E1.10 Fourier Series and Transforms

Problem Sheet 4 (Lectures 6, 7, 8)

Key: [A]= easy ... [E]=hard

Fourier Transform: $X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$ **Inverse Transform:** $x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df$

Questions from RBH textbook: 13.1, 13.2, 13.3, 13.5, 13.7, 13.9, 13.19, 13.20.

1. [B] Evaluate $\int_{-\infty}^{\infty} \delta(t-3)t^3e^{-t}dt$.
2. [B] (a) Evaluate $\int_{-\infty}^{\infty} \delta(t-6)t^2\,dt$. (b) Now make the substitution $t = 3\tau$ for the integration variable and show that the integral remains unchanged.
3. [B] Express $2x^2\delta(8-2x)$ in the form $a\delta(x-b)$
4. [C] (a) If $v(t) = e^{-|t|}$, show that its Fourier transform is $V(f) = \frac{2}{1+4\pi^2f^2}$.

   (b) Using the time shifting and scaling formulae from slides 6-9 and 6-10 and without doing any additional integrations, determine the Fourier transforms of (i) $v_1(t) = e^{-|at|}$, (ii) $v_2(t) = e^{-|t-b|}$, (iii) $v_3(t) = \frac{1}{1+t^2}$.
5. [D] Determine the Fourier transform, $X(f)$, when $x(t) = t^2e^{-|t|}$.
6. [C] If $x(t) = \delta(t)$ determine the Fourier transform, $X(f)$. Hence, by considering the inverse transform, show that $\int_{-\infty}^{\infty} e^{i\alpha ft}df = \frac{2\pi}{|\alpha|}\delta(t)$ where $\alpha \neq 0$ is a real constant.
7. [B] Determine the Fourier transform, $X(f)$, when $x(t)$ is a DC voltage: $x(t) = 10$.
8. [B] Determine the Fourier transform, $X(f)$, when $x(t) = 12\cos 200\pi t + 8\sin 400\pi t$.
9. [C] If $v(t)$ is a periodic signal with frequency F for which $v(t) = \delta(t)$ for $-\frac{1}{2F} \le t < \frac{1}{2F}$, determine the coefficients, V_n, of its complex Fourier series. Hence find the Fourier transform, $X(f)$, of the "impulse train" given by $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-\frac{n}{F})$.
10. [C] If the Fourier transform of $x(t)$ is $X(f) = \cos 100f$, determine $x(t)$ in two ways: (a) using the duality relation: $v(t) = U(t) \Leftrightarrow V(f) = u(-f)$ and (b) by directly evaluating the inverse transform integral and using the result of question 6.
11. [B] If $x(t) = \begin{cases} 1 & |t| \le 0.5 \\ 0 & |t| > 0.5 \end{cases}$ show that $X(f) = \frac{\sin \pi f}{\pi f}$. This function is often called a top-hat function or $\text{rect}(t)$.
12. [B] If $x(t) = \begin{cases} e^{-at} & t \ge 0 \\ 0 & t < 0 \end{cases}$ show that $X(f) = \frac{1}{i2\pi f + a}$ for $a > 0$.
13. [C] An electronic circuit, whose input and output signals are $x(t)$ and $y(t)$ respectively, has a frequency response given by $\frac{Y}{X}(i\omega) = \frac{2000}{i\omega+1000}$.

    (a) If $x(t) = \cos^2(1000t)$, use phasors to find an expression for $y(t)$. Give expressions for the Fourier transforms $X(f)$ and $Y(f)$.

    (b) If $x(t) = \begin{cases} e^{-500t} & t \ge 0 \\ 0 & t < 0 \end{cases}$ give an expression for $Y(f)$ (you may use without proof the result of question 12). Show that $Y(f)$ may be written as $\frac{c}{i2\pi f+500} + \frac{d}{i2\pi f+1000}$ and find the values of the constants c and d. Hence give an expression for $y(t)$.
14. [C] The triangle function is given by $y(t) = \begin{cases} 1-|t| & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$. Show that $y(t)$ may be obtained by convolving $x(t)$ with itself where $x(t) = \text{rect}(t)$ as defined in question 11, i.e. $y(t) = x(t) * x(t) \triangleq \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$. Hence use the convolution theorem and the result of question 11 to give the Fourier transform $Y(f)$.

15. [B] An "energy signal" has finite energy: $E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty$. A "power signal" has infinite energy but finite power: $\left\langle |x(t)|^2 \right\rangle = \lim_{A,B\to\infty} \frac{1}{B-A} \int_{-A}^{B} |x(t)|^2\, dt < \infty$. Say whether each of the following functions of time, t, is (i) an energy signal, (ii) a power signal or (iii) neither: (a) $2\cos\omega t$, (b) 10, (c) t, (d) $\sqrt{|t|}$, (e) e^t, (f) e^{-t}, (g) $e^{-|t|}$, (h) $\frac{1}{1+t^2}$, (i) $\cos t^2$, (i) $\frac{1}{1+|t|}$, (k) $\frac{1}{\sqrt{|t|}}$.

16. [C] Suppose the Fourier transform of $x(t)$ is $X(f) = \frac{1}{1+(2\pi f)^2} + 2i\left(\delta(f+4) - \delta(f-4)\right)$. Give expressions for the alternative versions of the Fourier transform: (a) $\widetilde{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$ and (b) $\widehat{X}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$. State the general formulae for the inverse transform integrals that give $x(t)$ in terms of $\widehat{X}(\omega)$ and $\widetilde{X}(\omega)$.