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The Basic Idea

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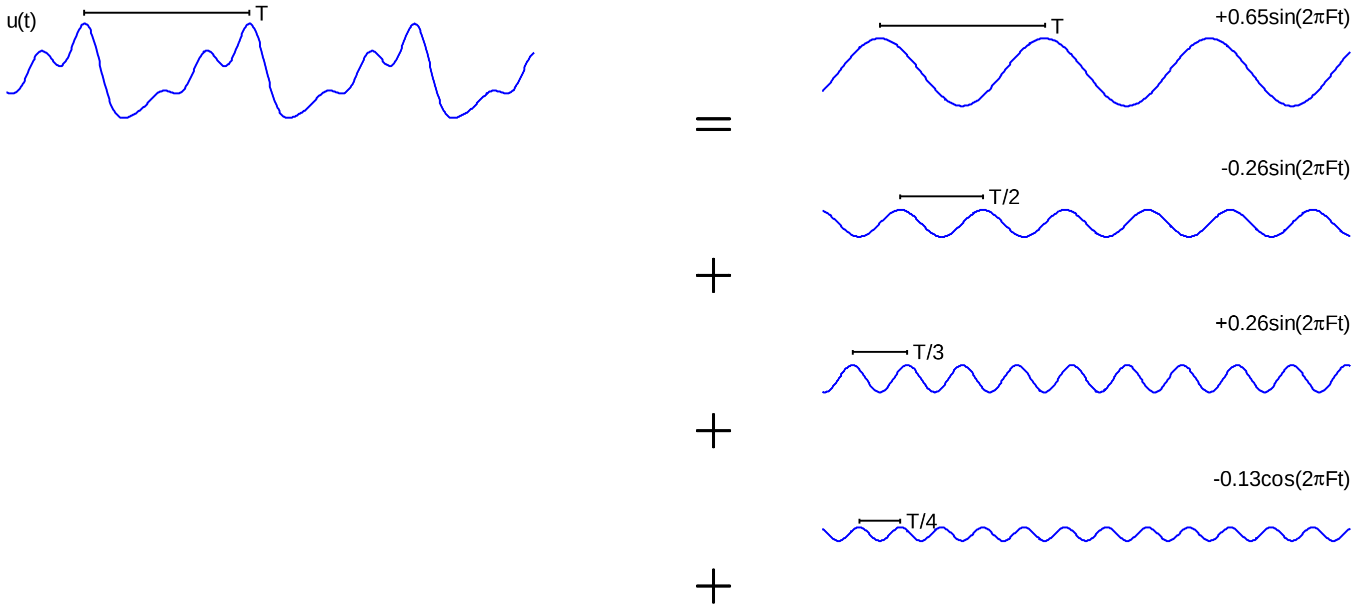
Periodic signals can be written as a sum of sine and cosine waves:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$



Fundamental Period: the smallest $T > 0$ for which $u(t+T) = u(t)$.
Fundamental Frequency: $F = \frac{1}{T}$. The n^{th} harmonic is at frequency nF.
Some waveforms need infinitely many harmonics (countable infinity).

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Real versus Complex Fourier Series

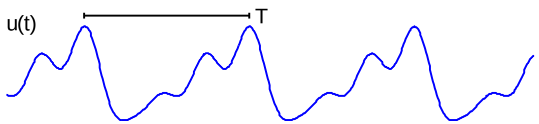
All the algebra is much easier if we use $e^{i\omega t}$ instead of $\cos \omega t$ and $\sin \omega t$

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi n F t + b_n \sin 2\pi n F t\right)$$

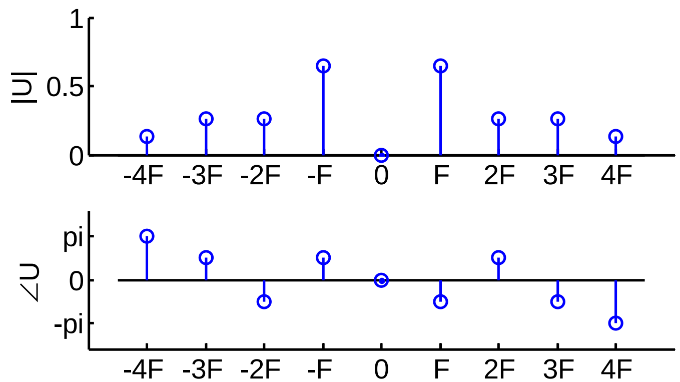
Substitute: $\cos \omega t = \frac{1}{2}e^{i\omega t} + \frac{1}{2}e^{-i\omega t}$ $\qquad \sin \omega t = \frac{-i}{2}e^{i\omega t} + \frac{i}{2}e^{-i\omega t}$

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\left(a_n - ib_n\right) e^{i2\pi n F t} + \frac{1}{2}\left(a_n + ib_n\right) e^{-i2\pi n F t}\right)$$
$$= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi n F t}$$

- $U_{+n} = \frac{1}{2}\left(a_n - ib_n\right)$ and $U_{-n} = \frac{1}{2}\left(a_n + ib_n\right)$.
- U_{+n} and U_{-n} are complex conjugates.
- U_{+n} is half the equivalent phasor in Analysis of Circuits.



Plot the magnitude spectrum and phase spectrum:



Fourier Series versus Fourier Transform

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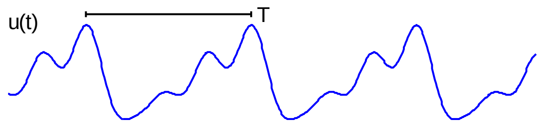
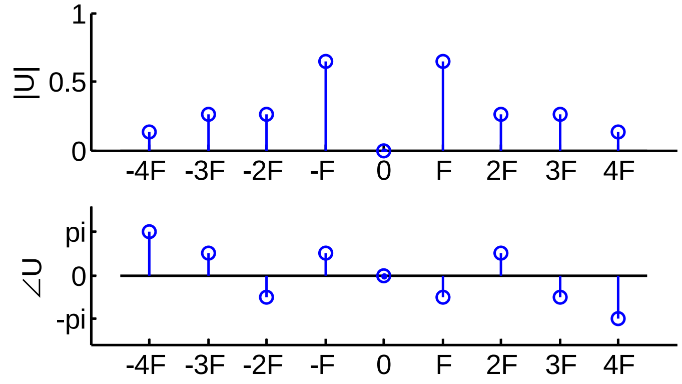
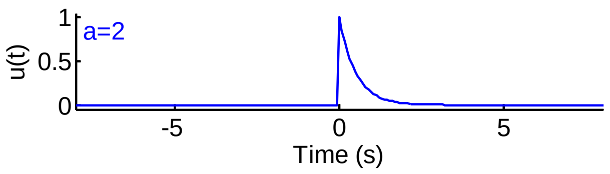
- Periodic signals → Fourier Series→ Discrete spectrum



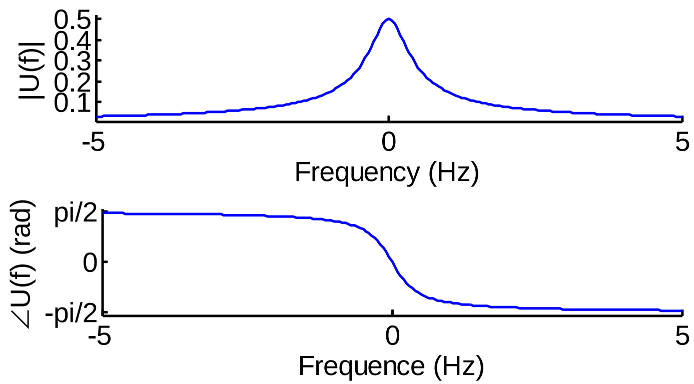
$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$



- Aperiodic signals → Fourier Transform→ Continuous Spectrum



$$u(t) = \int_{f=-\infty}^{\infty} U(f) e^{i2\pi ft} df$$



- Both types of spectrum are conjugate symmetric.
- If $u(t)$ is periodic, its Fourier transform consists of Dirac δ functions with amplitudes $\{U_n\}$.

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Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

Fourier Analysis = "how do you work out the Fourier coefficients, U_n ?"

Key idea: $\left\langle e^{i\omega t}\right\rangle = \langle\cos\omega t + i\sin\omega t\rangle = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow$ Orthogonality: $\left\langle e^{i2\pi nFt} \times e^{-i2\pi mFt}\right\rangle = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$

So, to find a particular coefficient, U_m, we work out

$$\begin{aligned}\left\langle u(t)e^{-i2\pi mFt}\right\rangle &= \left\langle \left(\sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}\right) e^{-i2\pi mFt}\right\rangle \\ &= \sum_{n=-\infty}^{\infty} U_n \left\langle e^{i2\pi nFt} e^{-i2\pi mFt}\right\rangle \\ &= U_m \qquad \text{[since all other terms are zero]}\end{aligned}$$

Calculate the average by integrating over any integer number of periods

$$U_m = \left\langle u(t)e^{-i2\pi mFt}\right\rangle = \frac{1}{T}\int_{t=0}^{T} u(t)e^{-i2\pi mFt}dt$$

Notice the negative sign in Fourier analysis: in order to extract the term in the series containing $e^{+i2\pi mFt}$ we need to multiply by $e^{-i2\pi mFt}$.

Power Conservation

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Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

Average power in $u(t)$: $P_u \triangleq \left\langle |u(t)|^2 \right\rangle = \frac{1}{T}\int_0^T u^2(t)dt$ [$u(t)$ real]

Average power in Fourier component n:

$$\left\langle \left|U_n e^{i2\pi nFt}\right|^2 \right\rangle = \left\langle |U_n|^2 \left|e^{i2\pi nFt}\right|^2 \right\rangle = |U_n|^2$$

Power conservation (Parseval's Theorem):

$$P_u = \left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} |U_n|^2$$

The average power in $u(t)$ is equal to the sum of the average powers in all the Fourier components.

This is a consequence of orthogonality:

$$\begin{aligned}
\left\langle |u(t)|^2 \right\rangle &= \left\langle \left(\sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}\right)\left(\sum_{m=-\infty}^{\infty} U_m^* e^{-i2\pi mFt}\right)\right\rangle \\
&= \left\langle \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} U_n U_m^* e^{i2\pi nFt} e^{-i2\pi mFt}\right\rangle \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} U_n U_m^* \left\langle e^{i2\pi nFt} e^{-i2\pi mFt}\right\rangle \\
&= \sum_{n=-\infty}^{\infty} |U_n|^2
\end{aligned}$$

Gibbs Phenomenon

Truncated Fourier Series: $u_N(t) = \sum_{n=-N}^{N} U_n e^{i2\pi nFt}$

Approximation error: $e_N(t) = u_N(t) - u(t)$

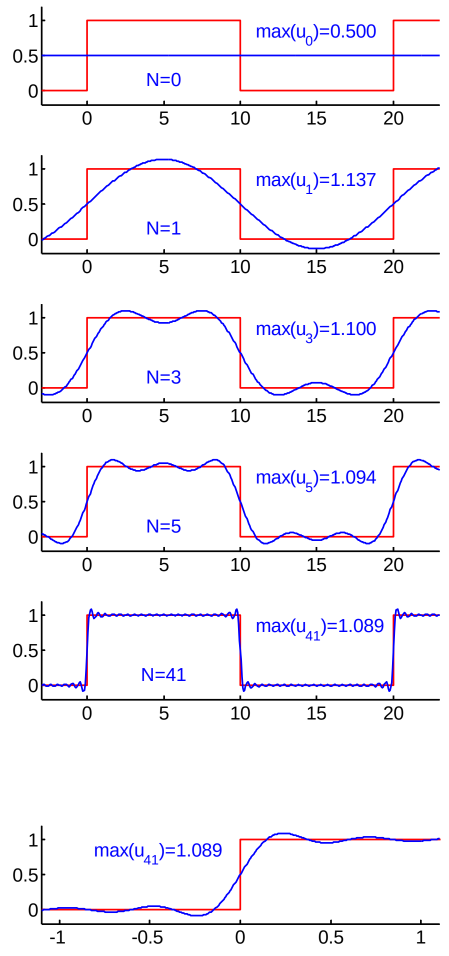
Average error power $P_{e_N} = \sum_{|n|>N} |U_n|^2$.

$P_{e_N} \to 0$ monotonically as $N \to \infty$.

Gibbs phenomenon

If $u(t_0)$ has a discontinuity of height h then:

- $u_N(t_0) \to$ the midpoint of the discontinuity as $N \to \infty$.
- $u_N(t)$ overshoots by $\approx \pm 9\% \times h$ at $t \approx t_0 \pm \frac{T}{2N+1}$.
- For large N, the overshoots move closer to the discontinuity but do not decrease in size.



[Enlarged View: $u_{41}(t)$]

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Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

Integration:

$$v(t) = \int_0^t u(\tau)d\tau \quad \Rightarrow \quad V_n = \frac{1}{i2\pi nF} U_n$$

provided $U_0 = V_0 = 0$.

Differentiation:

$$w(t) = \frac{du(t)}{dt} \quad \Rightarrow \quad W_n = i2\pi nF \times U_n$$

provided $w(t)$ satisfies the Dirichlet conditions.

Coefficient Decay Rate:

$u(t)$ has a discontinuity $\Rightarrow$ $|U_n|$ is $O\left(\frac{1}{n}\right)$ for large $|n|$

$\frac{d^k u(t)}{dt^k}$ is the lowest derivative with a discontinuity

$\Rightarrow$ $|U_n|$ is $O\left(\frac{1}{n^{k+1}}\right)$ for large $|n|$

> If the coefficients, U_n, decrease rapidly then only a few terms are needed for a good approximation.

Periodic Extension

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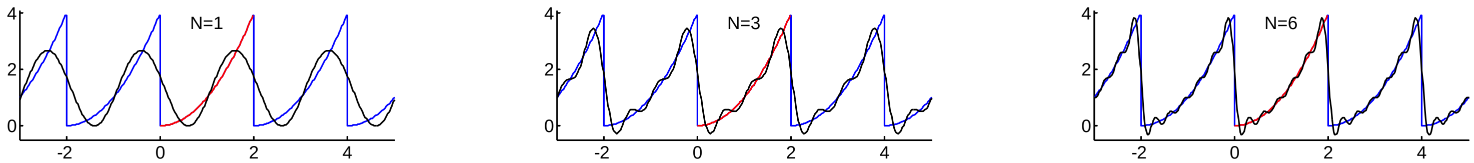
If $u(t)$ is only defined over a finite range, $[0, B]$, we can make it periodic by defining $u(t \pm B) = u(t)$.

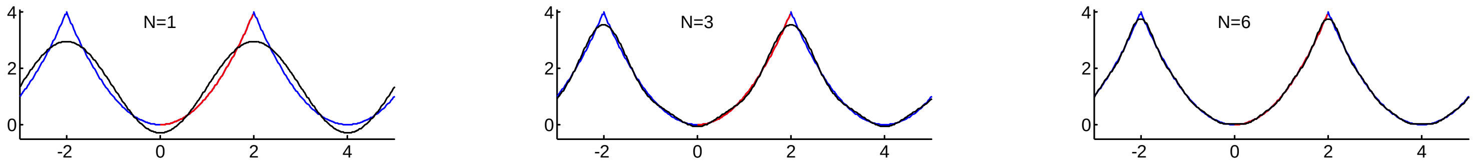
- Coefficients are given by $U_n = \frac{1}{B}\int_0^B u(t)e^{-i2\pi nFt}dt$.

Example: $u(t) = t^2$ for $0 \le t < 2$



Symmetric extension:

- To avoid a discontinuity at $t = T$, we can instead make the period $2B$ and define $u(-t) = u(+t)$.



- Symmetry around $t = 0$ means coefficients are real-valued and symmetric $(U_{-n} = U_n^* = U_n)$.
- Still have a first-derivative discontinuity at $t = B$ but now we have no Gibbs phenomenon and coefficients $\propto n^{-2}$ instead of $\propto n^{-1}$ so approximation error power decreases more quickly.

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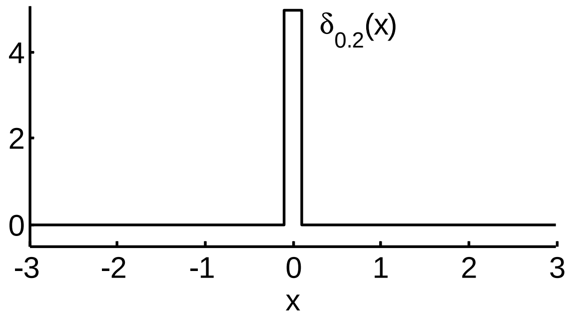
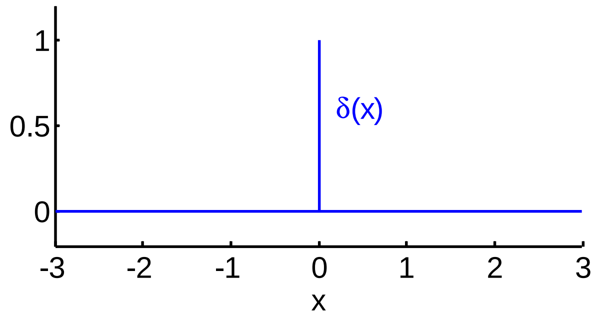
$\delta(x)$ is the limiting case as $w \to 0$ of a pulse w wide and $\frac{1}{w}$ high
It is an infinitely thin, infinitely high pulse at $x = 0$ with unit area.





- Area: $\int_{-\infty}^{\infty} \delta(x)dx = 1$
- Scaling: $\delta(cx) = \frac{1}{|c|}\delta(x)$
- Shifting: $\delta(x-a)$ is a pulse at $x = a$ and is zero everywhere else
- Multiplication: $f(x) \times \delta(x-a) = f(a) \times \delta(x-a)$
- Integration: $\int_{-\infty}^{\infty} f(x) \times \delta(x-a)dx = f(a)$
- Fourier Transform: $u(t) = \delta(t) \quad \Leftrightarrow \quad U(f) = 1$
- We plot $h\delta(x)$ as a pulse of height $|h|$ (instead of its true height of ∞)

Fourier Transform

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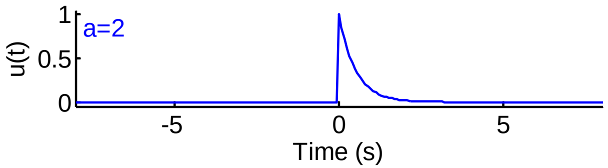
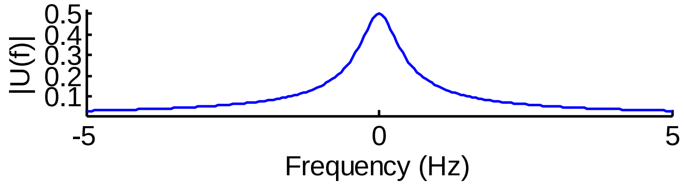
Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi f t} df$

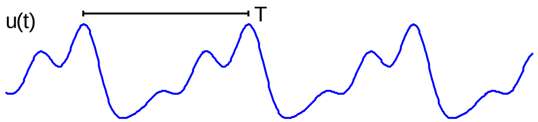
$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi f t} dt$$

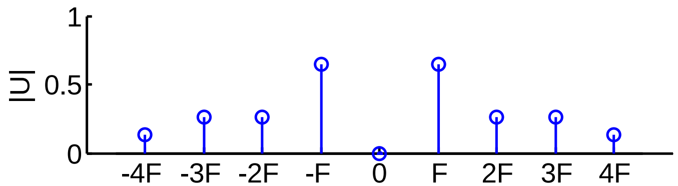
- An "Energy Signal" has finite energy $\Leftrightarrow E_u = \int_{-\infty}^{\infty} |u(t)|^2 \, dt < \infty$
  - Complex-valued spectrum, $U(f)$, decays to zero as $f \to \pm\infty$
  - Energy Conservation: $E_u = E_U$ where $E_U = \int_{-\infty}^{\infty} |U(f)|^2 \, df$





- Periodic Signals $\to$ Dirac δ functions at harmonics.
  Same complex-valued amplitudes as U_n from Fourier Series





  - $E_u = \infty$ but ave power is $P_u = \left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} |U_n|^2$

Convolution

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Convolution: $w(t) = u(t) * v(t) \quad \Leftrightarrow \quad w(t) = \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau$

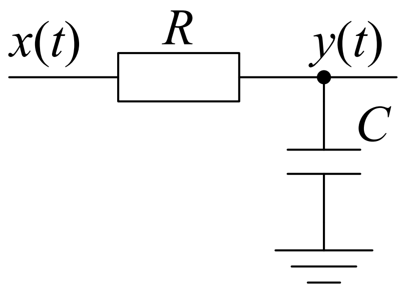
[In the integral, the arguments of $u(\ )$ and $v(\ )$ add up to t]

$*$ acts algebraically like $\times$: Commutative, Associative, Distributive over $+$.
Identity element is $\delta(t)$: $u(t) * \delta(t) = u(t)$

Multiplication in either the time or frequency domain is equivalent to convolution in the other domain:

$$w(t) = u(t) * v(t) \quad \Leftrightarrow \quad W(f) = U(f)V(f)$$
$$y(t) = u(t)v(t) \quad \Leftrightarrow \quad Y(f) = U(f) * V(f)$$

Example application:



- Impulse Response: [$\triangleq y(t)$ for $x(t) = \delta(t)$]
  $h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}$ for $t \geq 0$
- Frequency Response: $H(f) = \frac{1}{1+i2\pi fRC}$
- Convolution: $y(t) = h(t) * x(t)$
- Multiplication: $Y(f) = H(f)X(f)$

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Cross-correlation:

$$w(t) = u(t) \otimes v(t) \quad \Leftrightarrow \quad w(t) = \int_{-\infty}^{\infty} u^*(\tau - t) v(\tau) d\tau$$

[In the integral, the arguments of $u^*(\ )$ and $v(\ )$ differ by t]

$\otimes$ is not commutative or associative (unlike $*$)

Cauchy-Schwartz Inequality ⇒ Bound on $|w(t)|$

- For all values of t: $|w(t)|^2 \le E_u E_v$
- $u(t - t_0)$ is an exact multiple of $v(t) \Leftrightarrow |w(t_0)|^2 = E_u E_v$

Normalized cross-correlation: $\frac{w(t)}{\sqrt{E_u E_v}}$ has a maximum absolute value of 1

- Cross-correlation is used to find the time shift, t_0, at which two signals match and also how well they match.
- Auto-correlation is the cross-correlation of a signal with itself: used to find the period of a signal (i.e. the time shift where it matches itself).