E1.10 Fourier Series and Transforms

Problem Sheet 1 - Solutions

1. $\sum_{r=1}^{5} 3^r = 3 \times \frac{1-3^5}{1-3} = 3 \times \frac{-242}{-2} = 3 \times 121 = 363$. In the expression $3 \times \frac{1-3^5}{1-3}$, the "5" is the number of terms in the sum and the "$3\times$" is the first term (when $r = 1$).

2. (a) Each term is multiplied by a factor of x^2, so the standard formula gives $3x^2 \times \frac{1-x^{20}}{1-x^2}$ where $x^{20} = \left(x^2\right)^{10}$ since there are 10 terms.
(b) Each term is multiplied by a factor x^{-1} and, treating $x^0 = 1$, the first term equals 2, so the sum is $2 \times \frac{1-x^{-11}}{1-x^{-1}}$.
(c) Each term is multiplied by xy and the first term is y^{-2} so the sum is $y^{-2} \times \frac{1-(xy)^{R+1}}{1-xy}$.
(d) Each term is multiplied by -1 and the first term is $-1^0 = 1$ so the sum is
$$\frac{1-(-1)^{R+1}}{1-(-1)} = \frac{1+(-1)^R}{2} = \begin{cases} 1 & R \text{ even} \\ 0 & R \text{ odd} \end{cases}.$$

3. The substitution $r = m + 1 \Leftrightarrow m = r - 1$. So, making the substitution in both the limits and summand gives
$$\sum_{r=1}^{5} 3^r = \sum_{m=0}^{4} 3^{m+1} = 3\sum_{m=0}^{4} 3^m = 3 \times \frac{1-3^5}{1-3}.$$
So the answer is 363 as in question 1.

4. Each term is multiplied by $e^{j\omega}$ and the first of the $2N+1$ terms is $e^{-j\omega N}$ so the sum is
$$e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}} = \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1-e^{j\omega}}.$$
A very common trick when an expression includes the sum or difference of two exponentials is to take out a factor whose exponent is the average of the two original exponents; in this case the average exponent is $j0.5\omega$ in the denominator and also in the numerator since $\frac{-j\omega N + j\omega(N+1)}{2} = j0.5\omega$. This gives
$$\frac{e^{j0.5\omega}\left(e^{-j\omega(N+0.5)} - e^{j\omega(N+0.5)}\right)}{e^{j0.5\omega}\left(e^{-j0.5\omega} - e^{j0.5\omega}\right)} = \frac{e^{-j\omega(N+0.5)} - e^{j\omega(N+0.5)}}{e^{-j0.5\omega} - e^{j0.5\omega}} = \frac{-2j\sin((N+0.5)\omega)}{-2j\sin 0.5\omega} = \frac{\sin((N+0.5)\omega)}{\sin 0.5\omega}$$

5. Each term is multiplied by $e^{j2\pi R^{-1}}$ and the first term is $e^0 = 1$ so the sum formula gives
$$\frac{1-e^{j2\pi R R^{-1}}}{1-e^{j2\pi R^{-1}}} = \frac{1-e^{j2\pi}}{1-e^{j2\pi R^{-1}}} = \frac{0}{1-e^{j2\pi R^{-1}}} = 0.$$
However, when $R = 1$, the denominator is zero so the formula is invalid; in this case there is only one term in the summation and it equals $e^{j2\pi 0\times 1} = 1$. So the answer is 0 for all values of R except $R = 1$ when the answer is 1. We can write this compactly as
$$\delta[R-1] = \begin{cases} 1 & R = 1 \\ 0 & R > 1 \end{cases}$$
where the function $\delta[n]$ is the "Kroneker Delta function" and equals 1 if and only if its integer argument equals zero.

6. The summand in this question is "separable" because it can be expressed as the product of two factors that depend on m and n respectively. So we can write
$$\sum_{n=0}^{N}\sum_{m=1}^{M} 2x^{m-n} = 2\sum_{n=0}^{N} x^{-n}\sum_{m=1}^{M} x^m = 2\frac{1-x^{-(N+1)}}{1-x^{-1}}x\frac{1-x^M}{1-x} = \frac{2x^2\left(x^{-(N+1)}-1\right)\left(1-x^M\right)}{(1-x)^2}.$$

7. (a)The period is 2π, so we calculate the average by integrating over one period and dividing by the period: $\langle x(t)\rangle = \frac{1}{2\pi}\int_0^{2\pi}\sin t\,dt = \frac{1}{2\pi}\left[-\cos t\right]_0^{2\pi} = 0$.
(b) The period is now π, so we calculate the average as: $\langle |x(t)|\rangle = \frac{1}{\pi}\int_0^{\pi}|\sin t|\,dt = \frac{1}{\pi}\int_0^{\pi}\sin t\,dt = \frac{1}{\pi}\left[-\cos t\right]_0^{\pi} = \frac{2}{\pi}$.
(c) The period is still π:

$$\left\langle x^2(t)\right\rangle = \frac{1}{\pi}\int_0^{\pi}\sin^2 t\,dt = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\left(1-\cos 2t\right)\,dt = \frac{1}{2\pi}\int_0^{\pi}\left(1-\cos 2t\right)\,dt = \frac{1}{2\pi}\left[t-\frac{1}{2}\sin 2t\right]_0^{\pi} = \frac{\pi}{2\pi} = \frac{1}{2}.$$

An easier way of getting this answer is to write

$$\left\langle \sin^2 t\right\rangle = \frac{1}{2}\left(\langle 1\rangle - \langle\cos 2t\rangle\right) = \frac{1}{2}\left(1-0\right) = \frac{1}{2}.$$

8. (a)

$$\left\langle P_0^2(t)\right\rangle_{[-1,1]} = \frac{1}{2}\int_{-1}^{1}1^2 dt = \frac{1}{2}\left[t\right]_{-1}^{1} = 1$$

and

$$\left\langle P_1^2(t)\right\rangle_{[-1,1]} = \frac{1}{2}\int_{-1}^{1}3t^2 dt = \frac{1}{2}\left[t^3\right]_{-1}^{1} = 1.$$

Finally

$$\langle P_0(t)P_1(t)\rangle_{[-1,1]} = \frac{\sqrt{3}}{2}\int_{-1}^{1}t\,dt = \frac{\sqrt{3}}{4}\left[t^2\right]_{-1}^{1} = 0.$$

(b) The analysis is slightly easier if you do it in the right order.

$$\langle P_1(t)P_2(t)\rangle_{[-1,1]} = \frac{\sqrt{3}}{2}\int_{-1}^{1}at^3+bt^2+ct\,dt = \frac{\sqrt{3}}{2}\left[\frac{at^4}{4}+\frac{bt^3}{3}+\frac{ct^2}{2}\right]_{-1}^{1} = \frac{b}{\sqrt{3}} = 0$$

so $b=0$. Now

$$\langle P_0(t)P_2(t)\rangle_{[-1,1]} = \frac{1}{2}\int_{-1}^{1}at^2+c\,dt = \frac{1}{2}\left[\frac{at^3}{3}+ct\right] = \frac{a}{3}+c = 0$$

so $a=-3c$. Finally

$$\left\langle P_2^2(t)\right\rangle_{[-1,1]} = \frac{c^2}{2}\int_{-1}^{1}9t^4-6t^2+1\,dt = \frac{c^2}{2}\left[\frac{9t^5}{5}-2t^3+t\right]_{-1}^{1} = c^2\left(\frac{9}{5}-2+1\right) = \frac{4}{5}c^2 = 1$$

from which $c=\pm\frac{\sqrt{5}}{2}$. So the polynomial is $P_2(t) = \frac{\sqrt{5}}{2}\left(3t^2-1\right)$.