8: Correlation

The *cross-correlation* between two signals $u(t)$ and $v(t)$ is

$$w(t) = u(t) \otimes v(t) \triangleq \int_{-\infty}^{\infty} u^*(\tau)v(\tau + t)d\tau$$
$$= \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau \qquad \text{[sub: } \tau \to \tau - t\text{]}$$

The complex conjugate, $u^*(\tau)$ makes no difference if $u(t)$ is real-valued but makes the definition work even if $u(t)$ is complex-valued.

Correlation versus Convolution:

$$u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau)v(\tau + t)d\tau \qquad \text{[correlation]}$$
$$u(t) * v(t) = \int_{-\infty}^{\infty} u(\tau)v(t - \tau)d\tau \qquad \text{[convolution]}$$

Unlike convolution, the integration variable, τ, has the same sign in the arguments of $u(\cdots)$ and $v(\cdots)$ so the arguments have a constant difference instead of a constant sum (i.e. $v(t)$ is not time-flipped).

Notes: (a) The argument of $w(t)$ is called the "lag" (= delay of u versus v).
(b) Some people write $u(t) \star v(t)$ instead of $u(t) \otimes v(t)$.
(c) Some swap u and v and/or negate t in the integral.

It is all rather inconsistent ☹.

Signal Matching

Cross correlation is used to find where two signals match: $u(t)$ is the test waveform.

Example 1:

$v(t)$ contains $u(t)$ with an unknown delay and added noise.

$w(t) = u(t) \otimes v(t)$

$= \int u^*(\tau - t)v(\tau)dt$ gives a peak at the time lag where $u(\tau - t)$ best matches $v(\tau)$; in this case at $t = 450$

Example 2:

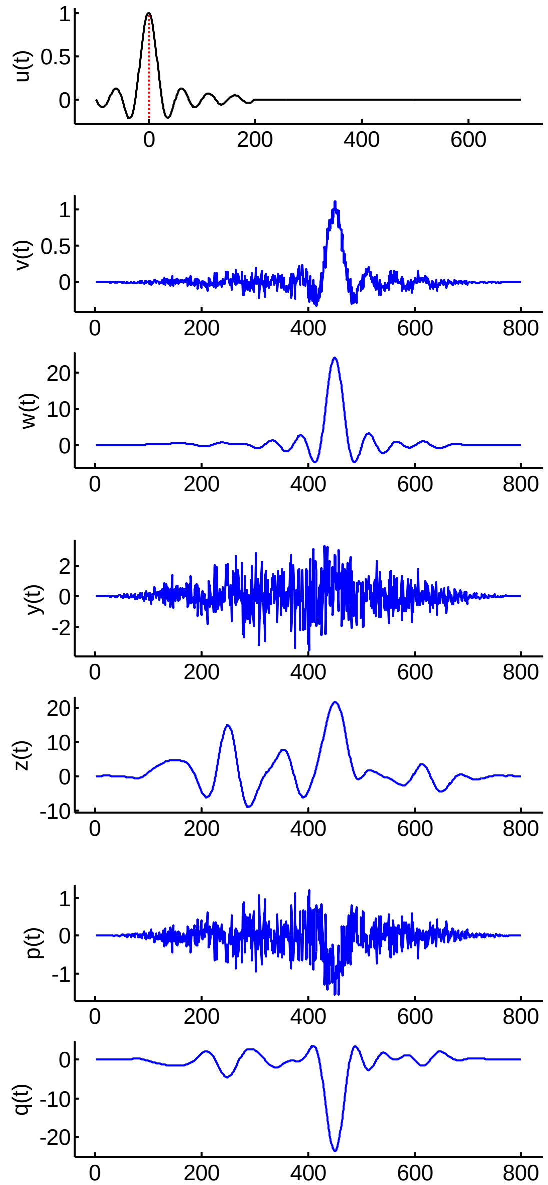
$y(t)$ is the same as $v(t)$ with more noise

$z(t) = u(t) \otimes y(t)$ can still detect the correct time delay (hard for humans)

Example 3:

$p(t)$ contains $-u(t)$ so that

$q(t) = u(t) \otimes p(t)$ has a negative peak



Correlation: $w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau$

If we define $x(t) = u^*(-t)$ then

$$x(t) * v(t) \triangleq \int_{-\infty}^{\infty} x(t-\tau)v(\tau)d\tau = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau$$
$$= u(t) \otimes v(t)$$

Fourier Transform of $x(t)$:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} u^*(-t)e^{-i2\pi ft}dt$$
$$= \int_{-\infty}^{\infty} u^*(t)e^{i2\pi ft}dt = \left(\int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt\right)^*$$
$$= U^*(f)$$

So $w(t) = x(t) * v(t) \quad \Rightarrow \quad W(f) = X(f)V(f) = U^*(f)V(f)$

Hence the Cross-correlation theorem:

$$w(t) = u(t) \otimes v(t) \qquad \Leftrightarrow \qquad W(f) = U^*(f)V(f)$$
$$= u^*(-t) * v(t)$$

Note that, unlike convolution, correlation is not associative or commutative:

$$v(t) \otimes u(t) = v^*(-t) * u(t) = u(t) * v^*(-t) = w^*(-t)$$

Normalized Cross-correlation

Correlation: $w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau$

If we define $y(t) = u(t - t_0)$ for some fixed t_0, then $E_y = E_u$:

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 \, dt = \int_{-\infty}^{\infty} |u(t - t_0)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} |u(\tau)|^2 \, d\tau = E_u \qquad [t \to \tau + t_0]$$

Cauchy-Schwarz inequality: $\left| \int_{-\infty}^{\infty} y^*(\tau)v(\tau)d\tau \right|^2 \le E_y E_v$

$$\Rightarrow |w(t_0)|^2 = \left| \int_{-\infty}^{\infty} u^*(\tau - t_0)v(\tau)d\tau \right|^2 \le E_y E_v = E_u E_v$$

but t_0 was arbitrary, so we must have $|w(t)| \le \sqrt{E_u E_v}$ for all t

We can define the *normalized cross-correlation*

$$z(t) = \frac{u(t) \otimes v(t)}{\sqrt{E_u E_v}}$$

with properties: (1) $|z(t)| \le 1$ for all t

(2) $|z(t_0)| = 1 \Leftrightarrow v(\tau) = \alpha u(\tau - t_0)$ with α constant

[Cauchy-Schwarz Inequality Proof]

You do not need to memorize this proof

We want to prove the Cauchy-Schwarz Inequality: $\left|\int_{-\infty}^{\infty} u^*(t)v(t)dt\right|^2 \le E_u E_v$

where $E_u \triangleq \int_{-\infty}^{\infty} |u(t)|^2 \, dt$.

Suppose we define $w \triangleq \int_{-\infty}^{\infty} u^*(t)v(t)dt$. Then,

$$
\begin{aligned}
0 &\le \int |E_v u(t) - w^* v(t)|^2 \, dt && [|\cdots|^2 \text{ always } \ge 0] \\
&= \int (E_v u^*(t) - w v^*(t)) (E_v u(t) - w^* v(t)) \, dt && [|z|^2 = z^* z] \\
&= E_v^2 \int u^*(t)u(t)dt + |w|^2 \int v^*(t)v(t)dt - w^* E_v \int u^*(t)v(t)dt - w E_v \int u(t)v^*(t)dt \\
&= E_v^2 \int |u(t)|^2 \, dt + |w|^2 \int |v(t)|^2 \, dt - E_v w^* w - E_v w w^* && [\text{definition of } w] \\
&= E_v^2 E_u + |w|^2 E_v - 2 |w|^2 E_v = E_v \left(E_u E_v - |w|^2\right) && [|z|^2 = z^* z]
\end{aligned}
$$

Unless $E_v = 0$ (in which case, $v(t) \equiv 0$ and the C-S inequality is true), we must have $|w|^2 \le E_u E_v$ which proves the C-S inequality.

Also, $E_u E_v = |w|^2$ only if we have equality in the first line,

that is, $\int |E_v u(t) - w^* v(t)|^2 \, dt = 0$ which implies that the integrand is zero for all t.

This implies that $u(t) = \frac{w^*}{E_v} v(t)$.

So we have shown that $E_u E_v = |w|^2$ if and only if $u(t)$ and $v(t)$ are proportional to each other.

Autocorrelation

The correlation of a signal with itself is its *autocorrelation*:

$$w(t) = u(t) \otimes u(t) = \int_{-\infty}^{\infty} u^*(\tau - t)u(\tau)d\tau$$

The autocorrelation at zero lag:

$$\begin{aligned} w(0) &= \int_{-\infty}^{\infty} u^*(\tau - 0)u(\tau)d\tau \\ &= \int_{-\infty}^{\infty} u^*(\tau)u(\tau)d\tau \\ &= \int_{-\infty}^{\infty} |u(\tau)|^2 d\tau = E_u \end{aligned}$$

The autocorrelation at zero lag, $w(0)$, is the energy of the signal.

The normalized autocorrelation: $z(t) = \frac{u(t)\otimes u(t)}{E_u}$

satisfies $z(0) = 1$ and $|z(t)| \leq 1$ for any t.

Wiener-Khinchin Theorem: [Cross-correlation theorem when $v(t) = u(t)$]

$$w(t) = u(t) \otimes u(t) \quad \Leftrightarrow \quad W(f) = U^*(f)U(f) = |U(f)|^2$$

The Fourier transform of the autocorrelation is the energy spectrum.

Cross-correlation is used to find when two different signals are similar.
Autocorrelation is used to find when a signal is similar to itself delayed.

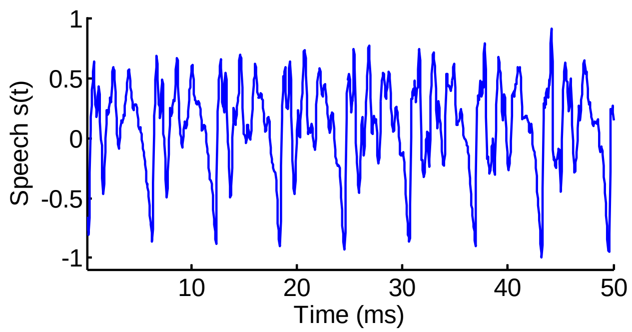
First graph shows $s(t)$ a segment of the microphone signal from the initial vowel of "early" spoken by me. The waveform is "quasi-periodic" = "almost periodic but not quite".

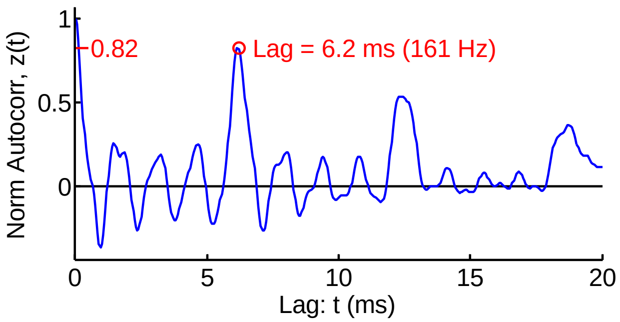
Second graph shows normalized autocorrelation, $z(t) = \frac{s(t) \otimes s(t)}{E_s}$.

$z(0) = 1$ for $t = 0$ since a signal always matches itself exactly.

$z(t) = 0.82$ for $t = 6.2\,\text{ms}$ = one period lag (not an exact match).

$z(t) = 0.53$ for $t = 12.4\,\text{ms}$ = two periods lag (even worse match).





Fourier Transform Variants

There are three different versions of the Fourier Transform in current use.

(1) Frequency version (we have used this in lectures)

$$U(f) = \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt \qquad u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft}df$$

- Used in the communications/broadcasting industry and textbooks.
- The formulae do not need scale factors of 2π anywhere. ☺☺☺

(2) Angular frequency version

$$\widetilde{U}(\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt \qquad u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{U}(\omega)e^{i\omega t}d\omega$$

Continuous spectra are unchanged: $\widetilde{U}(\omega) = U(f) = U(\frac{\omega}{2\pi})$

However δ-function spectral components are multiplied by 2π so that

$$U(f) = \delta(f - f_0) \quad \Rightarrow \quad \widetilde{U}(\omega) = 2\pi \times \delta(\omega - 2\pi f_0)$$

- Used in most signal processing and control theory textbooks.

(3) Angular frequency + symmetrical scale factor

$$\widehat{U}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt \qquad u(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{U}(\omega)e^{i\omega t}d\omega$$

In all cases $\widehat{U}(\omega) = \frac{1}{\sqrt{2\pi}}\widetilde{U}(\omega)$

- Used in many Maths textbooks (mathematicians like symmetry)

Fourier Transform using Angular Frequency:

$$\widetilde{U}(\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt \qquad u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{U}(\omega)e^{i\omega t}d\omega$$

Any formula involving $\int df$ will change to $\frac{1}{2\pi}\int d\omega$ [since $d\omega = 2\pi df$]

Parseval's Theorem:

$$\int u^*(t)v(t)dt = \frac{1}{2\pi}\int \widetilde{U}^*(\omega)\widetilde{V}(\omega)d\omega$$

$$E_u = \int |u(t)|^2 dt = \frac{1}{2\pi}\int \left|\widetilde{U}(\omega)\right|^2 d\omega$$

Waveform Multiplication: (convolution implicitly involves integration)

$$w(t) = u(t)v(t) \quad \Rightarrow \quad \widetilde{W}(\omega) = \frac{1}{2\pi}\widetilde{U}(\omega) * \widetilde{V}(\omega)$$

Spectrum Multiplication: (multiplication $\not\Rightarrow$ integration)

$$w(t) = u(t) * v(t) \quad \Rightarrow \quad \widetilde{W}(\omega) = \widetilde{U}(\omega)\widetilde{V}(\omega)$$

To obtain formulae for version (3) of the Fourier Transform, $\widehat{U}(\omega)$, substitute into the above formulae: $\widetilde{U}(\omega) = \sqrt{2\pi}\widehat{U}(\omega)$.

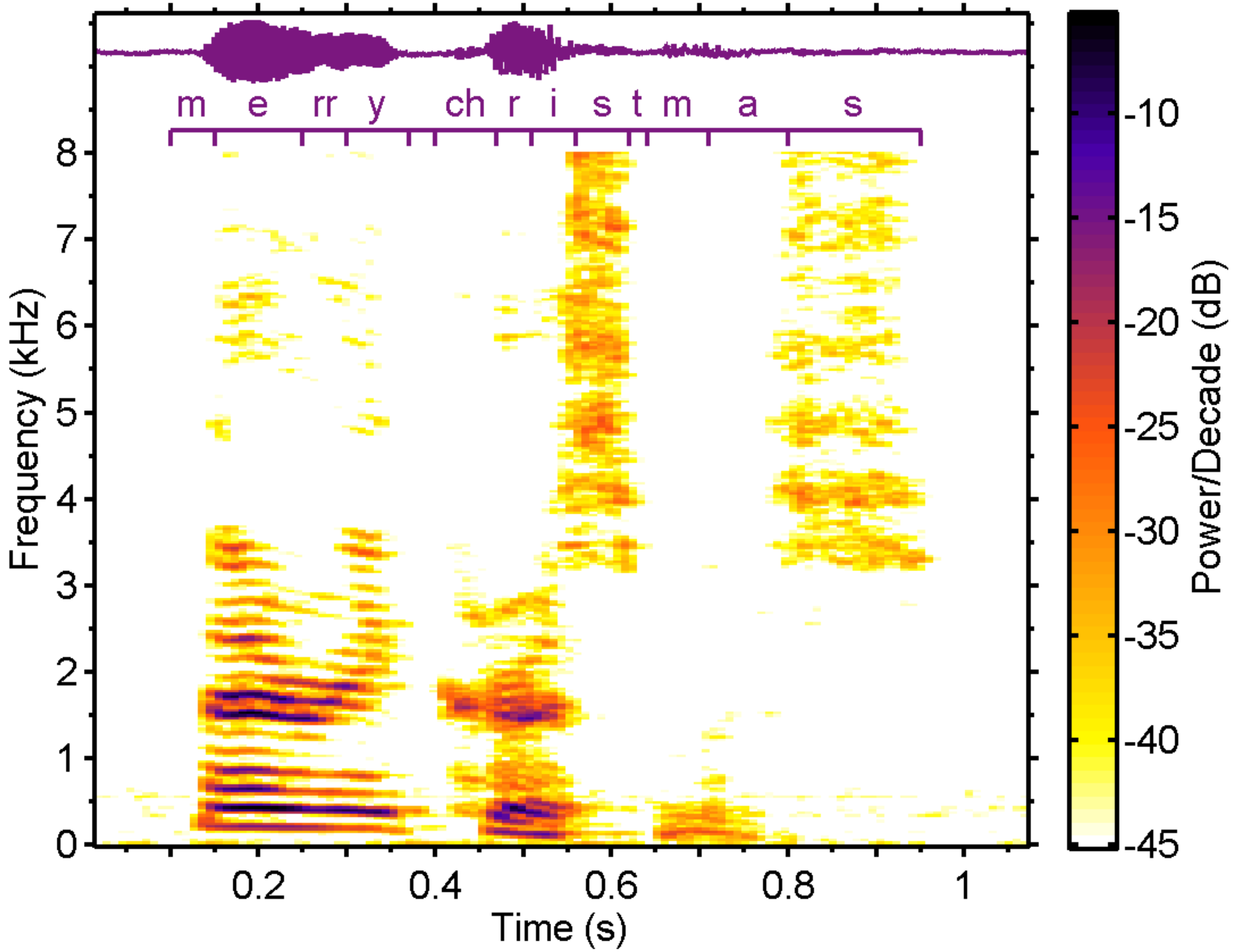
Summary

- Cross-Correlation: $w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau$
  - Used to find similarities between $v(t)$ and a delayed $u(t)$
  - Cross-correlation theorem: $W(f) = U^*(f)V(f)$
  - Cauchy-Schwarz Inequality: $|u(t) \otimes v(t)| \le \sqrt{E_u E_v}$
    - Normalized cross-correlation: $\left|\frac{u(t)\otimes v(t)}{\sqrt{E_u E_v}}\right| \le 1$
- Autocorrelation: $x(t) = u(t) \otimes u(t) = \int_{-\infty}^{\infty} u^*(\tau - t)u(\tau)d\tau \le E_u$
  - Wiener-Khinchin: $X(f)$ = energy spectral density, $|U(f)|^2$
  - Used to find periodicity in $u(t)$
- Fourier Transform using ω:
  - Continuous spectra unchanged; spectral impulses multiplied by 2π
  - In formulae: $\int df \to \frac{1}{2\pi}\int d\omega$; ω-convolution involves an integral

For further details see RHB Chapter 13.1

Spectrogram

Spectrogram of "Merry Christmas" spoken by Mike Brookes



[Complex Fourier Series]

All waveforms have period $T = 1$. $\delta_{condition}$ is 1 whenever "condition" is true and otherwise 0.

Waveform	$x(t)$ for $\|t\| < 0.5$	X_n
Square wave	$2\delta_{\|t\|<0.25} - 1$	$\frac{2\sin 0.5\pi n}{\pi n} \times \delta_{n\neq 0}$
Pulse of width d	$\delta_{\|t\|<0.5d}$	$\frac{\sin \pi d n}{\pi n}$
Sawtooth wave	$2t$	$\frac{i(-1)^n}{\pi n} \times \delta_{n\neq 0}$
Triangle wave	$1 - 4\|t\|$	$\frac{2(1-(-1)^n)}{\pi^2 n^2}$

[Fourier Transform Properties A]

You need not memorize these properties. All integrals are $\int_{-\infty}^{\infty}$

Property	$x(t)$	$X f)$
Forward	$x(t)$	$\int x(t)e^{-i2\pi ft}dt$
Inverse	$\int X(f)e^{i2\pi ft}df$	$X(f)$
Spectral Zero	$\int x(t)dt$	$= X(0)$
Temporal Zero	$x(0)$	$= \int X(f)df$
Duality	$X(t)$	$x(-f)$
Reversal	$x(-t)$	$X(-f)$
conjugate	$x^*(t)$	$X^*(-f)$
Temporal Derivative	$\frac{d^n}{dt^n}x(t)$	$(i2\pi f)^n X(f)$
Spectral Derivative	$(-i2\pi t)^n x(t)$	$\frac{d^n}{df^n}X(f)$
Integral	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{i2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$
Scaling	$x(\alpha t + \beta)$	$\frac{1}{\lvert\alpha\rvert}e^{\frac{2i\pi f\beta}{\alpha}}X(\frac{f}{\alpha})$
Time Shift	$x(t-T)$	$X(f)e^{-i2\pi fT}$
Frequency Shift	$x(t)e^{i2\pi Ft}$	$X(f-F)$

[Fourier Transform Properties B]

You need not memorize these properties. All integrals are $\int_{-\infty}^{\infty}$

Property	$x(t)$	$X f)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(f) + \beta Y(f)$
Multiplication	$x(t)y(t)$	$X(f) * Y(f)$
Convolution	$x(t) * y(t)$	$X(f)Y(f)$
Correlation	$x(t) \otimes y(t)$	$X^*(f)Y(f)$
Autocorrelation	$x(t) \otimes x(t)$	$\lvert X(f)\rvert^2$
Parseval or Plancherel	$\int x^*(t)y(t)dt$ $E_x = \int \lvert x(t)\rvert^2 dt$	$= \int X^*(f)Y(f)df$ $= \int \lvert X(f)\rvert^2 df$
Repetition	$\sum_n x(t - nT)$	$\left\lvert\frac{1}{T}\right\rvert \sum_k X\left(\frac{k}{T}\right) \delta\left(f - \frac{k}{T}\right)$
Sampling	$\sum_n x(nT)\delta(t - nT)$	$\left\lvert\frac{1}{T}\right\rvert \sum_k X\left(f - \frac{k}{T}\right)$
Modulation	$x(t)\cos(2\pi Ft)$	$\frac{1}{2}X(f - F) + \frac{1}{2}X(f + F)$

Convolution: $x(t) * y(t) = \int x(\tau)y(t - \tau)d\tau$

Cross-correlation: $x(t) \otimes y(t) = \int x^*(\tau)y(\tau + t)d\tau = \int x^*(\tau - t)y(\tau)d\tau$

[Fourier Transform Pairs]

You need not memorize these pairs.

$x(t)$	$X(f)$	$x(t)$	$X(f)$
$\delta(t)$	1	1	$\delta(f)$
$\text{rect}(t)$	$\frac{\sin(\pi f)}{\pi f}$	$\frac{\sin(t)}{t}$	$\pi\text{rect}(\pi f)$
$\text{tri}(t)$	$\frac{\sin^2(\pi f)}{\pi^2 f^2}$	$\frac{\sin^2(t)}{t^2}$	$\pi\text{tri}(\pi f)$
$\cos(2\pi\alpha t)$	$\frac{1}{2}\delta\left(f+\alpha\right)+\frac{1}{2}\delta\left(f-\alpha\right)$	$\sin(2\pi\alpha t)$	$\frac{i}{2}\delta\left(f+\alpha\right)-\frac{i}{2}\delta\left(f-\alpha\right)$
$e^{-\alpha t}u(t)$	$\frac{1}{\alpha+2\pi i f}$	$te^{-\alpha t}u(t)$	$\frac{1}{(\alpha+2\pi i f)^2}$
$e^{-\alpha\lvert t\rvert}$	$\frac{2\alpha}{\alpha^2+4\pi^2 f^2}$	$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\text{sgn}(t)$	$\frac{1}{i\pi f}$	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{2\pi i f}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\left\lvert\frac{1}{T}\right\rvert\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$		

Elementary Functions:

$$\text{rect}(t)=\begin{cases}1, & \lvert t\rvert<0.5\\ 0, & \text{elsewhere}\end{cases} \qquad \text{tri}(t)=\begin{cases}1-\lvert t\rvert, & \lvert t\rvert<1\\ 0, & \text{elsewhere}\end{cases}$$

$$\text{sgn}(t)=\begin{cases}-1, & t<0\\ 0, & t=0\\ 1, & t>0\end{cases} \qquad u(t)=\frac{1}{2}\left(1+\text{sgn}(t)\right)=\begin{cases}0, & x<0\\ 0.5, & x=0\\ 1, & x>0\end{cases}$$