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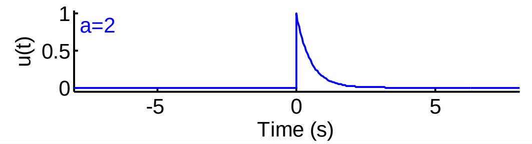
Multiplication Example

$$u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



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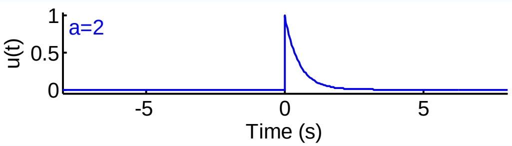

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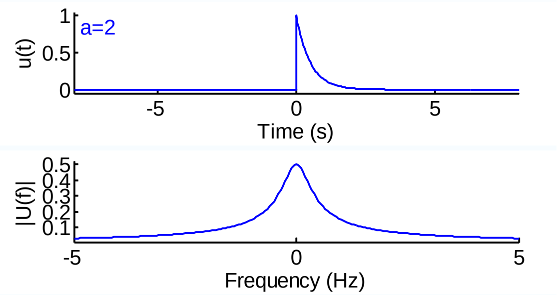

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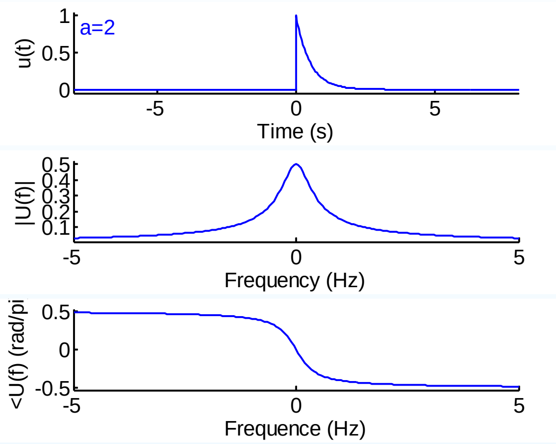

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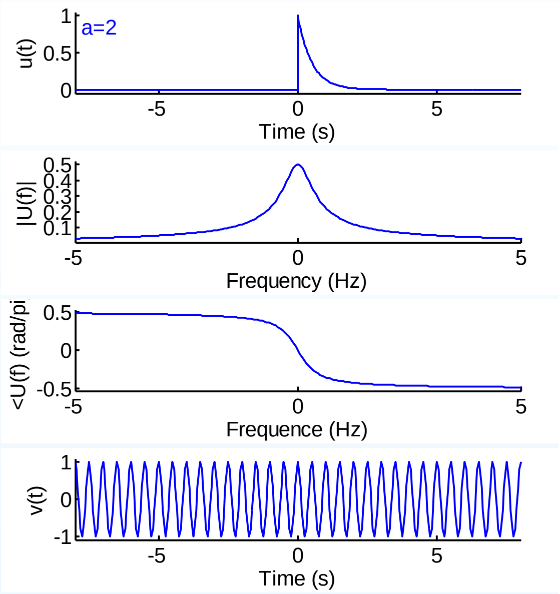

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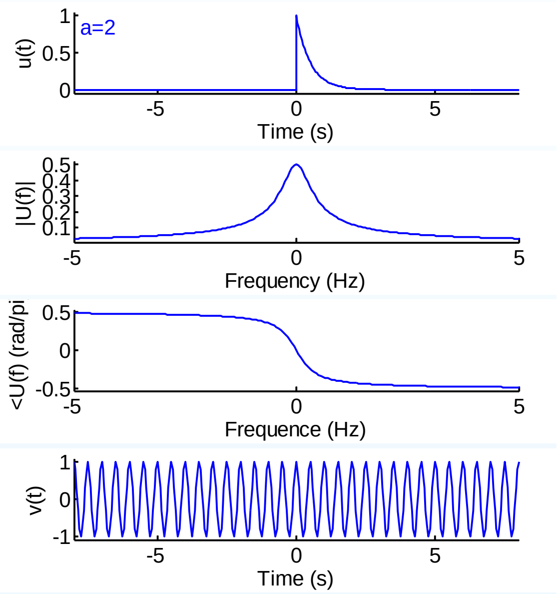

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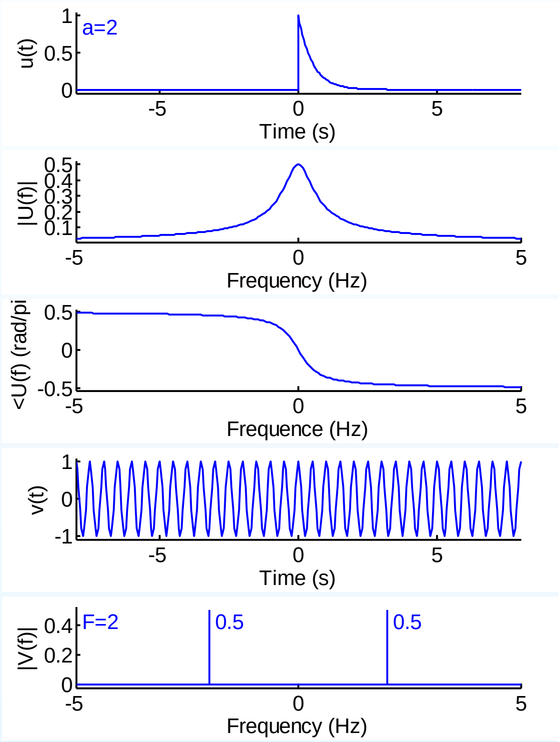

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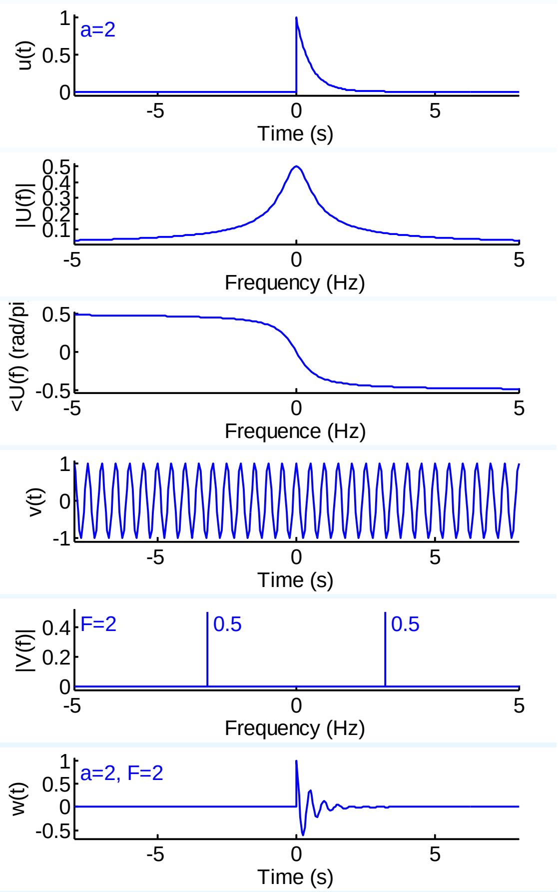

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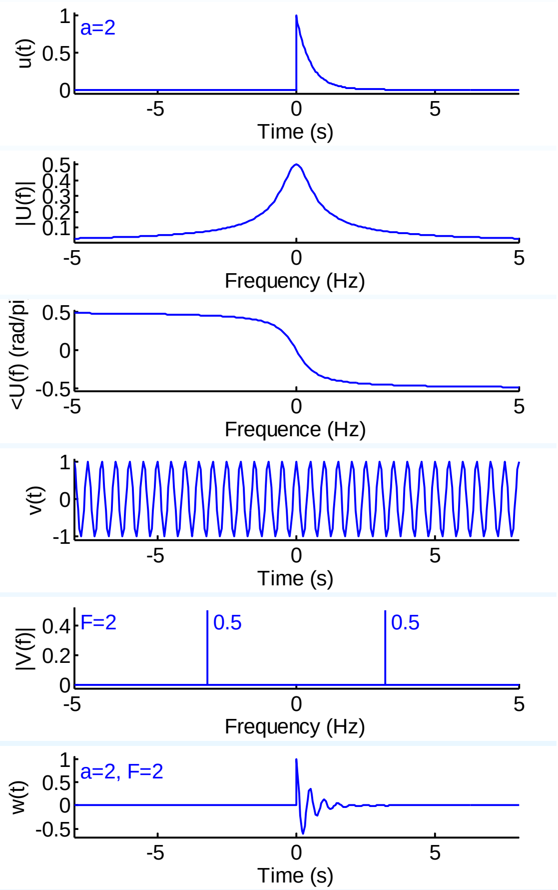

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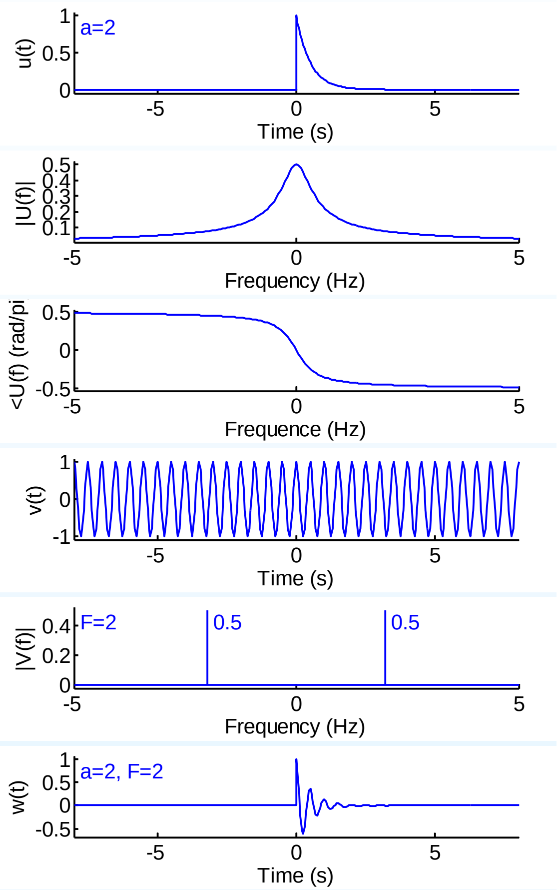

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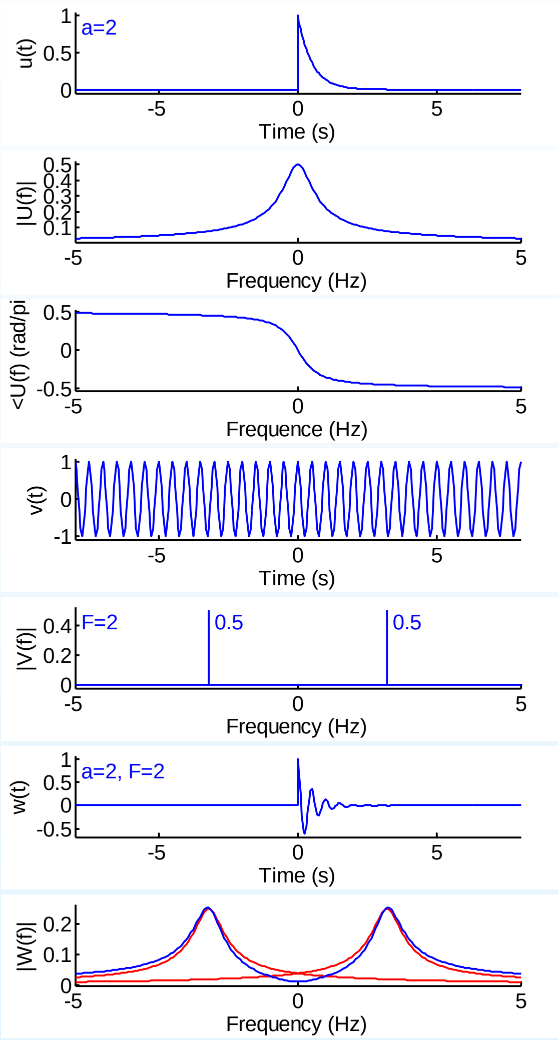

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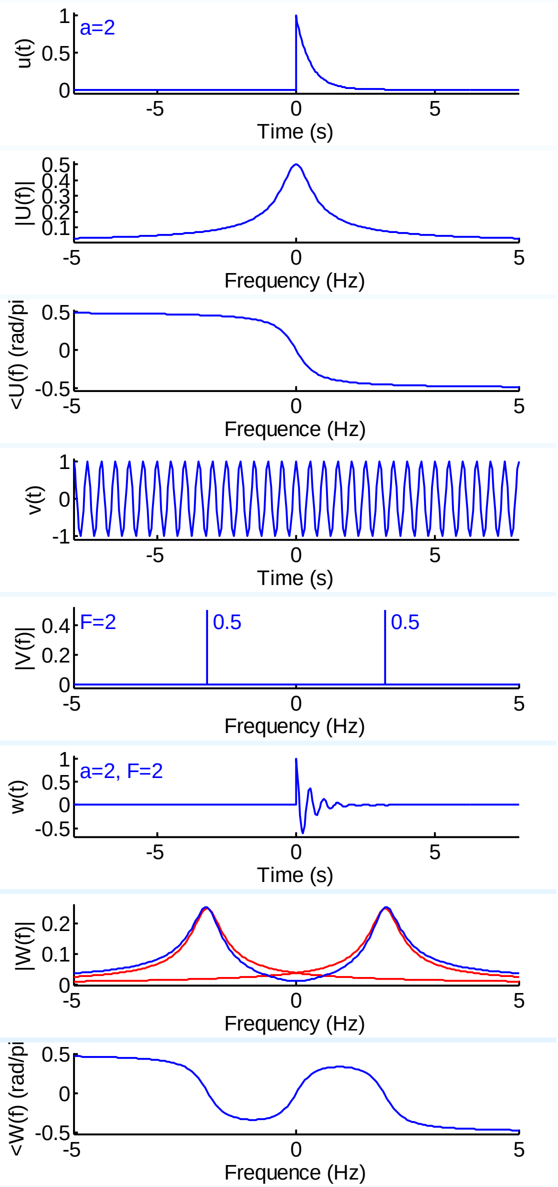

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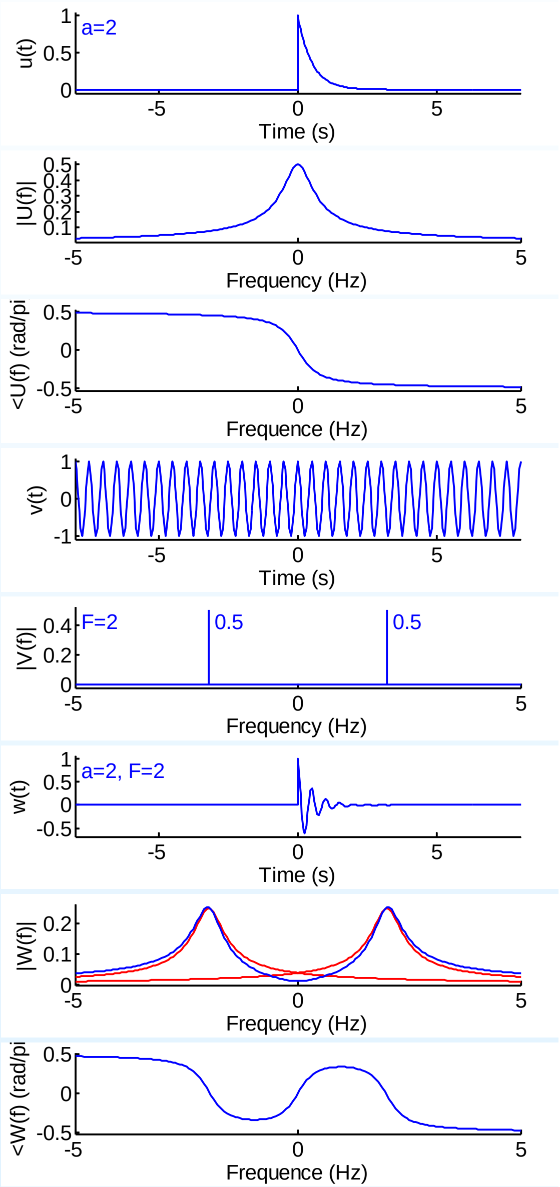
$$= \frac{0.5}{a+i2\pi(f+F)} + \frac{0.5}{a+i2\pi(f-F)}$$

If $V(f)$ consists entirely of Dirac impulses then $U(f) * V(f)$ just replaces each impulse with a complete copy of $U(f)$ scaled by the area of the impulse and shifted so that $0\ \mathrm{Hz}$ lies on the impulse. Then add the overlapping complex spectra.



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Given $u(t)$, $v(t)$ and $w(t)$ satisfying

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$$x(t) = U(t) \quad \Leftrightarrow \quad X(f) = u(-f)$$ [duality]

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Now the convolution property becomes:

$$w(-f) = u(-f)v(-f) \quad \Leftrightarrow \quad W(t) = U(t) * V(t)$$

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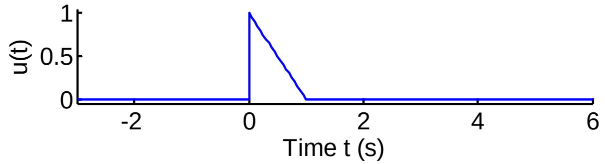

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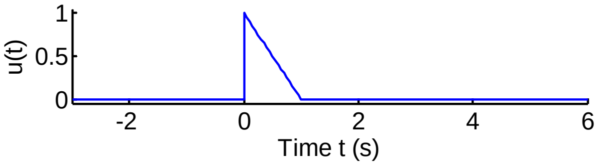

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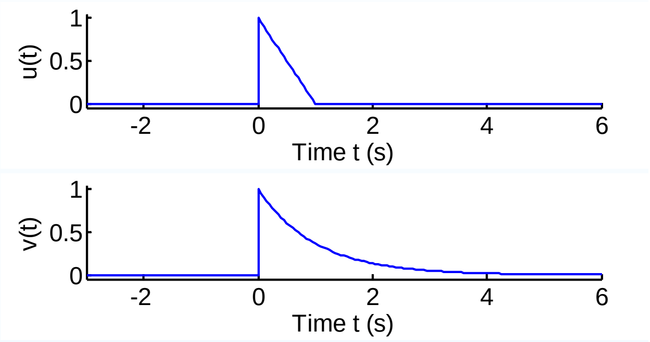

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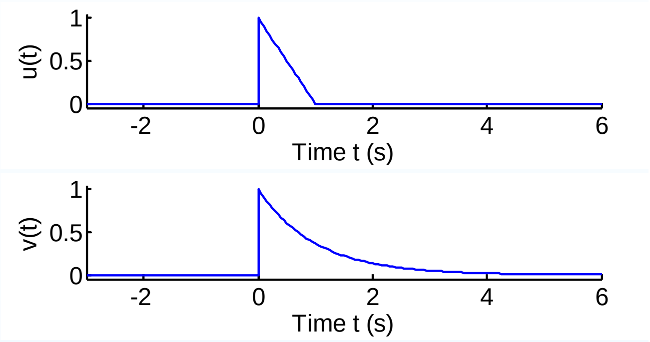

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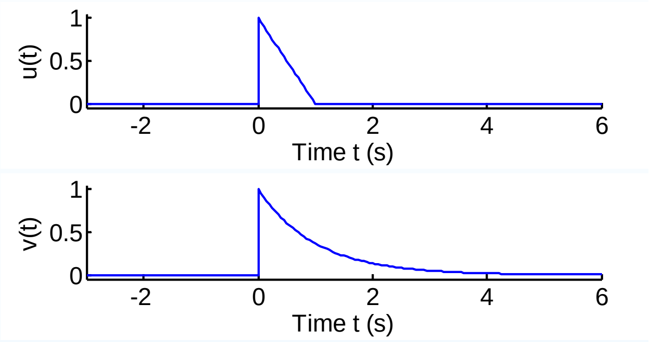

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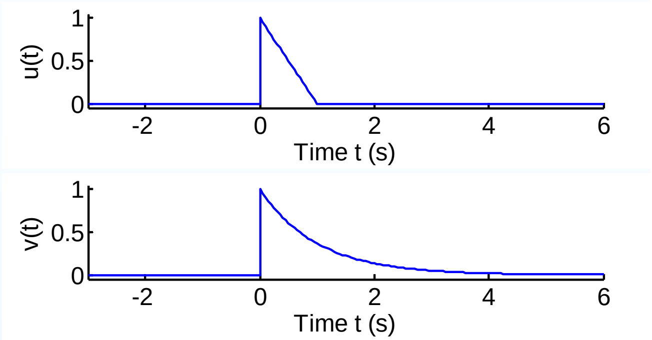

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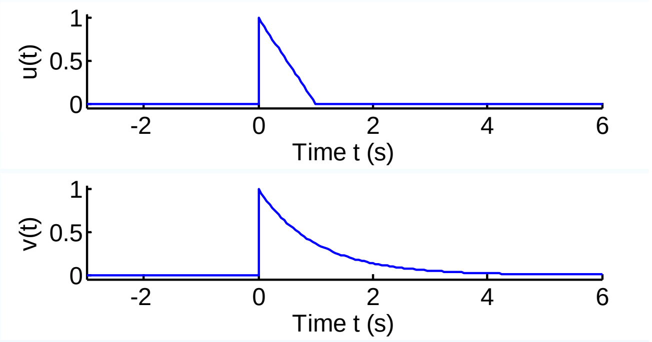

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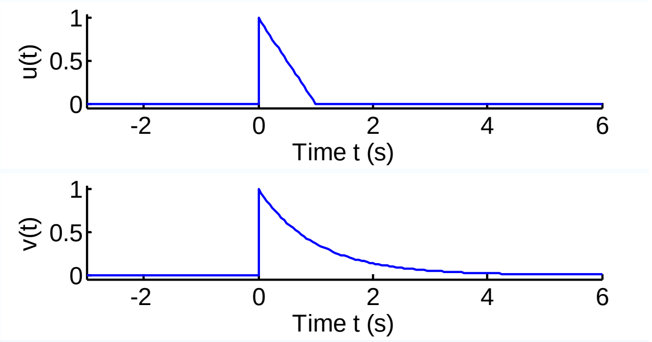

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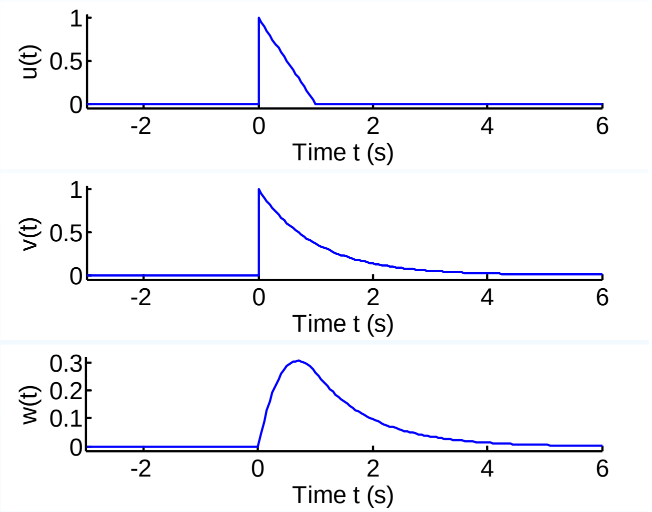

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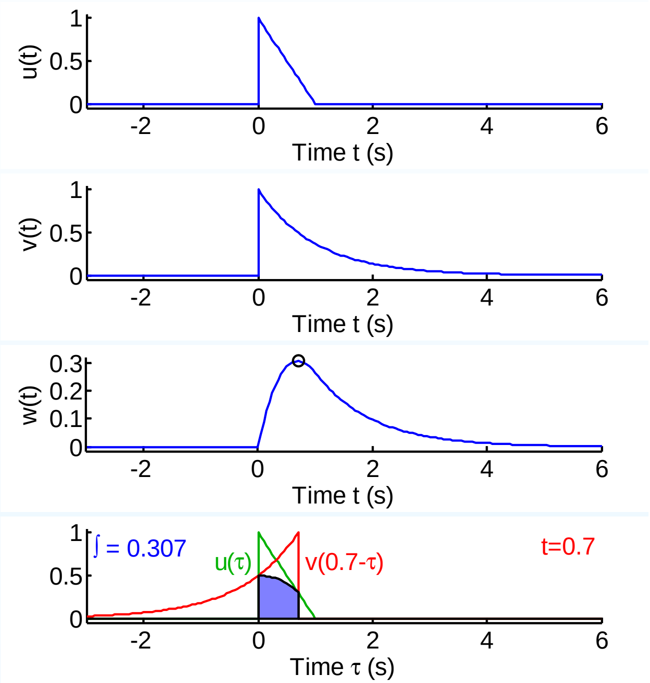

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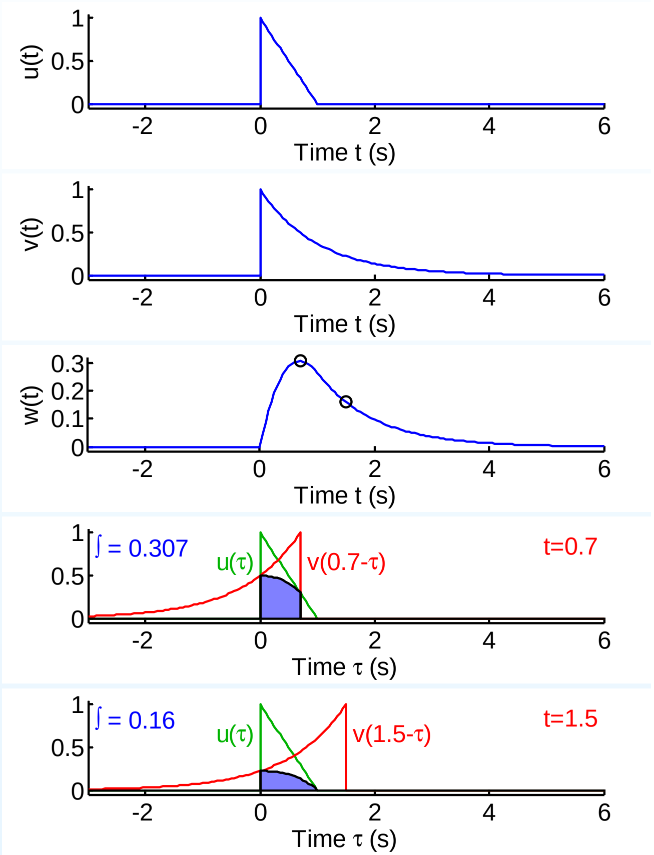

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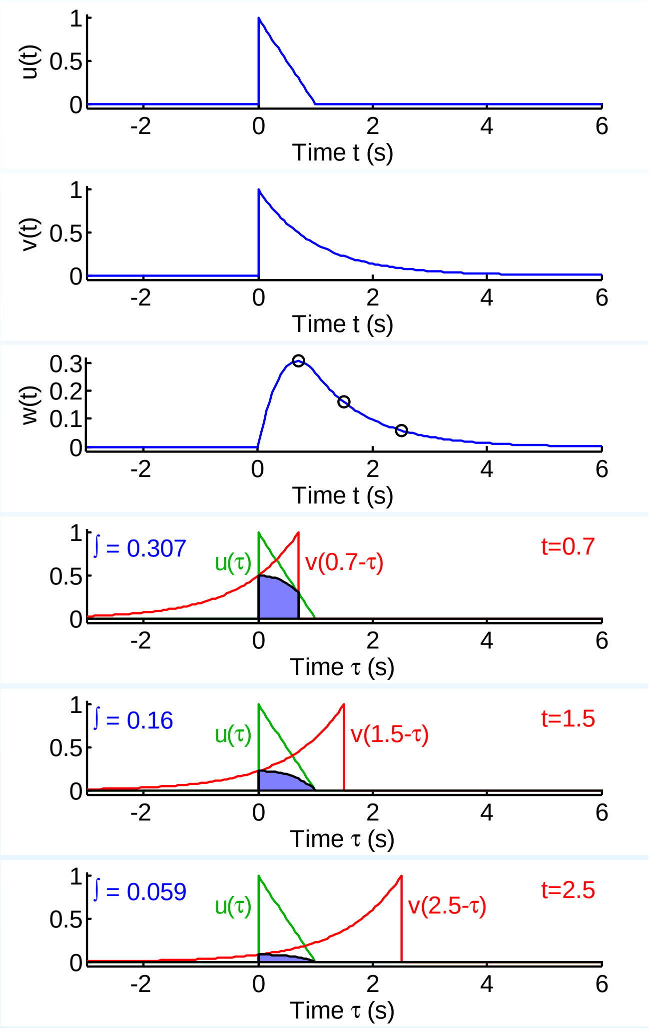

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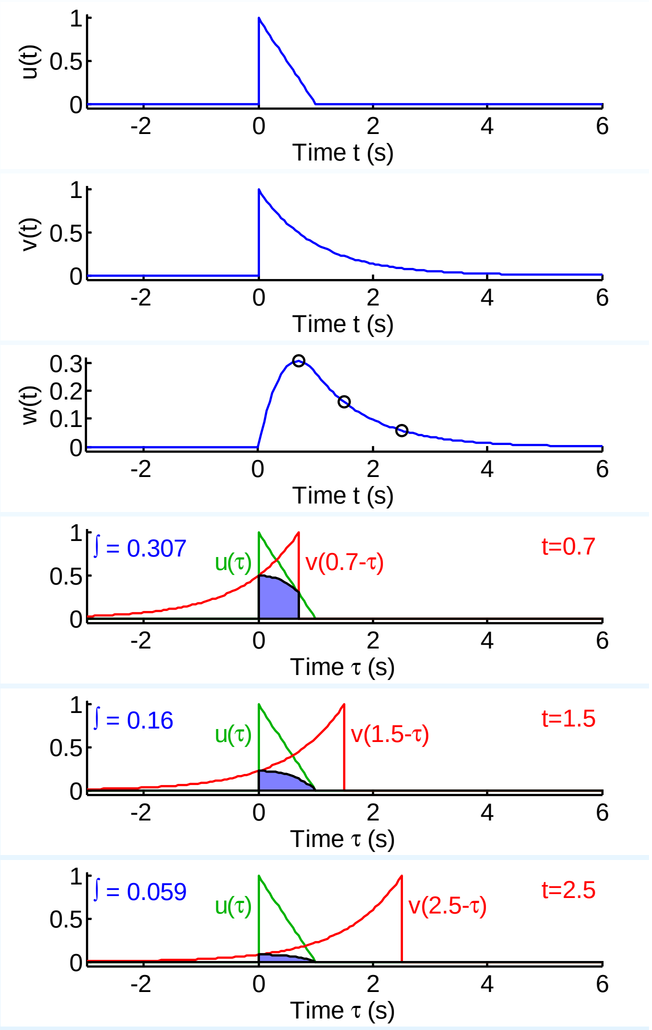

Note how $v(t-\tau)$ is time-reversed (because of the $-\tau$) and time-shifted to put the time origin at $\tau = t$.

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How to recognise a convolution integral:

the arguments of $u(\cdots)$ and $v(\cdots)$ sum to a constant.

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Lemma:

$$X(f) = \delta(f-g) \Rightarrow x(t) = \int \delta(f-g) e^{i2\pi ft} df = e^{i2\pi gt}$$

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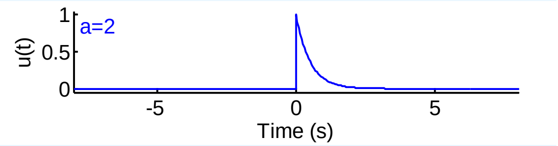

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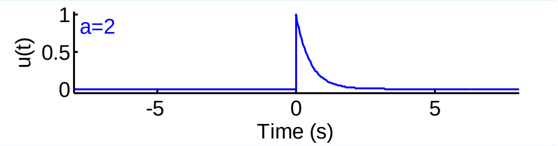

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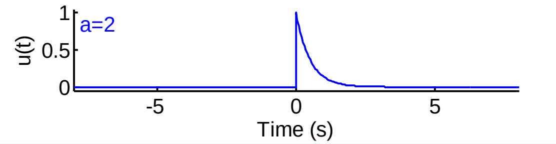

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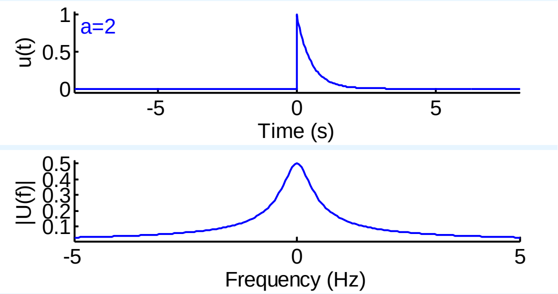

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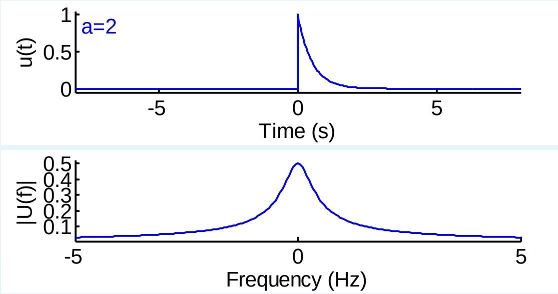

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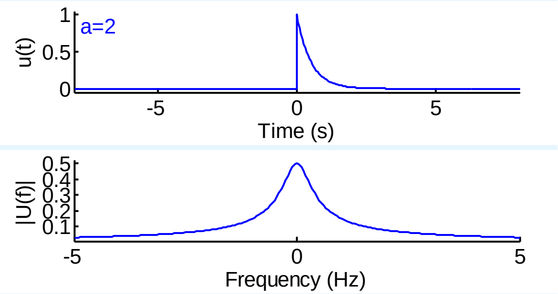

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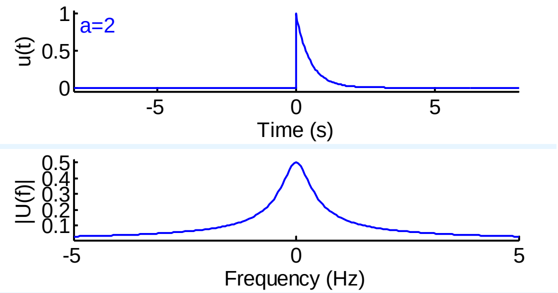

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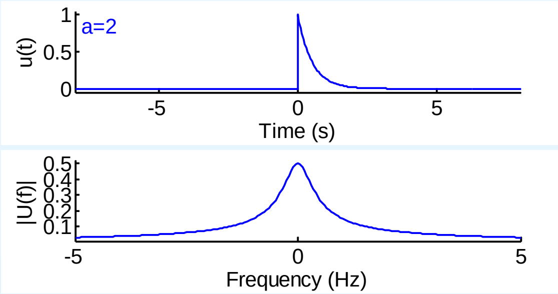

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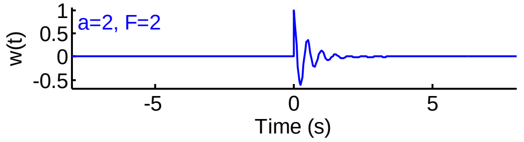

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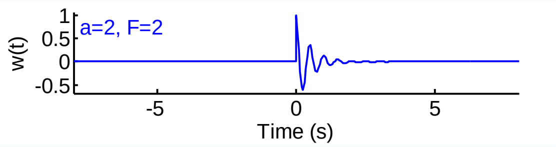

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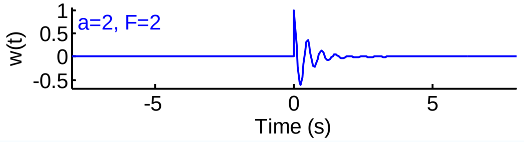

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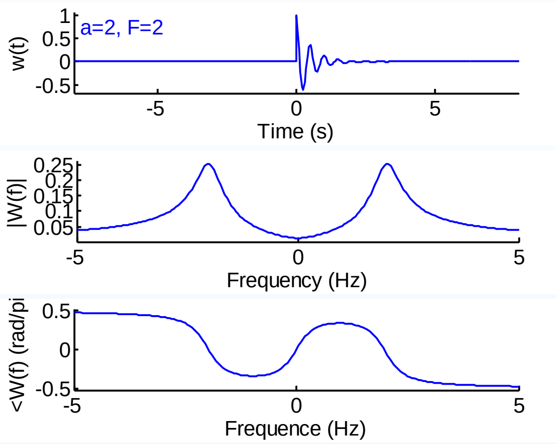

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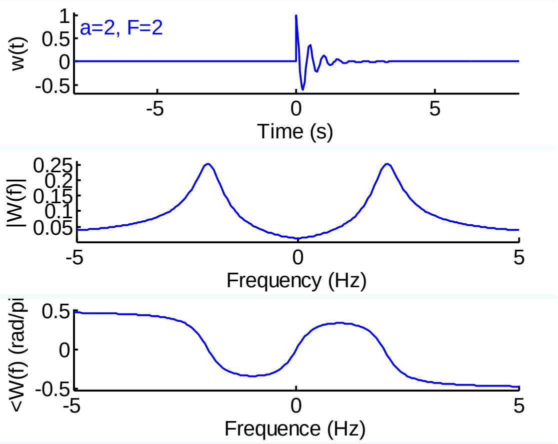

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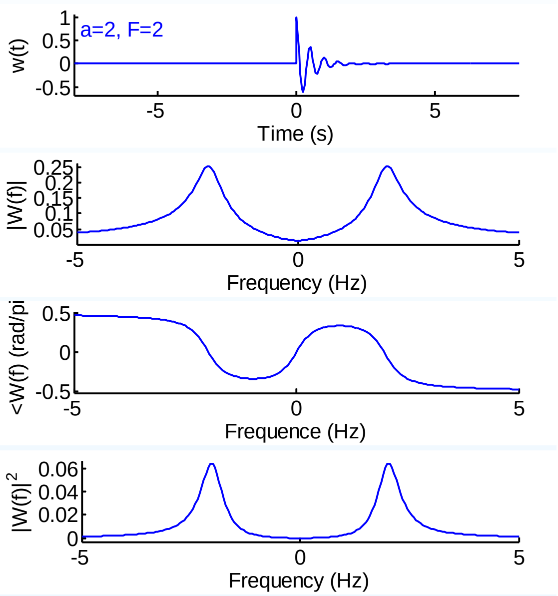

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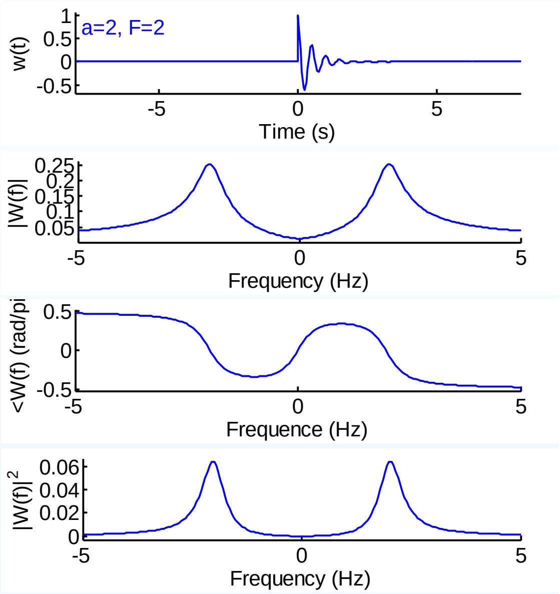

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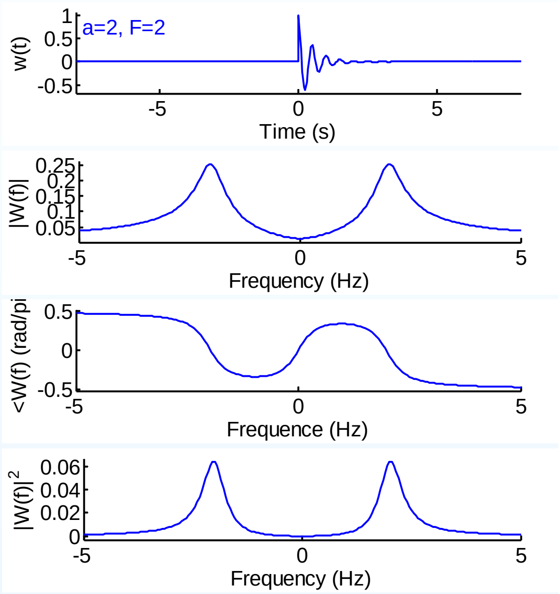

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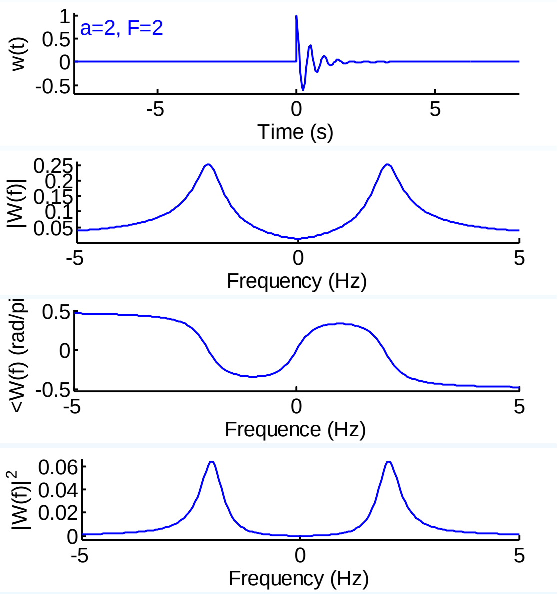

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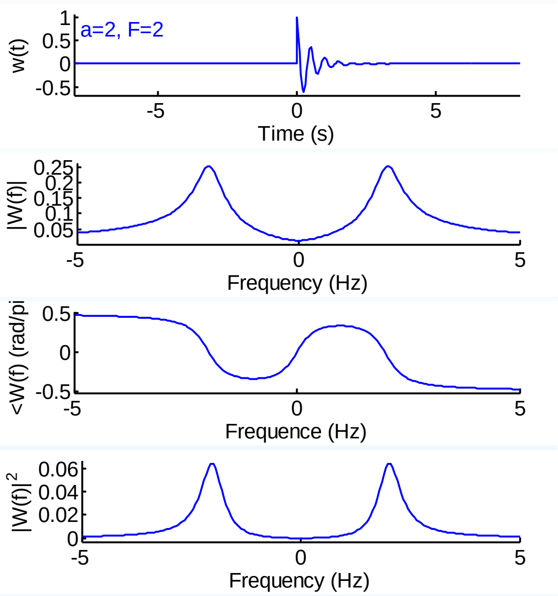

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For further details see RHB Chapter 13.1