6: Fourier Transform

Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

The harmonic frequencies are $nF \; \forall n$ and are spaced $F = \frac{1}{T}$ apart.

As T gets larger, the harmonic spacing becomes smaller.

e.g. $T = 1\,\text{s} \Rightarrow F = 1\,\text{Hz}$
$T = 1\,\text{day} \Rightarrow F = 11.57\,\mu\text{Hz}$

If $T \to \infty$ then the harmonic spacing becomes zero, the sum becomes an integral and we get the Fourier Transform:

$$u(t) = \int_{f=-\infty}^{+\infty} U(f) e^{i2\pi ft} df$$

Here, $U(f)$, is the *spectral density* of $u(t)$.

- $U(f)$ is a continuous function of f .
- $U(f)$ is complex-valued.
- $u(t)$ real $\Rightarrow U(f)$ is conjugate symmetric $\Leftrightarrow U(-f) = U(f)^*$.
- Units: if $u(t)$ is in volts, then $U(f)df$ must also be in volts
  $\Rightarrow U(f)$ is in volts/Hz (hence "spectral density").

Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

The summation is over a set of equally spaced frequencies $f_n = nF$ where the spacing between them is $\Delta f = F = \frac{1}{T}$.

$$U_n = \left\langle u(t) e^{-i2\pi nFt} \right\rangle = \Delta f \int_{t=-0.5T}^{0.5T} u(t) e^{-i2\pi nFt} dt$$

Spectral Density: If $u(t)$ has finite energy, $U_n \to 0$ as $\Delta f \to 0$. So we define a spectral density, $U(f_n) = \frac{U_n}{\Delta f}$, on the set of frequencies $\{f_n\}$:

$$U(f_n) = \frac{U_n}{\Delta f} = \int_{t=-0.5T}^{0.5T} u(t) e^{-i2\pi f_n t} dt$$

we can write [Substitute $U_n = U(f_n)\Delta f$]

$$u(t) = \sum_{n=-\infty}^{\infty} U(f_n) e^{i2\pi f_n t} \Delta f$$

Fourier Transform: Now if we take the limit as $\Delta f \to 0$, we get

$$u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df \quad \text{[Fourier Synthesis]}$$

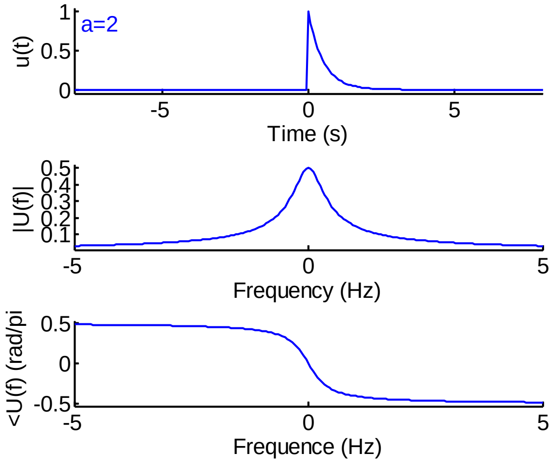
$$U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi ft} dt \quad \text{[Fourier Analysis]}$$

For non-periodic signals $U_n \to 0$ as $\Delta f \to 0$ and $U(f_n) = \frac{U_n}{\Delta f}$ remains finite. However, if $u(t)$ contains an exactly periodic component, then the corresponding $U(f_n)$ will become infinite as $\Delta f \to 0$. We will deal with it.

Fourier Transform Examples

Example 1:

$$u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi f t} dt$$

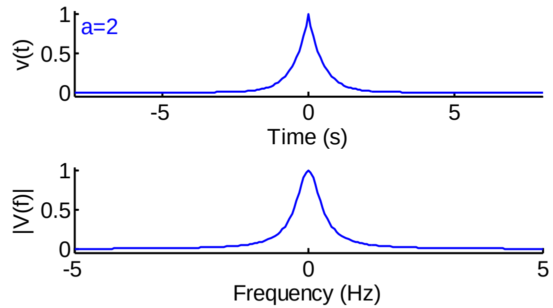
$$= \int_{0}^{\infty} e^{-at} e^{-i2\pi f t} dt$$

$$= \int_{0}^{\infty} e^{(-a-i2\pi f)t} dt$$

$$= \frac{-1}{a+i2\pi f} \left[e^{(-a-i2\pi f)t} \right]_0^{\infty} = \frac{1}{a+i2\pi f}$$

Example 2:

$$v(t) = e^{-a|t|}$$



$$V(f) = \int_{-\infty}^{\infty} v(t) e^{-i2\pi f t} dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-i2\pi f t} dt + \int_{0}^{\infty} e^{-at} e^{-i2\pi f t} dt$$

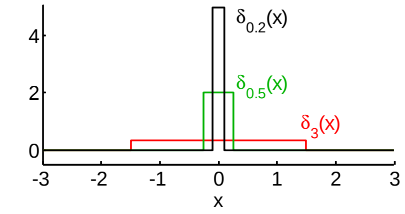
$$= \frac{1}{a-i2\pi f} \left[e^{(a-i2\pi f)t} \right]_{-\infty}^{0} + \frac{-1}{a+i2\pi f} \left[e^{(-a-i2\pi f)t} \right]_0^{\infty}$$

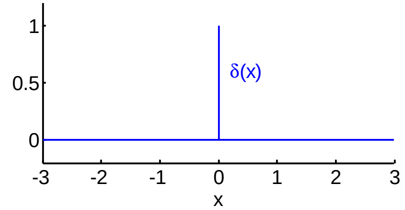
$$= \frac{1}{a-i2\pi f} + \frac{1}{a+i2\pi f} = \frac{2a}{a^2+4\pi^2 f^2}$$

[$v(t)$ real+symmetric
$\Rightarrow V(f)$ real+symmetric]

We define a unit area pulse of width w as

$$d_w(x) = \begin{cases} \frac{1}{w} & -0.5w \le x \le 0.5w \\ 0 & \text{otherwise} \end{cases}$$





This pulse has the property that its integral equals 1 for all values of w:

$$\int_{x=-\infty}^{\infty} d_w(x)dx = 1$$

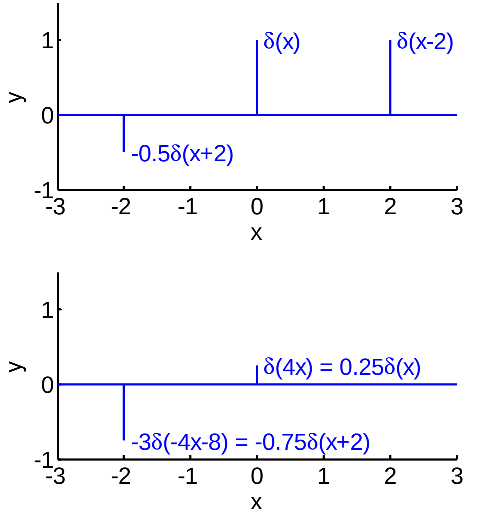
If we make w smaller, the pulse height increases to preserve unit area.

We define the Dirac delta function as $\delta(x) = \lim_{w \to 0} d_w(x)$

- $\delta(x)$ equals zero everywhere except at $x = 0$ where it is infinite.
- However its area still equals $1 \Rightarrow \int_{-\infty}^{\infty} \delta(x)dx = 1$
- We plot the height of $\delta(x)$ as its area rather than its true height of ∞.

$\delta(x)$ is not quite a proper function: it is called a generalized function.

Dirac Delta Function: Scaling and Translation



Translation: $\delta(x-a)$

$\delta(x)$ is a pulse at $x=0$
$\delta(x-a)$ is a pulse at $x=a$

Amplitude Scaling: $b\delta(x)$

$\delta(x)$ has an area of $1 \Leftrightarrow \int_{-\infty}^{\infty}\delta(x)dx = 1$

$b\delta(x)$ has an area of b since

$$\int_{-\infty}^{\infty}(b\delta(x))\,dx = b\int_{-\infty}^{\infty}\delta(x)dx = b$$

b can be a complex number (on a graph, we then plot only its magnitude)

Time Scaling: $\delta(cx)$

$c>0$: $\int_{x=-\infty}^{\infty}\delta(cx)dx = \int_{y=-\infty}^{\infty}\delta(y)\frac{dy}{c}$ [sub $y=cx$]

$$= \frac{1}{c}\int_{y=-\infty}^{\infty}\delta(y)dy = \frac{1}{c} = \frac{1}{|c|}$$

$c<0$: $\int_{x=-\infty}^{\infty}\delta(cx)dx = \int_{y=+\infty}^{-\infty}\delta(y)\frac{dy}{c}$ [sub $y=cx$]

$$= \frac{-1}{c}\int_{y=-\infty}^{+\infty}\delta(y)dy = \frac{-1}{c} = \frac{1}{|c|}$$

In general, $\delta(cx) = \frac{1}{|c|}\delta(x)$ for $c \neq 0$

If we multiply $\delta(x-a)$ by a function of x:

$$y = x^2 \times \delta(x-2)$$

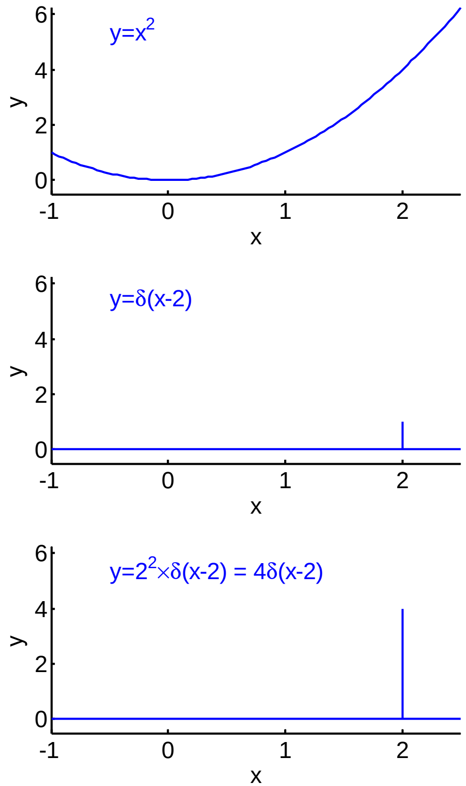
The product is 0 everywhere except at $x=2$.

So $\delta(x-2)$ is multiplied by the value taken by x^2 at $x=2$:

$$x^2 \times \delta(x-2) = \left[x^2\right]_{x=2} \times \delta(x-2)$$
$$= 4 \times \delta(x-2)$$

In general for any function, $f(x)$, that is continuous at $x=a$,

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$



Integrals:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(a)\delta(x-a)dx$$
$$= f(a)\int_{-\infty}^{\infty} \delta(x-a)dx$$
$$= f(a) \qquad \text{[if } f(x) \text{ continuous at } a\text{]}$$

Example: $\int_{-\infty}^{\infty} \left(3x^2 - 2x\right)\delta(x-2)dx = \left[3x^2 - 2x\right]_{x=2} = 8$

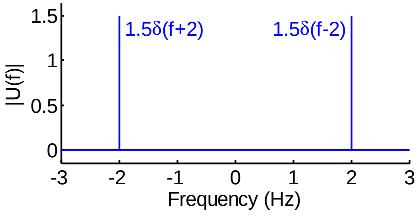
Periodic Signals

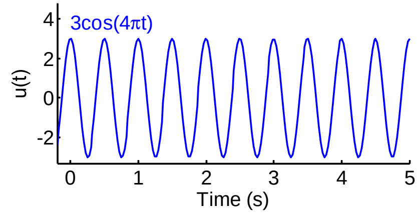
Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df$ [Fourier Synthesis]

$U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi ft} dt$ [Fourier Analysis]

Example: $U(f) = 1.5\delta(f+2) + 1.5\delta(f-2)$

$$\begin{aligned} u(t) &= \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df \\ &= \int_{-\infty}^{\infty} 1.5\delta(f+2) e^{i2\pi ft} df \\ &\quad + \int_{-\infty}^{\infty} 1.5\delta(f-2) e^{i2\pi ft} df \\ &= 1.5 \left[e^{i2\pi ft}\right]_{f=-2} + 1.5 \left[e^{i2\pi ft}\right]_{f=+2} \\ &= 1.5 \left(e^{i4\pi t} + e^{-i4\pi t}\right) = 3\cos 4\pi t \end{aligned}$$





If $u(t)$ is periodic then $U(f)$ is a sum of Dirac delta functions:

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} \quad \Rightarrow \quad U(f) = \sum_{n=-\infty}^{\infty} U_n \delta(f - nF)$$

Proof: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} U_n \delta(f - nF) e^{i2\pi ft} df \\ &= \sum_{n=-\infty}^{\infty} U_n \int_{-\infty}^{\infty} \delta(f - nF) e^{i2\pi ft} df \\ &= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} \end{aligned}$$

Duality

Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft}df$ [Fourier Synthesis]

$U(f) = \int_{t=-\infty}^{\infty} u(t)e^{-i2\pi ft}dt$ [Fourier Analysis]

Dual transform:

Suppose $v(t) = U(t)$, then

$V(f) = \int_{t=-\infty}^{\infty} v(t)e^{-i2\pi ft}d\tau$

$V(g) = \int_{t=-\infty}^{\infty} U(t)e^{-i2\pi gt}dt$ [substitute $f = g$, $v(t) = U(t)$]

$= \int_{f=-\infty}^{\infty} U(f)e^{-i2\pi gf}df$ [substitute $t = f$]

$= u(-g)$

So: $v(t) = U(t) \quad \Rightarrow \quad V(f) = u(-f)$

Example:

$u(t) = e^{-|t|} \quad \Rightarrow \quad U(f) = \frac{2}{1+4\pi^2 f^2}$ [from earlier]

$v(t) = \frac{2}{1+4\pi^2 t^2} \quad \Rightarrow \quad V(f) = e^{-|-f|} = e^{-|f|}$

Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df$ [Fourier Synthesis]

$U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi ft} dt$ [Fourier Analysis]

Time Shifting and Scaling:

Suppose $v(t) = u(at+b)$, then

$$V(f) = \int_{t=-\infty}^{\infty} u(at+b) e^{-i2\pi ft} dt$$ [now sub $\tau = at+b$]

$$= \operatorname{sgn}(a) \int_{\tau=-\infty}^{\infty} u(\tau) e^{-i2\pi f\left(\frac{\tau-b}{a}\right)} \frac{1}{a} d\tau$$

note that $\pm\infty$ limits swap if $a < 0$ hence $\operatorname{sgn}(a) = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}$

$$= \frac{1}{|a|} e^{i\frac{2\pi fb}{a}} \int_{\tau=-\infty}^{\infty} u(\tau) e^{-i2\pi \frac{f}{a}\tau} d\tau$$

$$= \frac{1}{|a|} e^{i\frac{2\pi fb}{a}} U\left(\frac{f}{a}\right)$$

So: $v(t) = u(at+b) \quad \Rightarrow \quad V(f) = \frac{1}{|a|} e^{i\frac{2\pi fb}{a}} U\left(\frac{f}{a}\right)$

Gaussian Pulse: $u(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$

This is a Normal (or Gaussian) probability distribution, so $\int_{-\infty}^{\infty} u(t)dt = 1$.

$$U(f) = \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-i2\pi ft}dt$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t^2+i4\pi\sigma^2 ft\right)}dt$$

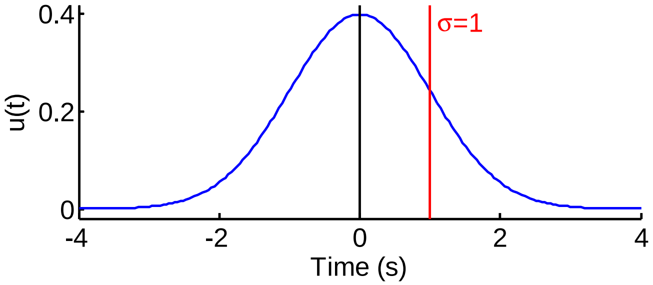
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t^2+i4\pi\sigma^2 ft+\left(i2\pi\sigma^2 f\right)^2-\left(i2\pi\sigma^2 f\right)^2\right)}dt$$

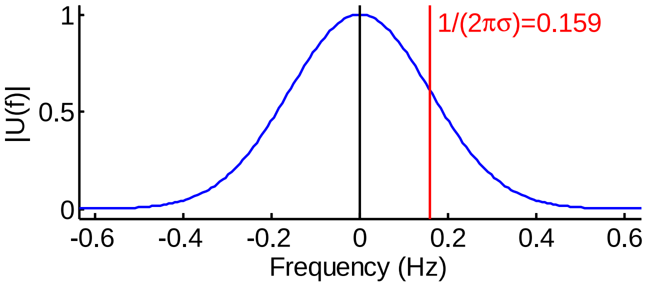
$$= e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t+i2\pi\sigma^2 f\right)^2}dt$$

$$\stackrel{(i)}{=} e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} = e^{-\frac{1}{2}(2\pi\sigma f)^2}$$

[(i) uses a result from complex analysis theory that:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t+i2\pi\sigma^2 f\right)^2}dt = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}t^2}dt = 1]$$





Uniquely, the Fourier Transform of a Gaussian pulse is a Gaussian pulse.

Summary

- Fourier Transform:
  - Inverse transform (synthesis): $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df$
  - Forward transform (analysis): $U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi ft} dt$
    - $U(f)$ is the spectral density function (e.g. Volts/Hz)
- Dirac Delta Function:
  - $\delta(t)$ is a zero-width infinite-height pulse with $\int_{-\infty}^{\infty} \delta(t) dt = 1$
  - Integral: $\int_{-\infty}^{\infty} f(t)\delta(t-a) = f(a)$
  - Scaling: $\delta(ct) = \frac{1}{|c|}\delta(t)$
- Periodic Signals: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

  $\Rightarrow U(f) = \sum_{n=-\infty}^{\infty} U_n \delta(f - nF)$
- Fourier Transform Properties:
  - $v(t) = U(t) \quad \Rightarrow \quad V(f) = u(-f)$
  - $v(t) = u(at+b) \quad \Rightarrow \quad V(f) = \frac{1}{|a|} e^{i\frac{2\pi fb}{a}} U\left(\frac{f}{a}\right)$
  - $v(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2} \quad \Rightarrow \quad V(f) = e^{-\frac{1}{2}(2\pi\sigma f)^2}$

For further details see RHB Chapter 13.1 (uses ω instead of f)