E1.10 Fourier Series and Transforms

Problem Sheet 2 - Solutions

1. (a) The fundamental frequency is $\frac{1000\pi}{2\pi} = 500\,\text{Hz}$ and the period is $\frac{2\pi}{1000\pi} = 2\,\text{ms}$.

   (b) For a mixture of cosine waves, the fundamental period is the lowest common multiple (LCM) of the periods of the constituent waves. In this case the frequencies of the two waves are 500 Hz and 625 Hz with periods 2 ms and 1.6 ms respectively. The LCM of 2 and 1.6 is 8 ms corresponding to a frequency of 125 Hz. Alternatively, you can obtain the same answer by finding the highest common factor (HCF) of the two frequencies. Notice that the addition of even a tiny amplitude at 625 Hz has quadrupled the fundamental period.

   (c) The periods of the two constituent waves are here $2\,ms$ and $\frac{2\pi}{1000} = 6.28\ldots\,ms$. Since the second of these is irrational, there is no LCM and the resultant waveform is not periodic at all (or equivalently its period is ∞).

2. (a) Yes. (b) No.
   We have $T = 2\pi$, so $\int_0^T \left|\frac{1}{\sin t}\right| dt = 2\int_0^\pi \frac{1}{\sin t} dt = 2\left[\ln\left(\tan\left(0.5t\right)\right)\right]_0^\pi = 2\ln\left(\frac{\tan 0.5\pi}{\tan 0}\right) = 2\ln\frac{\infty}{0} = \infty$.
   (c) Yes for a similar reason to the previous part since $\int_0^T \sqrt{\frac{1}{\tau}} d\tau = \left[\sqrt{\tau}\right]_0^T = \sqrt{T} < \infty$.
   (d) No since it is not periodic. (e) Yes; this function is a triangle wave with period 1.

3. The fundamental frequency is the highest common factor of the constituent frequencies, i.e. $2000\pi\,\text{rad/s} = 1\,\text{kHz}$. So, setting $F = 1000$, we have $u(t) = 1 + 2\cos\left(2\pi 3Ft\right) + 3\sin\left(2\pi 2Ft\right)$. So the Fourier coefficients are $a_0 = 2$, $b_2 = 3$, $a_3 = 2$ with all other coefficients zero.

4. (a) The non-zero Fourier coefficients are $a_1 = 2$ and $b_1 = -4$.

   (b) The non-zero complex Fourier coefficients are $U_{-1} = 1 - 2i$ and $U_1 = 1 + 2i$. We see that $U_1$is exactly half the value of the phasor and that $U_{-1}$is the complex conjugate of U_1.

5. We need to express $\cos^4\theta$ in terms of components of the form $\cos n\theta$. We can do this by writing

$$\begin{aligned}
\cos^4\theta &= \frac{1}{16}\left(e^{i\theta} + e^{-i\theta}\right)^4 \\
&= \frac{1}{16}\left(e^{i4\theta} + 4e^{i2\theta} + 6 + 4e^{-i2\theta} + e^{-i4\theta}\right) \\
&= \frac{1}{16}\left(2\cos 4\theta + 8\cos 2\theta + 6\right)
\end{aligned}$$

   From this we find that the fundamental frequency is actually $4000\pi\,\text{rad/s} = 2\,\text{kHz}$ and the non-zero Fourier coefficients are therefore $a_0 = \frac{3}{4}$, $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{8}$.

6. (a) From the formulae on slide 2-11 of the notes (and observing that $F = \frac{1}{T} = \frac{1}{2}$),

$$\begin{aligned}
a_n &= \frac{2}{T}\int_{-1}^{1} u(t)\cos\left(2\pi nFt\right) dt \\
&= \int_{-1}^{1} 3t\cos\left(\pi nt\right) dt \\
&= \frac{3}{\pi^2 n^2}\left[\pi nt\sin\left(\pi nt\right) + \cos\left(\pi nt\right)\right]_{t=-1}^{1} \\
&= 0 \\
b_n &= \int_{-1}^{1} 3t\sin\left(\pi nt\right) dt \\
&= \frac{3}{\pi^2 n^2}\left[-\pi nt\cos\left(\pi nt\right) + \sin\left(\pi nt\right)\right]_{t=-1}^{1} \\
&= \frac{-6}{\pi n}\cos\left(\pi n\right) = \frac{-6\left(-1\right)^n}{\pi n}
\end{aligned}$$

   Note that the b_n expression applies only for $n \geq 1$. Notice also that the a_n are all zero because $u(t)$ is a real-valued odd function and that the coefficient magnitudes are $\propto n^{-1}$ which is a characteristic of waveforms that include a discontinuity.

(b) We have $U_0 = \frac{1}{2}a_0 = 0$ and, for $n \geq 1$,

$$U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp ib_{|n|}\right) = \frac{\pm i3\,(-1)^n}{\pi\,|n|}$$

from which $U_n = \frac{i3(-1)^n}{\pi n}$. Again, we note that U_n is purely imaginary because $u(t)$ is a real-valued odd function.

(c) We can calculate the V_n directly by integrating over an interval that does not include a discontinuity.

$$\begin{aligned} V_0 &= \frac{1}{2}\int_0^2 v(t)dt = \frac{1}{2}\int_0^2 (t-1)dt = 0 \\ \text{for}\, n \neq 0: \; V_n &= \frac{1}{2}\int_0^2 v(t)e^{-i\pi nt}dt \\ &= \frac{3}{2}\int_0^2 (t-1)\,e^{-i\pi nt}dt \\ &= \frac{3}{2\pi^2 n^2}\left[(1+i\pi n\,(t-1))\,e^{-i\pi nt}\right]_{t=0}^{2} \\ &= \frac{3i\pi n}{\pi^2 n^2} = \frac{3i}{\pi n} \end{aligned}$$

An alternative way to calculate V_n is to use the time-shifting formula:

$$\begin{aligned} v(t) = u(t-1) \Rightarrow V_n &= U_n e^{-i2\pi nF} \\ &= U_n e^{-in\pi} \\ &= (-1)^n\,U_n. \end{aligned}$$

Notice that time-shifting a waveform changes the phases of the $V_n$but not the magnitudes.

(d) The complex Fourier transform of $x(t) = 4$ is iust $X_0 = 4$ with all other coefficients zero. So, since the Fourier transform is linear, if $w(t) = 2v(t) + x(t)$ we must have $W_n = 2V_n + X_n$ which means that $W_0 = 4$ and, for $n \neq 0$, $W_n = \frac{6i}{\pi n}$.

7. (a) $u(t) = 2.5 + 2\cos(2\pi Ft) + \sin(2\pi Ft) + 3\cos(4\pi Ft)$.

   (b) $U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp ib_{|n|}\right)$ from which $U_{-2:2} = [1.5,\, 1+0.5i,\, 2.5,\, 1-0.5i,\, 1.5]$. Notice that, since $u(t)$ is real, U_{-n} is the complex conjugate of U_{+n}.

8. The Fourier transform of a real-valued signal is purely real or purely imaginary if it is even or odd respectively. So we have the following: (a) real (b) imaginary (c) neither (d) real (e) neither (f) real (g) imaginary (g) imaginary.

9. All the even-numbered Fourier coefficients of a waveform are zero if it is anti-periodic which, in this case with $T = 2$, means that $u(t) = -u(t-1)$ for $0 \leq t < 1$; note that you only need to prove this relationship for half a period since the periodicity relationship, $u(t) = u(t+T)$, then means that it applies for the other half. So we have the following ("Yes" means it is anti-periodic):

   (a) Yes: $\sin \pi t = -\sin(\pi t - \pi)$ and of course the Fourier transform has only a single component with $b_1 = 1$ or, equivalently, $X_{\pm 1} = \mp i$.

   (b) Yes: for $0 \leq t < 1$, $u(t) = -t$ and $u(t-1) = (t-1) + 1 = t = -u(t)$

   (c) No: for $0 \leq t < 1$, $u(t) = 1 - t$ but $u(t-1) = (t-1) + 1 = t \neq -u(t)$. Note however that $v(t) = u(t) - 0.5$ is anti-periodic since, for $0 \leq t < 1$, $v(t) = u(t) - 0.5 = t + 0.5 = -((1-(t-1)) - 0.5) = -(u(t-1) - 0.5) = -v(t-1)$. Thus the only non-zero even harmonic of $u(t)$ is $U_0 = 0.5$.

   (d) Yes: for $0 \leq t < 1$, $u(t) = t\,(1-t)$ and $u(t-1) = (t-1)\,(1+(t-1)) = (t-1)\,t = -u(t)$

   (e) No: for $0 \leq t < 1$, $u(t) = t\left(t^2 - 1\right)$ but $u(t-1) = (t-1)^3 - (t-1) = t\left(t^2 - 3t + 2t\right) \neq -u(t)$.

10. (a) We note that the fundamental frequency is $F = \frac{1}{4}$.

$$\begin{aligned} U_n = \frac{1}{4}\int_0^1 e^{-i0.5\pi nt}dt &= \frac{i}{4\times 0.5\pi n}\left(e^{-i0.5\pi n}-1\right) \\ &= \frac{ie^{-i0.25\pi n}}{2\pi n}\left(e^{-i0.25\pi n}-e^{i0.25\pi n}\right) \\ &= \frac{ie^{-i0.25\pi n}}{2\pi n}\times -2i\sin 0.25\pi n \\ &= \frac{\sin 0.25\pi n}{\pi n}\times e^{-i0.25\pi n} \end{aligned}$$

We know that $\sin\theta = 0$ whenever θ is a multiple of π, so $U_n = 0$ whenever n is a non-zero multiple of 4.

(b) By the time-shift formula $V_n = U_n e^{i2\pi nF0.5} = U_n e^{i0.25\pi n} = \frac{\sin 0.25\pi n}{\pi n}$. Since $v(t)$ is real and symmetric, $V_n$will also be real and symmetric. The time-shifting affects only the phase and so $|V_n| = |U_n|$.

(c) By linearity and the time-shift formula, $W_n = V_n\left(1+e^{i2\pi nF2}\right) = V_n\left(1+e^{i\pi n}\right) = V_n\left(1+(-1)^n\right)$. The quantity $\left(1+(-1)^n\right)$ equals 2 for even values of n and 0 for odd values of n. Since in addition, $V_n = 0$ when n is a non-zero multiple of 4, W_n is only non-zero for $n = 0$ and for odd multiplies of 2. In fact, the period of $w(t)$ is 2 rather than 4 which explains why $W_n = 0$ for all odd values of n. In addition, when considered with a period of 2, $(w(t) - W_0)$ is anti-periodic and so all its even Fourier coefficients will be zero.