]

Dual transform:

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So: $v(t) = U(t) \Rightarrow V(f) = u(-f)$

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Example:

$$u(t) = e^{-|t|}$$

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Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df$ [Fourier Synthesis]

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Example:

$u(t) = e^{-|t|} \quad \Rightarrow \quad U(f) = \frac{2}{1+4\pi^2 f^2}$ [from earlier]

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So: $v(t) = U(t) \Rightarrow V(f) = u(-f)$

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Time Shifting and Scaling

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Time Shifting and Scaling:

Suppose $v(t) = u(at + b)$

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$$= \text{sgn}(a) \int_{\tau=-\infty}^{\infty} u(\tau) e^{-i2\pi f\left(\frac{\tau-b}{a}\right)} \frac{1}{a} d\tau$$

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note that $\pm\infty$ limits swap if $a < 0$ hence $\text{sgn}(a) = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}$

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So: $v(t) = u(at+b) \quad \Rightarrow \quad V(f) = \frac{1}{|a|} e^{i\frac{2\pi f b}{a}} U\left(\frac{f}{a}\right)$

Gaussian Pulse

Gaussian Pulse: $u(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$

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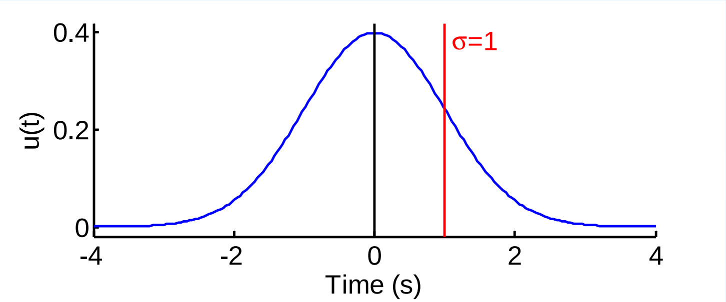

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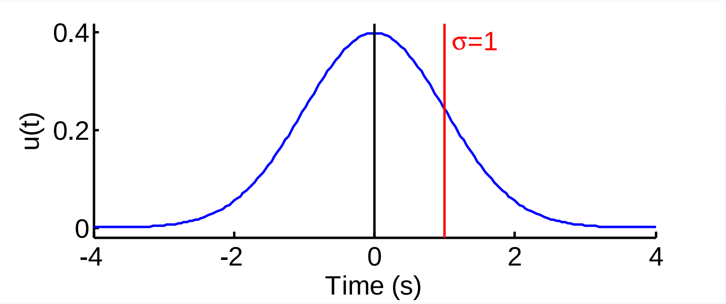
Gaussian Pulse: $u(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$

This is a Normal (or Gaussian) probability distribution, so $\int_{-\infty}^{\infty} u(t)dt = 1$.



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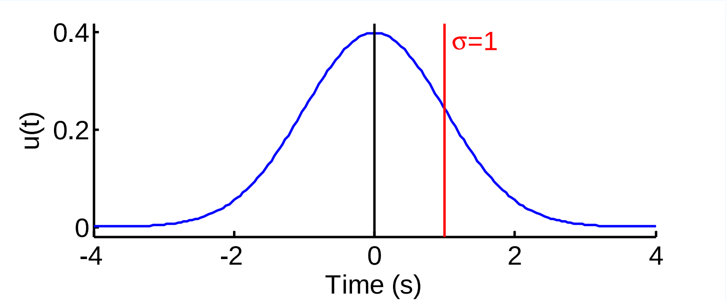
$U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi f t} dt$



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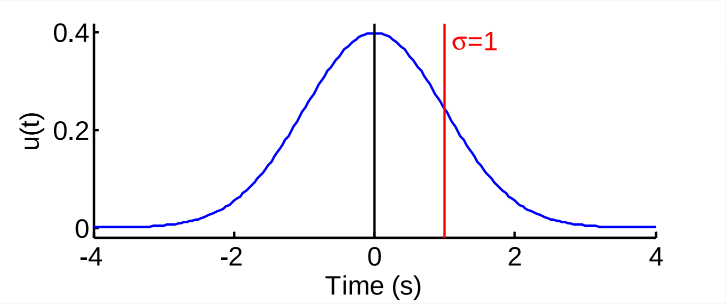

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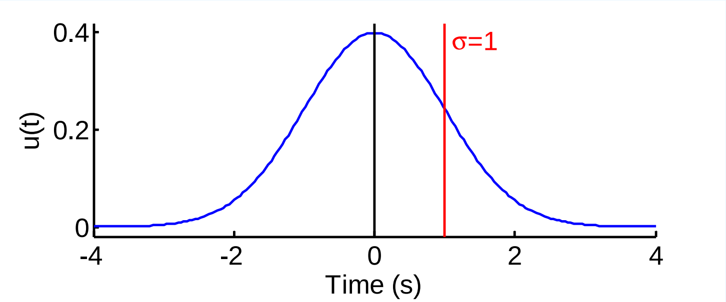

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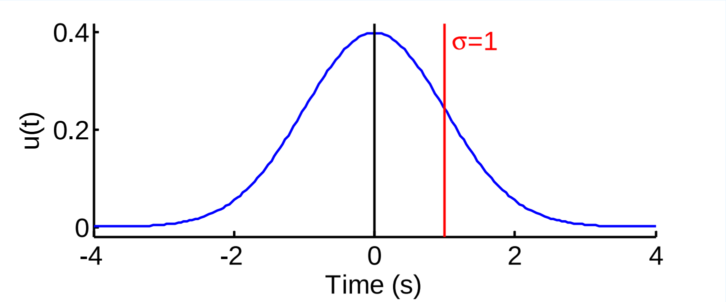

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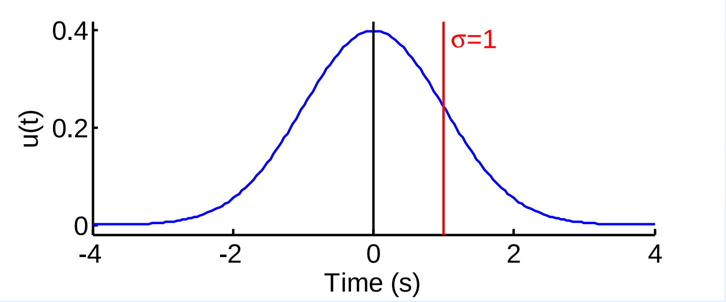
$$= e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t+i2\pi\sigma^2 f\right)^2}dt$$



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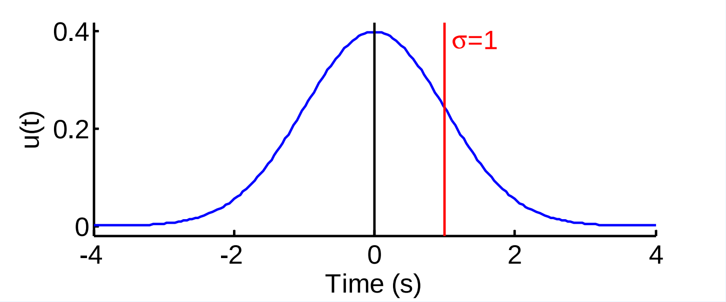
$$\begin{aligned}
U(f) &= \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-i2\pi ft}dt \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t^2+i4\pi\sigma^2 ft\right)}dt \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t^2+i4\pi\sigma^2 ft+\left(i2\pi\sigma^2 f\right)^2-\left(i2\pi\sigma^2 f\right)^2\right)}dt \\
&= e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t+i2\pi\sigma^2 f\right)^2}dt \\
&\stackrel{\text{(i)}}{=} e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2}
\end{aligned}$$



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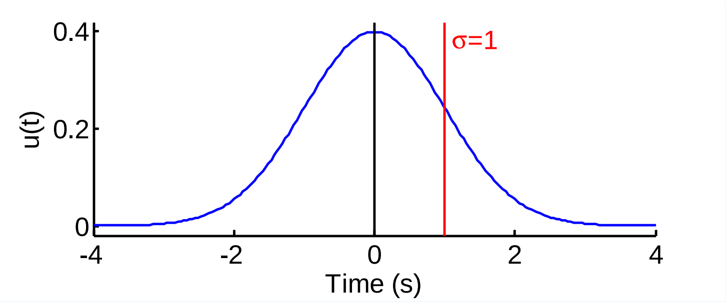
\end{aligned}$$

[(i) uses a result from complex analysis theory that:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t+i2\pi\sigma^2 f\right)^2}dt = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}t^2}dt = 1]$$



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$$= e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t+i2\pi\sigma^2 f\right)^2}dt$$

$$\stackrel{\text{(i)}}{=} e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} = e^{-\frac{1}{2}(2\pi\sigma f)^2}$$

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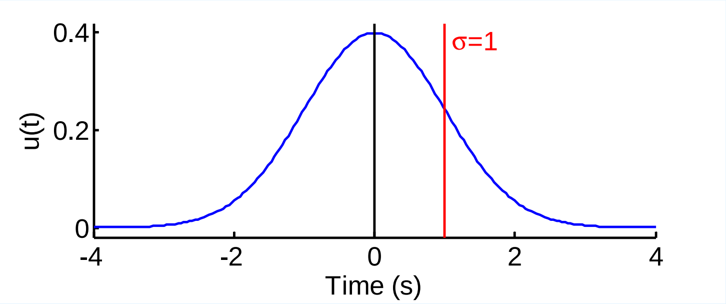

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$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t^2 + i4\pi\sigma^2 f t + \left(i2\pi\sigma^2 f\right)^2 - \left(i2\pi\sigma^2 f\right)^2\right)} dt$$

$$= e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t + i2\pi\sigma^2 f\right)^2} dt$$

$$\stackrel{(i)}{=} e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} = e^{-\frac{1}{2}\left(2\pi\sigma f\right)^2}$$

[(i) uses a result from complex analysis theory that:

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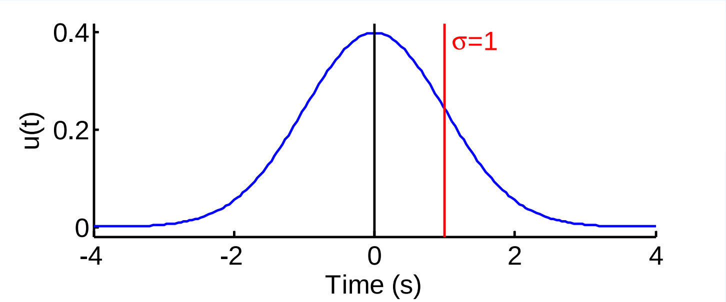

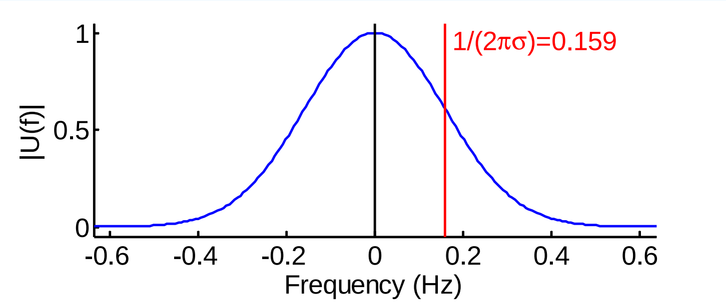

Gaussian Pulse

Gaussian Pulse: $u(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$

This is a Normal (or Gaussian) probability distribution, so $\int_{-\infty}^{\infty} u(t)dt = 1$.

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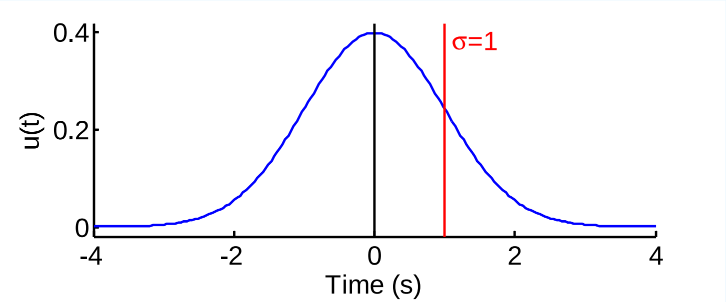

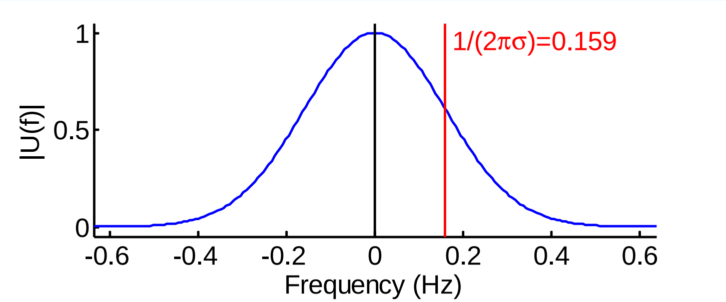

Uniquely, the Fourier Transform of a Gaussian pulse is a Gaussian pulse.

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For further details see RHB Chapter 13.1 (uses ω instead of f)