IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2014

EEE PART I: MEng, BEng and ACGI

MATHEMATICS 1A (E-STREAM AND I-STREAM)

Thursday, 29 May 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	E.M. Yeatman, A. Astolfi, D.M. Brookes
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MATHEMATICS 1

Information for Candidates:

Calculators are not permitted in this exam.

1. a) i) A function $s(x)$ has 4 distinct real roots and has a continuous derivative. If $g(x) = s'(x)$, what is the minimum number of real roots of $g(x)$? Justify your answer. [4 marks]

ii) If the order of a polynomial is known and its roots are known, when is this enough information to specify the polynomial uniquely – always, sometimes or never? Justify your answer. [4 marks]

iii) If $f(x)$ and $g(x)$ are 4^{th} and 3^{rd} order polynomials respectively, then solving $f(x) = g(x)$ is equivalent to finding the roots of a polynomial of what order? Justify your answer. [4 marks]

b) i) What is the name of the type of function which describes the locus of a point which is always equidistant from a given line and a given reference point? [2 marks]

ii) Find the function $y(x)$ which is equidistant from the line $y = 2$ and the point $(x, y) = (0, 4)$. [4 marks]

c) i) For a complex number X, where

$$X = \frac{a+ib}{a-ib}$$

and a and b are real, find expressions for the modulus and argument of X. [4 marks]

ii) For X as in (c)(i) above, if $a = b$, find the value of X in the form $u+iv$. [4 marks]

d) i) Euler's equation gives $e^{i\theta}$ in terms of trigonometric functions. Write Euler's equation. [2 marks]

ii) Using Euler's equation, show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. [2 marks]

iii) Using (d)(i) and (d)(ii) above, derive trigonometric identities for $\sin 3\theta$ and $\cos 3\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. [4 marks]

2. Consider the function

$$y = f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x,$$

defined for $x > 0$.

a) Compute the first and second derivatives of the function. Hence determine the only stationary point of the function. Show that $\frac{d^2 f}{dx^2} > 0$, for all $x > 0$, and hence that the stationary point is a minimum. [6 marks]

b) Plot the graph of the function for $x \in [0,3]$. Clearly indicate the stationary point and the value of the function for $x \to 0$. Note that the function f takes positive values for all $x > 0$. [4 marks]

c) Compute the indefinite integral

$$I = \int f(x)dx.$$

[5 marks]

d) Consider the region A in the (x,y)-plane defined as follows. The region is bounded from above by the graph of the function f and from below by the x-axis. The region is bounded from the left by the line $x = 1$ and from the right by the line $x = 2$.

i) Compute the area of the region A. [4 marks]

ii) Determine the length of the perimeter of the region A. Note that the perimeter is composed of four curves and that the length of each of these must be computed separately. [6 marks]

iii) Compute the volume of the solid of revolution obtained by rotating the region A around the x-axis. [8 marks]

3. The complex Fourier series for a periodic function, $u(t)$, with period $T = \frac{1}{F}$ is given by

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{j2\pi nFt}$$

where j denotes $\sqrt{-1}$.

a) Show that, if m and n are integers, then

$$\int_0^T e^{j2\pi mFt} e^{j2\pi nFt} dt = \begin{cases} T & \text{if } m = -n \\ 0 & \text{otherwise} \end{cases}.$$

[6 marks]

b) Hence show that

$$\frac{1}{T} \int_0^T u(t) e^{-j2\pi mFt} dt = U_m.$$

State clearly any assumptions you make. [8 marks]

c) Suppose $u(t)$ has period $T_u = 4$ and $u(t) = e^{-0.3t}$ for $0 \le t < 4$.

By evaluating the integral in part b), determine an expression for the complex Fourier coefficients U_m. Simplify the expression where possible. [10 marks]

d) Suppose $v(t)$ has period $T_v = 8$ and $v(t) = \begin{cases} u(t) & \text{for } 0 \le t < 4 \\ u(-t) & \text{for } -4 \le t < 0 \end{cases}$ where $u(t)$ is as defined in part c).

i) Sketch a graph showing both $u(t)$ and $v(t)$ on the same set of axes over the range $-5 \le t \le 5$. [4 marks]

ii) The partial Fourier series of order N are defined by

$$u_N(t) = \sum_{n=-N}^{N} U_n e^{j2\pi nFt}$$
$$v_N(t) = \sum_{n=-N}^{N} V_n e^{j2\pi nFt}$$

where U_n and V_n are respectively the complex Fourier coefficients of the functions $u(t)$ and $v(t)$ defined above.

Explain why, for any fixed value of N, $v_N(t)$ will generally be a better approximation of $u(t)$ over the range $0 \le t \le 4$ than $u_N(t)$.

You are not required to determine expressions for $u_N(t)$ and $v_N(t)$. [5 marks]

MATHEMATICS 1

*********** Solutions ***********

Information for Candidates:

Calculators are not permitted in this exam.

********* Questions and Solutions *********

1. a) i) A function $s(x)$ has 4 distinct real roots and has a continuous derivative. If $g(x) = s'(x)$, what is the minimum number of real roots of $g(x)$? Justify your answer. [4 marks]

By Rolle's theorem we know that there must be at least one x between any two $f(x)$ of equal value where $f'(x) = 0$, thus between two real roots of $f(x)$ there must be a real root of $g(x)$ so minimum number of roots of $g(x) = 3$.

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ii) If the order of a polynomial is known and its roots are known, when is this enough information to specify the polynomial uniquely – always, sometimes or never? Justify your answer. [4 marks]

Multiplying a polynomial by any non-zero real number changes its coefficients but leaves its roots unchanged. Knowing the roots is therefore never enough to specify the polynomial uniquely.

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iii) If $f(x)$ and $g(x)$ are 4^{th} and 3^{rd} order polynomials respectively, then solving $f(x) = g(x)$ is equivalent to finding the roots of a polynomial of what order? Justify your answer. [4 marks]

This is equivalent to $h(x) = f(x) - g(x) = 0$. Since $h(x)$ will have the largest order of f and g, the answer is 4^{th} order.

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b) i) What is the name of the type of function which describes the locus of a point which is always equidistant from a given line and a given reference point? [2 marks]

This is a parabola.

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ii) Find the function $y(x)$ which is equidistant from the line $y = 2$ and the point $(x, y) = (0, 4)$. [4 marks]

The distances to the line and point, r_1 and r_2, are $r_1 = \sqrt{x^2 + (y-4)^2}$ and $r_2 = |y-2|$.

Setting $r_1^2 = r_2^2$ gives $x^2 - 4y + 12 = 0$ or $y = \frac{x^2}{4} + 3$.

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c) i) For a complex number X, where

$$X = \frac{a+ib}{a-ib}$$

and a and b are real, find expressions for the modulus and argument of X. [4 marks]

The modulus of the fraction equals the ratio of the numerator and denominator moduli, and since they are equal the modulus of X is 1. The angle is the numerator angle minus the denominator angle, so $\phi = 2\arctan\left(\frac{b}{a}\right)$?

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ii) For X as in (c)(i) above, if $a = b$, find the value of X in the form $u + iv$. [4 marks]

Here $\phi = 2\arctan(1) = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$. So $X = \exp(\frac{i\pi}{2}) = i$.

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d) i) Euler's equation gives $e^{i\theta}$ in terms of trigonometric functions. Write Euler's equation. [2 marks]

$e^{i\theta} = \cos\theta + i\sin\theta$.

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ii) Using Euler's equation, show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. [2 marks]

$\cos\theta + i\sin\theta = e^{i\theta}$, *therefore* $(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta}$. *But* $e^{in\theta} = e^{i(n\theta)} = \cos n\theta + i\sin n\theta$

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iii) Using (d)(i) and (d)(ii) above, derive trigonometric identities for $\sin 3\theta$ and $\cos 3\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. [4 marks]

Applying the binomial expansion $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta\end{aligned}$$

Equating real and imaginary parts respectively gives

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

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2. Consider the function

$$y = f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x,$$

defined for $x > 0$.

a) Compute the first and second derivatives of the function. Hence determine the only stationary point of the function. Show that $\frac{d^2 f}{dx^2} > 0$, for all $x > 0$, and hence that the stationary point is a minimum. [6 marks]

The first and second derivatives of f are

$$\frac{df}{dx} = \frac{1}{2}\frac{(x-1)(x+1)}{x} \qquad \frac{d^2 f}{dx^2} = \frac{1}{2}\frac{x^2+1}{x^2}.$$

Note that $\frac{d^2 f}{dx^2}$ is clearly positive and that $\frac{df}{dx} = 0$ implies $x = 1$. As a result, $x = 1$ is the only stationary point of the function (for $x > 0$) and it is a minimum.

A few people got this wrong. Some students made a mistake in computing the easy derivative involved. Some students ignored the fact that the function has been defined only for positive x and considered the two stationary points. A few people were imprecise and did not state clearly that since the second derivative is positive then the stationary point is a minimum.

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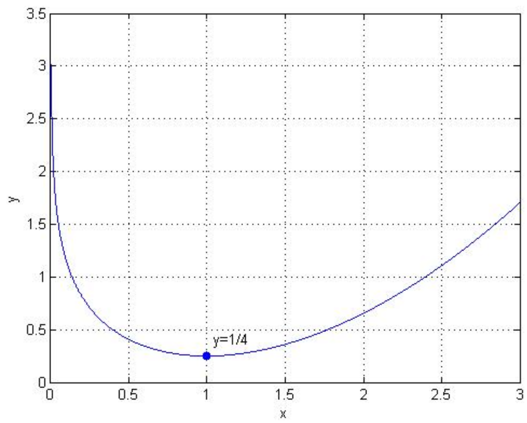
b) Plot the graph of the function for $x \in [0, 3]$. Clearly indicate the stationary point and the value of the function for $x \to 0$. Note that the function f takes positive values for all $x > 0$. [4 marks]

The graph of the function is indicated in the figure below. Note that as $x \to 0$, the function goes to $+\infty$, the stationary point is the point $x = 1$ and $y = 1/4$, and the function is



A few people failed to indicate the vertical asymptote at $x = 0$.

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c) Compute the indefinite integral

$$I = \int f(x)dx.$$

[5 marks]

The indefinite integral is

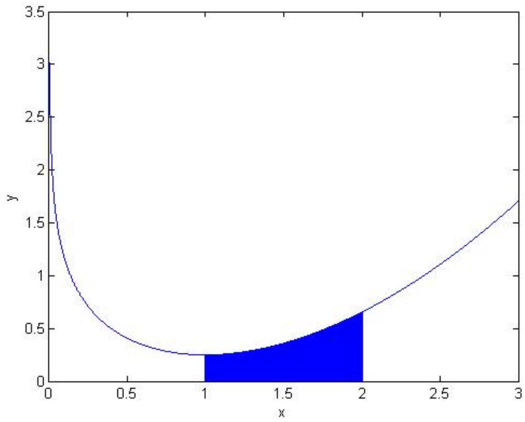
$$I = \frac{1}{12}x^3 - \frac{1}{2}x\ln x + \frac{1}{2}x + c$$

where c is an integration constant.

A few people forgot the integration constant.

d) Consider the region A in the (x,y)-plane defined as follows. The region is bounded from above by the graph of the function f and from below by the x-axis. The region is bounded from the left by the line $x = 1$ and from the right by the line $x = 2$.

The region A is the shaded area in the figure below.



i) Compute the area of the region A. [4 marks]

The area of the region is

$$A = \int_1^2 f(x)dx = \frac{13}{12} - \ln 2 \approx 0.39$$

Some students made mistakes in the integration of $\ln(x)$*, which can be computed by parts.*

ii) Determine the length of the perimeter of the region A. Note that the perimeter is composed of four curves and that the length of each of these must be computed separately. [6 marks]

The length of the perimeter is given by

$$\begin{aligned}
P &= 1+f(1)+f(2)+\int_1^2 \sqrt{1+\left(\frac{df}{dx}\right)^2}\,dx \\
&= 1+\frac{1}{4}+1-\frac{1}{2}\ln 2+\int_1^2 \sqrt{1+\frac{(x^2-1)^2}{4x^2}}\,dx \\
&= 2\frac{1}{4}-\frac{1}{2}\ln 2+\frac{1}{2}\int_1^2 \frac{x^2+1}{x}\,dx \\
&= 2\frac{1}{4}-\frac{1}{2}\ln 2+\frac{1}{2}\left[\frac{x^2}{2}+\ln x\right]_1^2 \\
&= 2\frac{1}{4}-\frac{1}{2}\ln 2+\frac{3}{4}+\frac{1}{2}\ln 2 = 3
\end{aligned}$$

This is where most of the students made mistakes, in particular in the computation of the length of the function. People failed to understand that the argument of the square root is a perfect square and consequently tried various different approaches that made the computation harder or wrong. Computation errors were quite common, although they received little penalty. A few people forgot to compute one or more of the four curves and some did not understand that $f(1)$ and $f(2)$ were the length of the two side lines.

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iii) Compute the volume of the solid of revolution obtained by rotating the region A around the x-axis. [8 marks]

The volume of the surface of revolution is given by

$$\begin{aligned}
V &= \pi\int_1^2 (f(x))^2 dx \\
&= \pi\left[\frac{1}{80}\,x^5-\frac{1}{12}\,x^3\,\ln x+\frac{1}{36}\,x^3+\frac{1}{4}\,x\,(\ln x)^2-\frac{1}{2}\,x\,\ln x+\frac{1}{2}\,x\right]_1^2 \\
&= \pi\left(\frac{779}{720}-\frac{5}{3}\ln 2+\frac{1}{2}(\ln 2)^2\right)\approx 0.1669\pi
\end{aligned}$$

Some students were unable to compute the integration by parts.

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3. The complex Fourier series for a periodic function, $u(t)$, with period $T=\frac{1}{F}$ is given by

$$u(t)=\sum_{n=-\infty}^{\infty}U_n e^{j2\pi nFt}$$

where j denotes $\sqrt{-1}$.

a) Show that, if m and n are integers, then

$$\int_0^T e^{j2\pi mFt}e^{j2\pi nFt}dt=\begin{cases}T & \text{if } m=-n\\ 0 & \text{otherwise}\end{cases}.$$

[6 marks]

For the special case of $m+n=0$, $I=\int_0^T e^{j2\pi(m+n)Ft}dt=\int_0^T e^{j0}dt=T$.

Otherwise (assuming $m+n\neq 0$),

$$\begin{aligned}I=\int_0^T e^{j2\pi mFt}e^{j2\pi nFt}dt&=\int_0^T e^{j2\pi(m+n)Ft}dt\\&=\frac{1}{j2\pi(m+n)F}\left[e^{j2\pi(m+n)Ft}\right]_0^T\\&=\frac{1}{j2\pi(m+n)F}\left(e^{j2\pi(m+n)FT}-1\right)\end{aligned}$$

.

However $FT=1$ and, since $m+n$ is an integer, $e^{j2\pi(m+n)FT}=e^{j2\pi(m+n)}=\left(e^{j2\pi}\right)^{m+n}=1^{m+n}=1$.

Hence $I=\frac{0}{j2\pi(m+n)F}$ which is zero since the denominator is non-zero (because we are dealing with the case $m+n\neq 0$).

Several people stated that $e^{j2\pi(m+n)FT}=1$ without giving any justification at all; it wasn't always evident that they understood that this is only true when $(m+n)FT$ is an integer.

Several people split up $e^{j2\pi(m+n)Ft}=\cos(2\pi(m+n)Ft)+j\sin(2\pi(m+n)Ft)$ which leads to the same answer with quite a lot more work (algebra is almost always easier using complex exponentials). Splitting it up as $e^{j2\pi mFt}e^{j2\pi nFt}=(\cos(2\pi mFt)+j\sin(2\pi mFt))(\cos(2\pi nFt)+j\sin(2\pi nFt))$ is an even worse idea unless you really love trigonometry; a few people split it up like this and then ignored the imaginary part completely.

Some just stated that the averages of cos and sin were zero rather than doing the integration to prove it; although the statement is true, it comes very close to assuming the result that you are being asked to prove. Most people did not explicitly say why the original integral is invalid when $m+n=0$ (i.e. because the denominator is zero).

It is easiest to deal separately with the special case of $m+n=0$ as is done above. Alternatively, you can use L'Hôpital's rule treating $m+n$ as a continuous variable, x. Thus if we substitute $x=m+n$, we have $I(x)=\frac{e^{j2\pi xFT}-1}{j2\pi xF}$; differentiating numerator and denominator with respect to x gives $I(0)=\left.\frac{j2\pi FTe^{j2\pi xFT}}{j2\pi F}\right|_{x=0}=T$. Most who tried this method got it wrong; commonly they wrote the equivalent of $I(x)=\frac{\cos(2\pi xFT)+j\sin(2\pi xFT)-1}{j2\pi xF}$ and then said $I(0)=\frac{\cos(2\pi 0FT)+j\sin(2\pi 0FT)-1}{j2\pi 0F}=$

$\frac{j\sin(2\pi 0FT)}{j2\pi 0F}$ and finally applied L'Hôpital's rule to the latter expression. To apply L'Hôpital's rule correctly, you need to differentiate with respect to x before substituting $x = 0$ in any of the expression.

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b) Hence show that

$$\frac{1}{T}\int_0^T u(t)e^{-j2\pi mFt}dt = U_m.$$

State clearly any assumptions you make. [8 marks]

$$\begin{aligned}
\frac{1}{T}\int_0^T u(t)e^{-j2\pi mFt}dt &= \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty} U_n e^{j2\pi nFt}e^{-j2\pi mFt}dt \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} U_n \int_0^T e^{j2\pi(n-m)Ft}dt \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} U_n T\delta_{n-m} \\
&= \sum_{n=-\infty}^{\infty} U_n\delta_{n-m} = U_m
\end{aligned}$$

where $\delta_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$.

In the second line, we assume that we can interchange the order of the summation and integral, i.e. that the integral of the sum is equal to the sum of the integrals of the individual terms. In the final line we are summing an infinite number of terms but they are all zero except for the term with $m = n$; we assume that the sum of an infinite number of zero terms equals zero.

A surprising number of people were unable to do this part even though it follows directly from the previous part. A few assumed the formulae for the real Fourier series coefficients which is almost the same as assuming the result they were asked to prove.

In the first line above, we substitute $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{j2\pi nFt}$; several people instead substituted $u(t) = \sum_{m=-\infty}^{\infty} U_m e^{j2\pi mFt}$ where they used "m" for the dummy variable even though "m" is used elsewhere in the expression. This is not allowed: dummy variables must not conflict with variables used outside the summation.

Some people were confused by the "-" sign in the equation given in the questions which was not there in part a). In this part, the exponents cancel when $m = n$ and not when $m = -n$. A few said "assuming $m = n$" or "assuming $m = -n$" and then quietly removed the summation.

Practically no one stated either of the two assumptions given above: (i) that interchanging integration and summation is OK and (ii) that the sum of an infinite number of zero terms is zero. Neither of these assumptions is true under all circumstances. Many people did not explicitly swap the integral and summation but instead just stated that you could ignore most of the summation terms.

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c) Suppose $u(t)$ has period $T_u = 4$ and $u(t) = e^{-0.3t}$ for $0 \le t < 4$.

By evaluating the integral in part b), determine an expression for the complex Fourier coefficients U_m. Simplify the expression where possible. [10 marks]

$$
\begin{aligned}
U_m &= \frac{1}{T_u}\int_0^{T_u} u(t)e^{-j2\pi mFt}dt \\
&= \frac{1}{4}\int_0^{4} e^{-0.3t}e^{-j0.5\pi mt}dt \\
&= \frac{1}{4}\int_0^{4} e^{(-0.3-j0.5\pi m)t}dt \\
&= \frac{1}{4(-0.3-j0.5\pi m)}\left[e^{(-0.3-j0.5\pi m)t}\right]_0^4 \\
&= \frac{1}{-1.2-j2\pi m}\left(e^{-1.2-j2\pi m}-1\right)
\end{aligned}
$$

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However, $e^{-j2\pi m} \equiv 1$ since its exponent is an integer multiple of $j2\pi$. It follows that $e^{-1.2-j2\pi m} = e^{-1.2} \times e^{-j2\pi m} = e^{-1.2}$.

Hence $U_m = \frac{e^{-1.2}-1}{-1.2-j2\pi m} = \frac{1-e^{-1.2}}{1.2+j2\pi m}$.

In this integral, decomposing $e^{-j2\pi mFt} = \cos(2\pi mFt) - j\sin(2\pi mFt)$ is a bad idea. Another popular but bad idea is to multiply the numerator and denominator by $1.2 - j2\pi m$ (i.e. the complex conjugate of the denominator); doing this almost always makes algebra more complicated. A few factorized $e^{-1.2-j2\pi m} - 1 = e^{-0.6-j\pi m}\left(e^{-0.6-j\pi m} - e^{0.6+j\pi m}\right)$; this trick is often useful, but only when the exponent is purely imaginary which is not true in this case. There were quite a lot of algebra errors: the commonest was omitting minus signs and the second commonest was the factor $\frac{1}{4}$ being forgotten or else only being applied to one of the two terms in the denominator. A few people did not substitute $F = 0.25$ and so ended up with an unsimplified expression.

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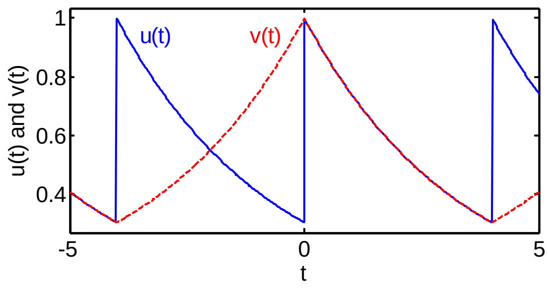
d) Suppose $v(t)$ has period $T_v = 8$ and $v(t) = \begin{cases} u(t) & \text{for } 0 \le t < 4 \\ u(-t) & \text{for } -4 \le t < 0 \end{cases}$ where $u(t)$ is as defined in part c).

i) Sketch a graph showing both $u(t)$ and $v(t)$ on the same set of axes over the range $-5 \le t \le 5$. [4 marks]

$u(t)$ and $v(t)$ are identical for $0 \le t < 4$ but $v(t)$ is an even function. So the graphs look like



The two functions are identical for $t \in [-5, -4) \cup [0, 4)$.

Quite a few people only drew one period of each function instead of making them periodic. Several made $v(t) = -u(t)$ in the range $[-4, 0)$ instead of $v(t) = u(-t)$. Some people showed $u(t)$ as periodic with period 8 instead of 4. Some of the graphs had $u(t)$ and/or $v(t)$ going to infinity at $t = 0$. Several people only drew a graph of $v(t)$ even though they were asked for $u(t)$ as well.

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ii) The partial Fourier series of order N are defined by

$$u_N(t) = \sum_{n=-N}^{N} U_n e^{j2\pi nFt}$$

$$v_N(t) = \sum_{n=-N}^{N} V_n e^{j2\pi nFt}$$

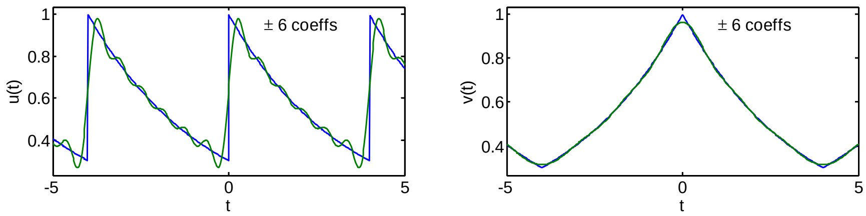
where U_n and V_n are respectively the complex Fourier coefficients of the functions $u(t)$ and $v(t)$ defined above.

Explain why, for any fixed value of N, $v_N(t)$ will generally be a better approximation of $u(t)$ over the range $0 \le t \le 4$ than $u_N(t)$.

You are not required to determine expressions for $u_N(t)$ and $v_N(t)$.

[5 marks]

$u(t)$ has a discontinuity whenever t is a multiple of 4 whereas $v(t)$ has no discontinuities. This has three consequences: (i) $u_N(t)$ will suffer from Gibbs phenomenon and will overshoot $u(t)$ by about 9% either side of the discontinuity, (ii) $u_N(0) \to 0.5(1 + e^{-1.2})$ whereas $u(0) = 1$ and (iii) for large n we will have $U_n \propto n^{-1}$ whereas we will have $V_n \propto n^{-2}$. This means that the energy of the error will decrease faster with N for $v_N(t)$ than for $u_N(t)$. Although not requested from candidates, graphs of $u_6(t)$ and $v_6(t)$ are shown below.



Several people also noted that, since $v(t)$ is an even function, the V_n will all be real-valued. This is true but not relevant to the convergence which would, for example, be unaffected by a slight time delay *(e.g. $v(t) \to v(t-1)$) even though this would destroy the evenness. Also irrelevant is the fact that $v(t)$ has a longer period than $u(t)$. Some people said that $v(t)$ had some discontinuities (or had "fewer discontinuities than $u(t)$"): it doesn't have any discontinuities (although its derivative does).*