E1.10 Fourier Series and Transforms

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Main fact: Complicated time waveforms can be expressed as a sum of sine and cosine waves.

Why bother? Sine/cosine are the only bounded waves that stay the same when differentiated.

Any electronic circuit:
sine wave in $\Rightarrow$ sine wave out (same frequency).

Joseph Fourier
1768-1830

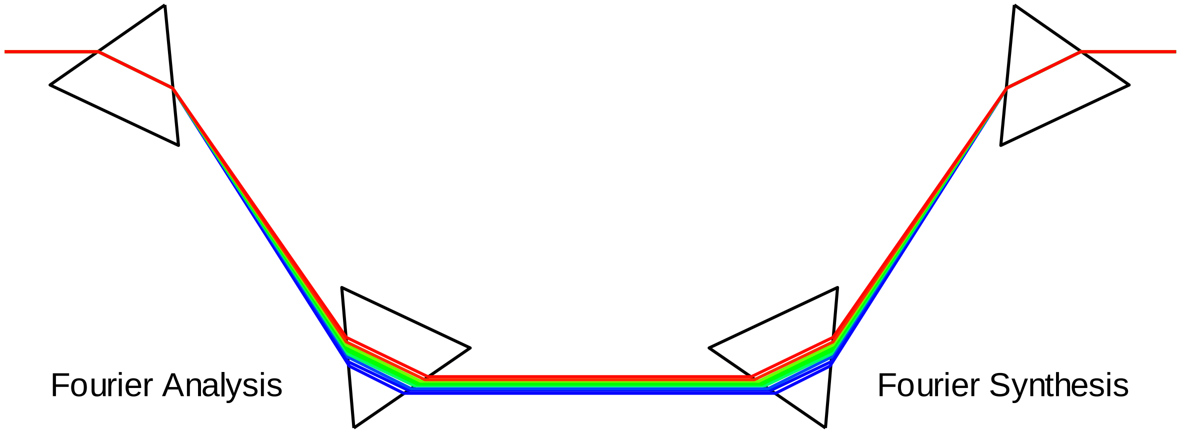
Hard problem: Complicated waveform $\rightarrow$ electronic circuit$\rightarrow$ output = ?

Easier problem: Complicated waveform $\rightarrow$ sum of sine waves
$\rightarrow$ linear electronic circuit ($\Rightarrow$ obeys superposition)
$\rightarrow$ add sine wave outputs $\rightarrow$ output = ?

Syllabus: Preliminary maths (1 lecture)
Fourier series for periodic waveforms (4 lectures)
Fourier transform for aperiodic waveforms (3 lectures)

Optical Fourier Transform

A pair of prisms can split light up into its component frequencies (colours).
This is called Fourier Analysis.
A second pair can re-combine the frequencies.
This is called Fourier Synthesis.



We want to do the same thing with mathematical signals instead of light.

Organization

- 8 lectures: feel free to ask questions
- Textbook: Riley, Hobson & Bence "Mathematical Methods for Physics and Engineering", ISBN:978052167971-8, Chapters [4], 12 & 13
- Lecture slides (including animations) and problem sheets + answers available via Blackboard or from my website: http://www.ee.ic.ac.uk/hp/staff/dmb/courses/E1Fourier/E1Fourier.htm
- Email me with any errors in slides or problems and if answers are wrong or unclear

1: Sums and Averages

A geometric series is a sum of terms that increase or decrease by a constant factor, x:

$$S = a + ax + ax^2 + \ldots + ax^n$$

The sequence of terms themselves is called a geometric progression.

We use a trick to get rid of most of the terms:

$$\begin{aligned} S &= a + ax + ax^2 + \ldots + ax^{n-1} + ax^n \\ xS &= \quad\; ax + ax^2 + ax^3 + \ldots \quad + ax^n + ax^{n+1} \end{aligned}$$

Now subtract the lines to get: $S - xS = (1-x)\,S = a - ax^{n+1}$

Divide by $1-x$ to get: $a =$ first term $\quad n+1 =$ number of terms

$$S = a \times \frac{1-x^{n+1}}{1-x}$$

Example:

$$S = 3 + 6 + 12 + 24 \qquad [a = 3,\ x = 2,\ n+1 = 4]$$

$$= 3 \times \frac{1-2^4}{1-2} = 3 \times \frac{-15}{-1} = 45$$

Infinite Geometric Series

A finite geometric series: $S_n = a + ax + ax^2 + \cdots + ax^n = a\frac{1-x^{n+1}}{1-x}$

What is the limit as $n \to \infty$?

If $|x| < 1$ then $x^{n+1} \underset{n\to\infty}{\longrightarrow} 0$ which gives

$$S_\infty = a + ax + ax^2 + \cdots = a\frac{1}{1-x} = \frac{a}{1-x}$$

$a =$ first term

$x =$ factor

Example 1:

$$0.4 + 0.04 + 0.004 + \ldots = \frac{0.4}{1-0.1} = 0.\dot{4} \qquad [a = 0.4,\ x = 0.1]$$

Example 2: (alternating signs)

$$2 - 1.2 + 0.72 - 0.432 + \ldots = \frac{2}{1-(-0.6)} = 1.25 \qquad [a = 2,\ x = -0.6]$$

Example 3:

$$1 + 2 + 4 + \ldots \neq \frac{1}{1-2} = \frac{1}{-1} = -1 \qquad [a = 1,\ x = 2]$$

The formula $S = a + ax + ax^2 + \ldots = \frac{a}{1-x}$ is only valid for $|x| < 1$

Dummy Variables

Using a $\sum$ sign, we can write the geometric series more compactly:

$$S_n = a + ax + ax^2 + \ldots + ax^n = \sum_{r=0}^{n} ax^r$$

[Note: $x^0 \triangleq 1$ in this context even when $x = 0$]

Here r is a dummy variable: you can replace it with anything else

$$\sum_{r=0}^{n} ax^r \quad = \quad \sum_{k=0}^{n} ax^k \quad = \quad \sum_{\alpha=0}^{n} ax^\alpha$$

Dummy variables are undefined outside the summation so they sometimes get re-used elsewhere in an expression:

$$\sum_{r=0}^{3} 2^r + \sum_{r=1}^{2} 3^r = \left(1 \times \frac{1-2^4}{1-2}\right) + \left(3 \times \frac{1-3^2}{1-3}\right) = 15 + 12 = 27$$

The two dummy variables are both called r but they have no connection with each other at all (or with any other variable called r anywhere else).

We can derive the formula for the geometric series using $\sum$ notation:

$$S_n = \sum_{r=0}^{n} ax^r \text{ and } xS_n = \sum_{r=0}^{n} ax^{r+1}$$

We need to manipulate the second sum to involve x^r.

Use the substitution $s = r + 1 \Leftrightarrow r = s - 1$.
Substitute for r everywhere it occurs (including both limits)

$$xS_n = \sum_{s=1}^{n+1} ax^s = \sum_{r=1}^{n+1} ax^r$$

It is essential to sum over exactly the same set of values when substituting for dummy variables.

Subtracting gives $(1-x)S_n = S_n - xS_n = \sum_{r=0}^{n} ax^r - \sum_{r=1}^{n+1} ax^r$

$r \in [1, n]$ is common to both sums, so extract the remaining terms:

$$(1-x)S_n = ax^0 - ax^{n+1} + \sum_{r=1}^{n} ax^r - \sum_{r=1}^{n} ax^r$$
$$= ax^0 - ax^{n+1} = a\left(1 - x^{n+1}\right)$$

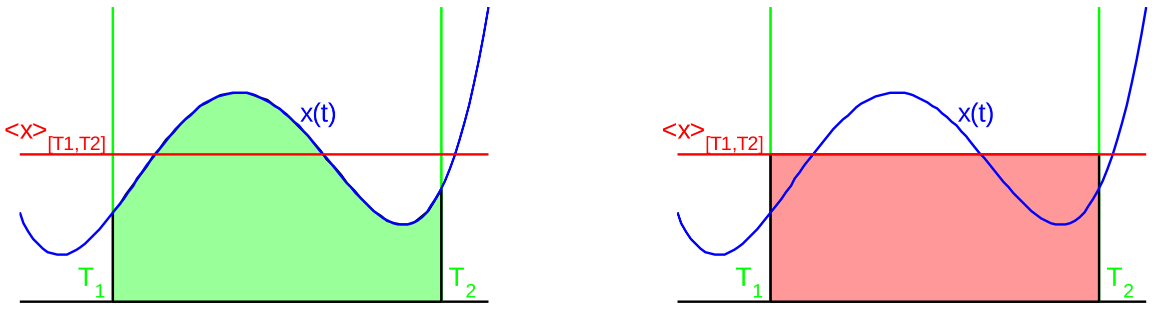
Hence: $S_n = a\frac{1-x^{n+1}}{1-x}$

Averages

If a signal varies with time, we can plot its waveform, $x(t)$.

The average value of $x(t)$ in the range $T_1 \le t \le T_2$ is

$$\langle x \rangle_{[T_1,T_2]} = \frac{1}{T_2 - T_1} \int_{t=T_1}^{T_2} x(t)dt$$



The area under the curve $x(t)$ is equal to the area of the rectangle defined by 0 and $\langle x \rangle_{[T_1,T_2]}$.

Angle brackets alone, $\langle x \rangle$, denotes the average value over all time

$$\langle x(t) \rangle = \lim_{A,B \to \infty} \langle x(t) \rangle_{[-A,+B]}$$

The properties of averages follow from the properties of integrals:

Addition: $\langle x(t) + y(t) \rangle = \langle x(t) \rangle + \langle y(t) \rangle$

Add a constant: $\langle x(t) + c \rangle = \langle x(t) \rangle + c$

Constant multiple: $\langle a \times x(t) \rangle = a \times \langle x(t) \rangle$

where the constants a and c do not depend on time.

For example:

$$\begin{aligned}\langle x(t) + y(t) \rangle_{[T_1,T_2]} &= \tfrac{1}{T_2 - T_1} \int_{t=T_1}^{T_2} (x(t) + y(t))\, dt \\ &= \tfrac{1}{T_2 - T_1} \int_{t=T_1}^{T_2} x(t) dt + \tfrac{1}{T_2 - T_1} \int_{t=T_1}^{T_2} y(t) dt \\ &= \langle x(t) \rangle_{[T_1,T_2]} + \langle y(t) \rangle_{[T_1,T_2]}\end{aligned}$$

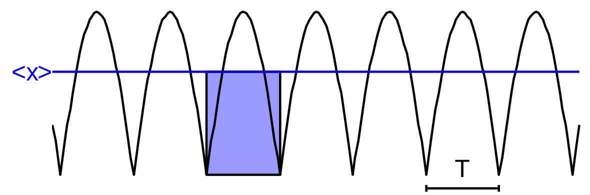
But beware: $\langle x(t) \times y(t) \rangle \neq \langle x(t) \rangle \times \langle y(t) \rangle$.

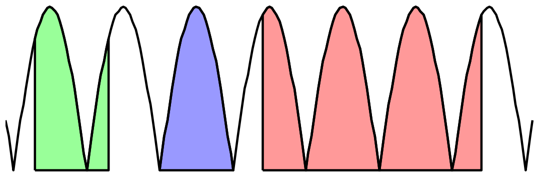
Periodic Waveforms

A periodic waveform with period T repeats itself at intervals of T:

$$x(t+T) = x(t) \quad \Rightarrow \quad x(t \pm kT) = x(t) \text{ for any integer } k.$$

The smallest $T > 0$ for which $x(t+T) = x(t)\ \forall t$ is the fundamental period. The fundamental frequency is $F = \frac{1}{T}$.





For a periodic waveform, $\langle x(t) \rangle$ equals the average over one period.
It doesn't make any difference where in a period you start or how many whole periods you take the average over.

Example:

$$x(t) = |\sin t|$$

$$\langle x \rangle = \frac{1}{\pi}\int_{t=0}^{\pi} |\sin t|\ dt = \frac{1}{\pi}\int_{t=0}^{\pi} \sin t\, dt$$

$$= \frac{1}{\pi}\left[-\cos t\right]_0^{\pi} = \frac{1}{\pi}(1+1) = \frac{2}{\pi} \approx 0.637$$

[proof that $x(t \pm kT) = x(t)$]

Proof that $x(t+T) = x(t)\ \forall t \ \Rightarrow\ x(t \pm kT) = x(t)\ \forall t, \forall k \in \mathbb{Z}$

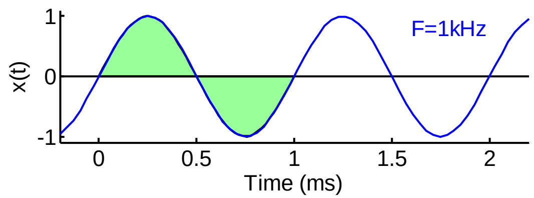
We use induction. Let H_k be the hypothesis that $x(t+kT) = x(t)\ \forall t$. Under the assumption that $x(t+T) = x(t)\ \forall t$, we will show that if H_k is true, then so are H_{k+1} and H_{k-1}. Since we know that H_0 is definitely true, this implies that H_k is true for all integers k, i.e. for all $k \in \mathbb{Z}$.

- ☐ Suppose H_k is true, i.e. $x(\tau + kT) = x(\tau)\ \forall \tau$. Now set $\tau = t + T$. This gives $x(t + T + kT) = x(t+T)\ \forall t$. But, we assume that $x(t+T) = x(t)$, so $x\left(t + (k+1)T\right) = x(t+T+kT) = x(t+T) = x(T)\ \forall t$. Hence $H_{k+1}$is true.
- ☐ Now suppose H_k is true as before but this time set $\tau = t - T$. Substituting this into $u(\tau + kT) = u(\tau)$ gives $u(t-T+kT) = u(t-T)$. Substituting it also into $u(\tau+T) = u(\tau)$ gives $u(t) = u(t-T)$. Finally, combining these two identities gives $u\left(t + (k-1)T\right) = u(t)$ which is H_{k-1}.

A sine wave, $x(t) = \sin 2\pi Ft$, has a frequency F and a period $T = \frac{1}{F}$

so that, $\sin\left(2\pi F\left(t + \frac{1}{F}\right)\right) = \sin\left(2\pi Ft + 2\pi\right) = \sin 2\pi Ft$.

$$\begin{aligned}\langle \sin 2\pi Ft \rangle &= \frac{1}{T}\int_{t=0}^{T} \sin\left(2\pi Ft\right)\, dt \\ &= 0\end{aligned}$$



Also, $\langle \cos 2\pi Ft \rangle = 0$ except for the case $F = 0$ since $\cos 2\pi 0t \equiv 1$.

Hence: $\langle \sin 2\pi Ft \rangle = 0$ and $\langle \cos 2\pi Ft \rangle = \begin{cases} 0 & F \neq 0 \\ 1 & F = 0 \end{cases}$

Also:
$$\begin{aligned}\left\langle e^{i2\pi Ft} \right\rangle &= \langle \cos 2\pi Ft + i \sin 2\pi Ft \rangle \\ &= \langle \cos 2\pi Ft \rangle + i \langle \sin 2\pi Ft \rangle \\ &= \begin{cases} 0 & F \neq 0 \\ 1 & F = 0 \end{cases}\end{aligned}$$

Summary

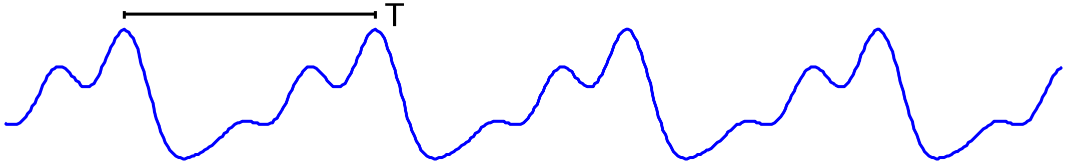
- Sum of geometric series (see RHB Chapter 4)
  - Finite series: $S = a \times \frac{1-x^{n+1}}{1-x}$
  - Infinite series: $S = \frac{a}{1-x}$ but only if $|x| < 1$
- Dummy variables
  - Commonly re-used elsewhere in expressions
  - Substitutions must cover exactly the same set of values
- Averages: $\langle x \rangle_{[T_1,T_2]} = \frac{1}{T_2 - T_1} \int_{t=T_1}^{T_2} x(t)dt$
- Periodic waveforms: $x(t \pm kT) = x(t)$ for any integer k
  - Fundamental period is the smallest T
  - Fundamental frequency is $F = \frac{1}{T}$
  - For periodic waveforms, $\langle x \rangle$ is the average over any integer number of periods
  - $\langle \sin 2\pi Ft \rangle = 0$
  - $\langle \cos 2\pi Ft \rangle = \left\langle e^{i2\pi Ft} \right\rangle = \begin{cases} 0 & F \neq 0 \\ 1 & F = 0 \end{cases}$

2: Fourier Series

A function, $u(t)$, is periodic with period T if $u(t+T) = u(t)\ \forall t$

- Periodic with period $T \quad \Rightarrow \quad$ Periodic with period $kT\ \forall k \in \mathbb{Z}^+$

The fundamental period is the smallest $T > 0$ for which $u(t+T) = u(t)$

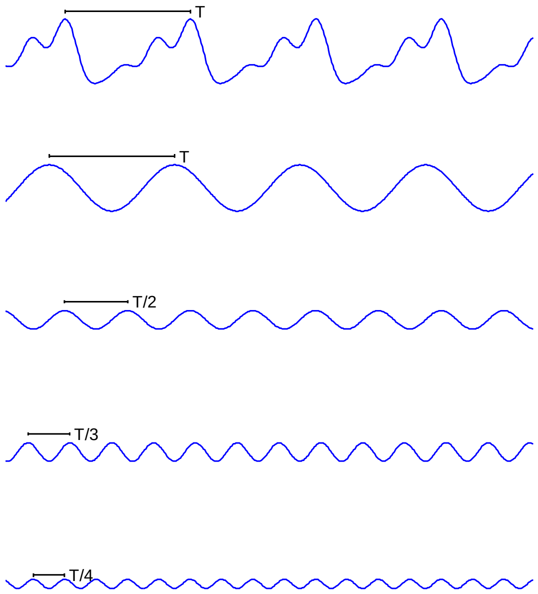


If you add together functions with different periods the fundamental period is the lowest common multiple (LCM) of the individual fundamental periods.

Example:

- $u(t) = \cos 4\pi t \quad \Rightarrow \quad T_u = \frac{2\pi}{4\pi} = 0.5$
- $v(t) = \cos 5\pi t \quad \Rightarrow \quad T_v = \frac{2\pi}{5\pi} = 0.4$
- $w(t) = u(t) + 0.1v(t) \quad \Rightarrow \quad T_w = \text{lcm}(0.5,\ 0.4) = 2.0$

If $u(t)$ has fundamental period T and fundamental frequency $F = \frac{1}{T}$ then, in most cases, we can express it as a (possibly infinite) sum of sine and cosine waves with frequencies 0, F, $2F$, $3F$, $\cdots$.



$u(t) =$

$\sin 2\pi Ft$ $[b_1 = 1]$

$-0.4 \ \sin 2\pi 2Ft$ $[b_2 = -0.4]$

$+0.4 \ \sin 2\pi 3Ft$ $[b_3 = 0.4]$

$-0.2 \ \cos 2\pi 4Ft$ $[a_4 = -0.2]$

The Fourier series for $u(t)$ is

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

The Fourier coefficients of $u(t)$ are a_0, a_1, $\cdots$ and b_1, b_2, $\cdots$.

The n^{th} harmonic of the fundamental is the component at a frequency nF.

Why are engineers obsessed with sine waves?
Answer: Because ...

1. A sine wave remains a sine wave of the same frequency when you
   (a) multiply by a constant,
   (b) add onto to another sine wave of the same frequency,
   (c) differentiate or integrate or shift in time
2. Almost any function can be expressed as a sum of sine waves
   - Periodic functions $\to$ Fourier Series
   - Aperiodic functions $\to$ Fourier Transform
3. Many physical and electronic systems are
   (a) composed entirely of constant-multiply/add/differentiate
   (b) linear: $u(t) \to x(t)$ and $v(t) \to y(t)$
   means that $u(t) + v(t) \to x(t) + y(t)$
   $\Rightarrow$ sum of sine waves $\to$ sum of sine waves

In these lectures we will use T for the fundamental period and $F = \frac{1}{T}$ for the fundamental frequency.

Dirichlet Conditions

Not all $u(t)$ can be expressed as a Fourier Series.

Peter Dirichlet derived a set of sufficient conditions.

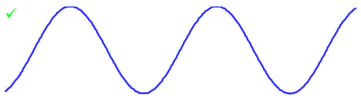
The function $u(t)$ must satisfy:

- periodic and single-valued
- $\int_0^T |u(t)|\, dt < \infty$
- finite number of maxima/minima per period
- finite number of finite discontinuities per period

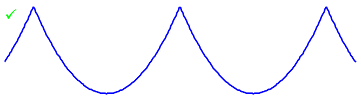
Peter Dirichlet

1805-1859

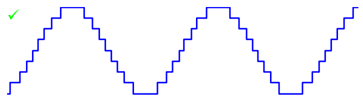
Good:



$\sin(t)$

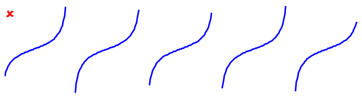


t^2

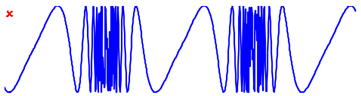


quantized

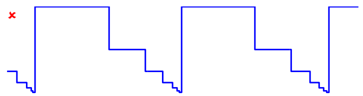
Bad:



$\tan(t)$



$\sin\left(\frac{1}{t}\right)$



∞ halving steps

Suppose that $u(t)$ satisfies the Dirichlet conditions so that

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Question: How do we find a_n and b_n?

Answer: We use a clever trick that relies on taking averages.

$\langle x(t) \rangle$ equals the average of $x(t)$ over any integer number of periods:

$$\langle x(t) \rangle = \frac{1}{T} \int_{t=0}^{T} x(t)dt$$

Remember, for any integer n, $\langle \sin 2\pi nFt \rangle = 0$

$$\langle \cos 2\pi nFt \rangle = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Finding a_n and b_n is called Fourier analysis.

Trigonometric Products

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\Rightarrow \sin x \cos y = \tfrac{1}{2}\sin(x+y) + \tfrac{1}{2}\sin(x-y)$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\Rightarrow \cos x \cos y = \tfrac{1}{2}\cos(x+y) + \tfrac{1}{2}\cos(x-y)$$
$$\sin x \sin y = \tfrac{1}{2}\cos(x-y) - \tfrac{1}{2}\cos(x+y)$$

Set $x = 2\pi mFt$, $y = 2\pi nFt$ (with $m + n \neq 0$) and take time-averages:

- $\langle \sin(2\pi mFt)\cos(2\pi nFt)\rangle$
  $= \left\langle \tfrac{1}{2}\sin(2\pi(m+n)Ft)\right\rangle + \left\langle \tfrac{1}{2}\sin(2\pi(m-n)Ft)\right\rangle = 0$
- $\langle \cos(2\pi mFt)\cos(2\pi nFt)\rangle$
  $= \left\langle \tfrac{1}{2}\cos(2\pi(m+n)Ft)\right\rangle + \left\langle \tfrac{1}{2}\cos(2\pi(m-n)Ft)\right\rangle = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}$
- $\langle \sin(2\pi mFt)\sin(2\pi nFt)\rangle$
  $= \left\langle \tfrac{1}{2}\cos(2\pi(m-n)Ft)\right\rangle - \left\langle \tfrac{1}{2}\cos(2\pi(m+n)Ft)\right\rangle = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}$

Summary: $\langle \sin\cos\rangle = 0$ [provided that $m + n \neq 0$]
$\langle \sin\sin\rangle = \langle \cos\cos\rangle = \frac{1}{2}$ if $m = n$ or otherwise $= 0$.

Proof that $\cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$

We know that

$$\begin{aligned}\cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y\end{aligned}$$

Adding these two gives

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y$$

From which: $\cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$

Subtracting instead of adding gives: $\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$

Proof that $\left\langle \frac{1}{2}\cos(2\pi(m+n)Ft)\right\rangle + \left\langle \frac{1}{2}\cos(2\pi(m-n)Ft)\right\rangle = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}$

We are assuming that m and n are integers with $m+n \neq 0$ and we use the result that $\langle \cos 2\pi f t \rangle$ is zero unless $f = 0$ in which case $\langle \cos 2\pi 0 t \rangle = 1$. The frequency of the first term, $\cos(2\pi(m+n)Ft)$, is $(m+n)F$ which is definitely non-zero (because of our assumption that $m+n \neq 0$) and so the average of this cosine wave is zero. The frequency of the second term is $(m-n)F$ and this is zero only if $m=n$. So it follows that the entire expression is zero unless $m=n$ in which case the second term gives a value of $\frac{1}{2}$. Since m and n are integers, we can take the averages over a time interval T and be sure that this includes an integer number of periods for both terms.

Find a_n and b_n in $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$

Answer: $a_n = 2\langle u(t)\cos(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt$

$b_n = 2\langle u(t)\sin(2\pi nFt)\rangle \triangleq \frac{2}{T}\int_0^T u(t)\sin(2\pi nFt)\,dt$

Proof $[a_0]$: $2\langle u(t)\cos(2\pi 0Ft)\rangle = 2\langle u(t)\rangle = 2 \times \frac{a_0}{2} = a_0$

Proof $[a_n,\ n > 0]$:

$$2\langle u(t)\cos(2\pi nFt)\rangle$$
$$= 2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle + \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle$$
$$+ \sum_{r=1}^{\infty} 2\langle b_r \sin(2\pi rFt)\cos(2\pi nFt)\rangle$$

Term 1: $2\left\langle \frac{a_0}{2}\cos(2\pi nFt)\right\rangle = 0$

Term 2: $2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = \begin{cases} a_n & r = n \\ 0 & r \neq n \end{cases}$

$$\Rightarrow \sum_{r=1}^{\infty} 2\langle a_r \cos(2\pi rFt)\cos(2\pi nFt)\rangle = a_n$$

Term 3: $2\langle b_r \sin 2\pi rFt \cos(2\pi nFt)\rangle = 0$

Proof $[b_n,\ n > 0]$: same method as for a_n

Fourier Analysis Example

Truncated Series:

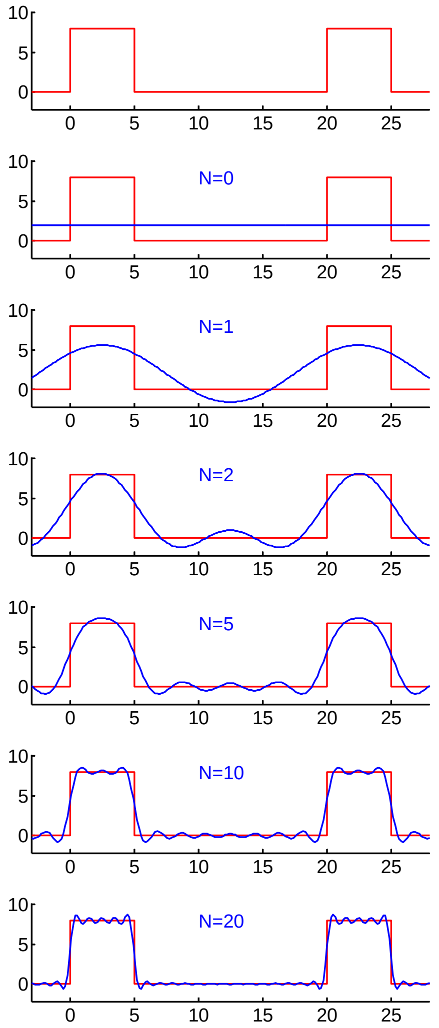
$$u_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$

Pulse: $T = 20$, width $W = \frac{T}{4}$, height $A = 8$

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T u(t)\cos\frac{2\pi nt}{T}dt \\ &= \frac{2}{T}\int_0^W A\cos\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[\sin\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\sin\frac{2\pi nW}{T} = \frac{A}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T u(t)\sin\frac{2\pi nt}{T}dt \\ &= \frac{2AT}{2\pi nT}\left[-\cos\frac{2\pi nt}{T}\right]_0^W \\ &= \frac{A}{n\pi}\left(1-\cos\frac{n\pi}{2}\right) \end{aligned}$$

n	0	1	2	3	4	5	6
a_n	4	$\frac{8}{\pi}$	0	$\frac{-8}{3\pi}$	0	$\frac{8}{5\pi}$	0
b_n		$\frac{8}{\pi}$	$\frac{16}{2\pi}$	$\frac{8}{3\pi}$	0	$\frac{8}{5\pi}$	$\frac{16}{6\pi}$



[Small Angle Approximation]

In the previous example, we can obtain a_0 by noting that $\frac{a_0}{2} = \langle u(t) \rangle$, the average value of the waveform which must be $\frac{AW}{T} = 2$. From this, $a_0 = 4$. We can, however, also derive this value from the general expression.
The expression for a_m is $a_m = \frac{A}{n\pi} \sin \frac{n\pi}{2}$. For the case, $n = 0$, this is difficult to evaluate because both the numerator and denominator are zero. The general way of dealing with this situation is L'Hôpital's rule (see section 4.7 of RHB) but here we can use a simpler and very useful technique that is often referred to as the "small angle approximation". For small values of θ we can approximate the standard trigonometrical functions as: $\sin\theta \approx \theta$, $\cos\theta \approx 1 - 0.5\theta^2$ and $\tan\theta \approx \theta$. These approximations are obtained by taking the first three terms of the Taylor series; they are accurate to $O(\theta^3)$ and are exactly correct when $\theta = 0$. When $m = 0$ we can therefore make an exact approximation to a_n by writing $a_n = \frac{A}{n\pi} \sin \frac{n\pi}{2} \approx \frac{A}{n\pi} \times \frac{n\pi}{2} = \frac{A}{2} = 4$. What we have implicitly done here is to assume that n is a real number (instead of an integer) and then taken the limit of a_n as $n \to 0$.

Linearity

Fourier analysis maps a function of time onto a set of Fourier coefficients:

$$u(t) \to \{a_n, b_n\}$$

This mapping is linear which means:

(1) For any γ, if $u(t) \to \{a_n, b_n\}$ then $\gamma u(t) \to \{\gamma a_n, \gamma b_n\}$

(2) If $u(t) \to \{a_n, b_n\}$ and $u'(t) \to \{a'_n, b'_n\}$ then

$$(u(t) + u'(t)) \to \{a_n + a'_n, b_n + b'_n\}$$

Proof for a_n: (proof for b_n is similar)

(1) If $\frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt = a_n$, then

$$\frac{2}{T}\int_0^T (\gamma u(t))\cos(2\pi nFt)\,dt$$
$$= \gamma\frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt = \gamma a_n$$

(2) If $\frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt = a_n$ and

$\frac{2}{T}\int_0^T u'(t)\cos(2\pi nFt)\,dt = a'_n$ then

$$\frac{2}{T}\int_0^T (u(t) + u'(t))\cos(2\pi nFt)\,dt$$
$$= \frac{2}{T}\int_0^T u(t)\cos(2\pi nFt)\,dt + \frac{2}{T}\int_0^T u'(t)\cos(2\pi nFt)\,dt$$
$$= a_n + a'_n$$

- Fourier Series:
  $$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$
- Dirichlet Conditions: sufficient conditions for FS to exist
  - Periodic, Single-valued, Bounded absolute integral
  - Finite number of (a) max/min and (b) finite discontinuities
- Fourier Analysis = "finding a_n and b_n"
  - $a_n = 2\left\langle u(t)\cos\left(2\pi nFt\right)\right\rangle \triangleq \frac{2}{T}\int_0^T u(t)\cos\left(2\pi nFt\right)dt$
  - $b_n = 2\left\langle u(t)\sin\left(2\pi nFt\right)\right\rangle \triangleq \frac{2}{T}\int_0^T u(t)\sin\left(2\pi nFt\right)dt$
- The mapping $u(t) \to \{a_n, b_n\}$ is linear

For further details see RHB 12.1 and 12.2.

3: Complex Fourier Series

Euler's Equation: $e^{i\theta} = \cos\theta + i\sin\theta$ [see RHB 3.3]

Hence: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = -\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}$$

Most maths becomes simpler if you use $e^{i\theta}$ instead of $\cos\theta$ and $\sin\theta$

The Complex Fourier Series is the Fourier Series but written using $e^{i\theta}$

Examples where using $e^{i\theta}$ makes things simpler:

Using $e^{i\theta}$	Using $\cos\theta$ and $\sin\theta$
$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$	$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$	$\cos\theta\cos\phi = \frac{1}{2}\cos(\theta+\phi) + \frac{1}{2}\cos(\theta-\phi)$
$\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}$	$\frac{d}{d\theta}\cos\theta = -\sin\theta$

Fourier Series: $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$

Substitute: $\cos\theta = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$ and $\sin\theta = -\frac{1}{2}ie^{i\theta} + \frac{1}{2}ie^{-i\theta}$

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \left(\tfrac{1}{2}e^{i\theta} + \tfrac{1}{2}e^{-i\theta}\right) + b_n \left(-\tfrac{1}{2}ie^{i\theta} + \tfrac{1}{2}ie^{-i\theta}\right)\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\left(\tfrac{1}{2}a_n - \tfrac{1}{2}ib_n\right) e^{i2\pi nFt}\right) \qquad [\theta = 2\pi nFt]$$

$$+ \sum_{n=1}^{\infty} \left(\left(\tfrac{1}{2}a_n + \tfrac{1}{2}ib_n\right) e^{-i2\pi nFt}\right)$$

$$= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

where $[b_0 \triangleq 0]$

$$U_n = \begin{cases} \frac{1}{2}a_n - \frac{1}{2}ib_n & n \geq 1 \\ \frac{1}{2}a_0 & n = 0 \\ \frac{1}{2}a_{|n|} + \frac{1}{2}ib_{|n|} & n \leq -1 \end{cases} \quad \Leftrightarrow \quad U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp ib_{|n|}\right)$$

The U_n are normally complex except for U_0 and satisfy $U_n = U_{-n}^*$

Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$ [simpler ☺]

Averaging Complex Exponentials

If $x(t)$ has period $\frac{T}{n}$ for some integer n (i.e. frequency $\frac{n}{T} = nF$):

$$\langle x(t) \rangle \triangleq \frac{1}{T} \int_{t=0}^{T} x(t) dt$$

This is the average over an integer number of cycles.

For a complex exponential:

$$\begin{aligned} \left\langle e^{i2\pi nFt} \right\rangle &= \langle \cos(2\pi nFt) + i \sin(2\pi nFt) \rangle \\ &= \langle \cos(2\pi nFt) \rangle + i \langle \sin(2\pi nFt) \rangle \\ &= \begin{cases} 1 + 0i & n = 0 \\ 0 + 0i & n \neq 0 \end{cases} \end{aligned}$$

Hence:

$$\left\langle e^{i2\pi nFt} \right\rangle = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

[☺]

Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

To find the coefficient, U_n, we multiply by something that makes all the terms involving the other coefficients average to zero.

$$\begin{aligned}\left\langle u(t)e^{-i2\pi nFt}\right\rangle &= \left\langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi rFt} e^{-i2\pi nFt}\right\rangle \\ &= \left\langle \sum_{r=-\infty}^{\infty} U_r e^{i2\pi(r-n)Ft}\right\rangle \\ &= \sum_{r=-\infty}^{\infty} U_r \left\langle e^{i2\pi(r-n)Ft}\right\rangle\end{aligned}$$

All terms in the sum are zero, except for the one when $n = r$ which equals U_n:

$$U_n = \left\langle u(t)e^{-i2\pi nFt}\right\rangle$$ [☺]

This shows that the Fourier series coefficients are unique: you cannot have two different sets of coefficients that result in the same function $u(t)$.

Note the sign of the exponent: "$+$" in the Fourier Series but "$-$" for Fourier Analysis (in order to cancel out the "$+$").

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos 2\pi nFt + b_n \sin 2\pi nFt\right)$$
$$= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

We can easily convert between the two forms.

Fourier Coefficients $\rightarrow$ Complex Fourier Coefficients:

$$U_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp ib_{|n|}\right) \qquad [U_n = U_{-n}^*]$$

Complex Fourier Coefficients $\rightarrow$ Fourier Coefficients:

$$a_n = U_n + U_{-n} = 2\Re\left(U_n\right) \qquad [\Re = \text{“real part”}]$$
$$b_n = i\left(U_n - U_{-n}\right) = -2\Im\left(U_n\right) \qquad [\Im = \text{“imaginary part”}]$$

The formula for a_n works even for $n = 0$.

[Complex functions of time]

In these lectures, we are assuming that $u(t)$ is a periodic real-valued function of time. In this case we can represent $u(t)$ using either the Fourier Series or the Complex Fourier Series:

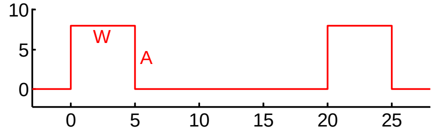
$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

We have seen that the U_n coefficients are complex-valued and that U_n and U_{-n} are complex conjugates so that we can write $U_{-n} = U_n^*$.

In fact, the complex Fourier series can also be used when $u(t)$ is a complex-valued function of time (this is sometimes useful in the fields of communications and signal processing). In this case, it is still true that $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$, but now U_n and U_{-n} are completely independent and normally $U_{-n} \neq U_n^*$.

Complex Fourier Analysis Example

$T = 20$, width $W = \frac{T}{4}$, height $A = 8$



Method 1:

$$U_{\pm n} = \tfrac{1}{2}a_n \mp i\tfrac{1}{2}b_n$$

Method 2:

$$\begin{aligned}
U_n &= \left\langle u(t)e^{-i2\pi nFt}\right\rangle \\
&= \tfrac{1}{T}\int_0^T u(t)e^{-i2\pi nFt}dt \\
&= \tfrac{1}{T}\int_0^W Ae^{-i2\pi nFt}dt \\
&= \tfrac{A}{-i2\pi nFT}\left[e^{-i2\pi nFt}\right]_0^W \\
&= \tfrac{A}{i2\pi n}\left(1 - e^{-i2\pi nFW}\right) \\
&= \tfrac{Ae^{-i\pi nFW}}{i2\pi n}\left(e^{i\pi nFW} - e^{-i\pi nFW}\right) \\
&= \tfrac{Ae^{-i\pi nFW}}{n\pi}\sin\left(n\pi FW\right) \\
&= \tfrac{8}{n\pi}\sin\left(\tfrac{n\pi}{4}\right)e^{-i\frac{n\pi}{4}}
\end{aligned}$$

n	a_n	b_n	U_n
-6			$i\frac{8}{6\pi}$
-5			$\frac{4}{5\pi} + i\frac{4}{5\pi}$
-4			0
-3			$\frac{-4}{3\pi} + i\frac{4}{3\pi}$
-2			$i\frac{8}{2\pi}$
-1			$\frac{4}{\pi} + i\frac{4}{\pi}$
0	4		2
1	$\frac{8}{\pi}$	$\frac{8}{\pi}$	$\frac{4}{\pi} + i\frac{-4}{\pi}$
2	0	$\frac{16}{2\pi}$	$i\frac{-8}{2\pi}$
3	$\frac{-8}{3\pi}$	$\frac{8}{3\pi}$	$\frac{-4}{3\pi} + i\frac{-4}{3\pi}$
4	0	0	0
5	$\frac{8}{5\pi}$	$\frac{8}{5\pi}$	$\frac{4}{5\pi} + i\frac{-4}{5\pi}$
6	0	$\frac{16}{6\pi}$	$i\frac{-8}{6\pi}$

Time Shifting

Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

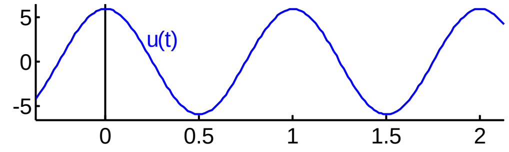
If $v(t)$ is the same as $u(t)$ but delayed by a time τ: $v(t) = u(t-\tau)$

$$v(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nF(t-\tau)} = \sum_{n=-\infty}^{\infty} \left(U_n e^{-i2\pi nF\tau}\right) e^{i2\pi nFt}$$
$$= \sum_{n=-\infty}^{\infty} V_n e^{i2\pi nFt}$$

where $V_n = U_n e^{-i2\pi nF\tau}$

Example:

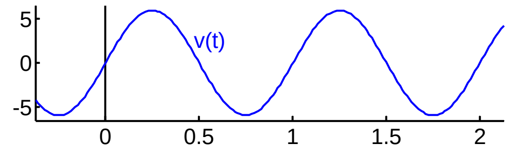
$u(t) = 6\cos(2\pi Ft)$



Fourier: $a_1 = 6,\ b_1 = 0$

Complex: $U_{\pm1} = \frac{1}{2}a_1 \mp \frac{1}{2}ib_1 = 3$

$v(t) = 6\sin(2\pi Ft) = u(t-\tau)$



Time delay: $\tau = \frac{T}{4} \Rightarrow F\tau = \frac{1}{4}$

Complex: $V_1 = U_1 e^{-i\frac{\pi}{2}} = -3i$

$V_{-1} = U_{-1} e^{i\frac{\pi}{2}} = +3i$

Note: If $u(t)$ is a sine wave, U_1 equals half the corresponding phasor.

Even/Odd Symmetry

(1) $u(t)$ real-valued $\Leftrightarrow$ U_n conjugate symmetric $[U_n = U_{-n}^*]$
(2) $u(t)$ even $[u(t) = u(-t)]$ $\Leftrightarrow$ U_n even $[U_n = U_{-n}]$
(3) $u(t)$ odd $[u(t) = -u(-t)]$ $\Leftrightarrow$ U_n odd $[U_n = -U_{-n}]$
(1)+(2) $u(t)$ real & even $\Leftrightarrow$ U_n real & even $[U_n = U_{-n}^* = U_{-n}]$
(1)+(3) $u(t)$ real & odd $\Leftrightarrow$ U_n imaginary & odd $[U_n = U_{-n}^* = -U_{-n}]$

Proof of (2): $u(t)$ even $\Rightarrow$ U_n even

$$U_{-n} = \frac{1}{T}\int_0^T u(t)e^{-i2\pi(-n)Ft}dt$$

$$= \frac{1}{T}\int_{x=0}^{-T} u(-x)e^{-i2\pi nFx}(-dx) \quad \text{[substitute } x=-t\text{]}$$

$$= \frac{1}{T}\int_{x=-T}^{0} u(-x)e^{-i2\pi nFx}dx \quad \text{[reverse the limits]}$$

$$= \frac{1}{T}\int_{x=-T}^{0} u(x)e^{-i2\pi nFx}dx = U_n \quad \text{[even: } u(-x)=u(x)\text{]}$$

Proof of (3): $u(t)$ odd $\Rightarrow$ U_n odd

Same as before, except for the last line:

$$= \frac{1}{T}\int_{x=-T}^{0} -u(x)e^{-i2\pi nFx}dx = -U_n \quad \text{[odd: } u(-x)=-u(x)\text{]}$$

A waveform, $u(t)$, is anti-periodic if $u(t+\frac{1}{2}T) = -u(t)$.
If $u(t)$ is anti-periodic then $U_n = 0$ for n even.

Proof:

Define $v(t) = u(t+\frac{T}{2})$, then

(1) $v(t) = -u(t) \Rightarrow V_n = -U_n$

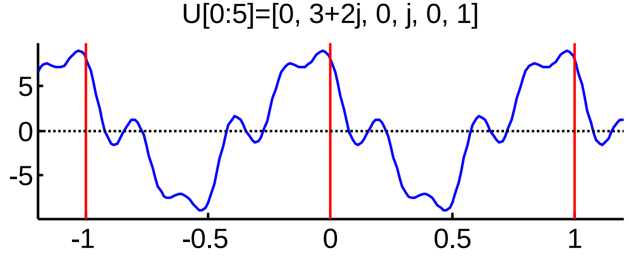
(2) $v(t)$ equals $u(t)$ but delayed by $-\frac{T}{2}$

$$\Rightarrow V_n = U_n e^{i2\pi n F \frac{T}{2}} = U_n e^{in\pi} = \begin{cases} U_n & n \text{ even} \\ -U_n & n \text{ odd} \end{cases}$$

Hence for n even: $V_n = -U_n = U_n \Rightarrow U_n = 0$

Example:

$U_{0:5} = [0,\ 3+2i,\ 0,\ i,\ 0,\ 1]$
Odd harmonics only ⇔
Second half of each period is the negative of the first half.

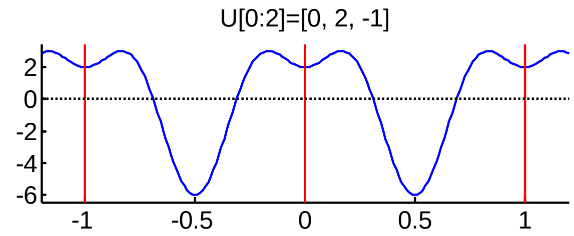


Symmetry Examples

All these examples assume that $u(t)$ is real-valued $\Leftrightarrow U_{-n} = U_{+n}^{*}$.

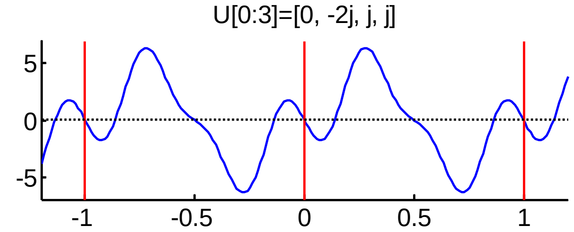
(1) Even $u(t) \Leftrightarrow$ real U_n

$$U_{0:2} = [0,\ 2,\ -1]$$



(2) Odd $u(t) \Leftrightarrow$ imaginary U_n

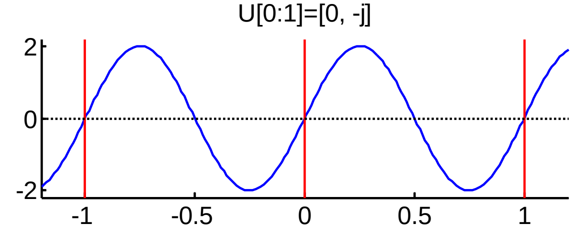
$$U_{0:3} = [0,\ -2i,\ i,\ i]$$



(3) Anti-periodic $u(t)$

$\Leftrightarrow$ odd harmonics only

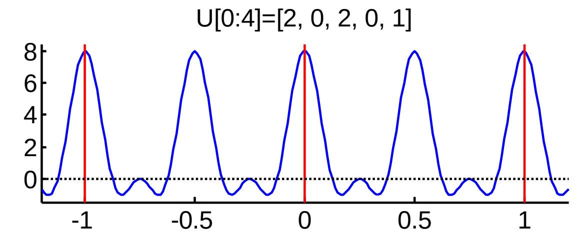
$$U_{0:1} = [0,\ -i]$$



(4) Even harmonics only

$\Leftrightarrow$ period is really $\frac{1}{2}T$

$$U_{0:4} = [2,\ 0,\ 2,\ 0,\ 1]$$



- Fourier Series:
  $u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nFt + b_n \sin 2\pi nFt)$
- Complex Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$
  - $U_n = \langle u(t) e^{-i2\pi nFt} \rangle \triangleq \frac{1}{T} \int_0^T u(t) e^{-i2\pi nFt} dt$
  - Since $u(t)$ is real-valued, $U_n = U_{-n}^*$
  - FS→CFS: $U_{\pm n} = \frac{1}{2} a_{|n|} \mp i \frac{1}{2} b_{|n|}$
  - CFS→FS: $a_n = U_n + U_{-n}$
    $b_n = i\,(U_n - U_{-n})$
- $u(t)$ real and even $\Leftrightarrow u(-t) = u(t)$
  $\Leftrightarrow U_n$ is real-valued and even $\Leftrightarrow b_n = 0\ \forall n$
- $u(t)$ real and odd $\Leftrightarrow u(-t) = -u(t)$
  $\Leftrightarrow U_n$ is purely imaginary and odd $\Leftrightarrow a_n = 0\ \forall n$
- $u(t)$ anti-periodic $\Leftrightarrow u(t + \frac{T}{2}) = -u(t)$
  $\Leftrightarrow$ odd harmonics only $\Leftrightarrow a_{2n} = b_{2n} = U_{2n} = 0\ \forall n$

For further details see RHB 12.3 and 12.7.

4: Parseval's Theorem and Convolution

Parseval's Theorem (a.k.a. Plancherel's Theorem)

Suppose we have two signals with the same period, $T = \frac{1}{F}$,

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$

$$\Rightarrow \quad u^*(t) = \sum_{n=-\infty}^{\infty} U_n^* e^{-i2\pi nFt} \qquad [u(t) = u^*(t) \text{ if real}]$$

$$v(t) = \sum_{n=-\infty}^{\infty} V_n e^{i2\pi nFt}$$

Now multiply $u^*(t)$ and $v(t)$ together and take the average over $[0, T]$.
[Use different "dummy variables", n and m, so they don't get mixed up]

$$\begin{aligned}\langle u^*(t)v(t)\rangle &= \left\langle \sum_{n=-\infty}^{\infty} U_n^* e^{-i2\pi nFt} \sum_{m=-\infty}^{\infty} V_m e^{i2\pi mFt} \right\rangle \\ &= \sum_{n=-\infty}^{\infty} U_n^* \sum_{m=-\infty}^{\infty} V_m \left\langle e^{-i2\pi nFt} e^{i2\pi mFt} \right\rangle \\ &= \sum_{n=-\infty}^{\infty} U_n^* \sum_{m=-\infty}^{\infty} V_m \left\langle e^{i2\pi(m-n)Ft} \right\rangle\end{aligned}$$

The quantity $\langle\cdots\rangle$ equals 1 if $m = n$ and 0 otherwise, so the only non-zero element in the second sum is when $m = n$, so the second sum equals V_n.

Hence Parseval's theorem: $\langle u^*(t)v(t)\rangle = \sum_{n=-\infty}^{\infty} U_n^* V_n$

If $v(t) = u(t)$ we get: $\left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} U_n^* U_n = \sum_{n=-\infty}^{\infty} |U_n|^2$

[Manipulating sums]

If you have a multiplicative expression involving two or more sums, then you must use different dummy variables for each of the sums:

$$\sum_n af(n) \sum_m bg(m)$$

(1) You can always move any quantities to the right

$$\begin{aligned}\sum_n af(n) \sum_m bg(m) &= \sum_n a \sum_m bf(n)g(m) \\ &= \sum_n \sum_m abf(n)g(m)\end{aligned}$$

(2) You can move quantities to the left past a summation provided that they do not involve the dummy variable of the summation:

$$\begin{aligned}\sum_n \sum_m abf(n)g(m) &= \sum_n af(n) \sum_m bg(m) \\ &\neq \sum_n af(n)g(m) \sum_m b\end{aligned}$$

The last expression doesn't make sense in any case since m is undefined outside $\sum_m$

(3) You can swap the summation order if the sum converges absolutely

$$\sum_n \sum_m h(n,m) = \sum_m \sum_n h(n,m) \quad \text{provided that } \sum_n \sum_m |h(n,m)| < \infty$$

The equality on the left is not necessarily true if the sum does not converge absolutely. Of course, if the sum has only a finite number of terms, it is bound to converge absolutely.

Power Conservation

The average power of a periodic signal is given by $P_u \triangleq \left\langle |u(t)|^2 \right\rangle$.

This is the average electrical power that would be dissipated if the signal represents the voltage across a $1\,\Omega$ resistor.

Parseval's Theorem: $P_u = \left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} |U_n|^2$

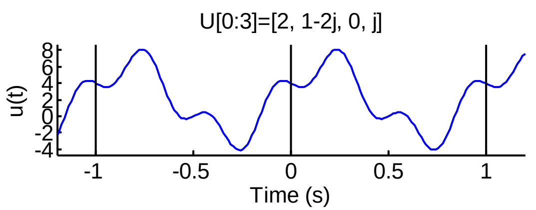
$= |U_0|^2 + 2\sum_{n=1}^{\infty} |U_n|^2$ [assume $u(t)$ real]

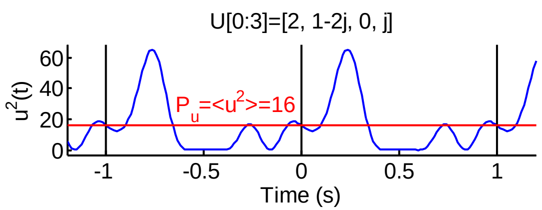
$= \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$ $\left[U_{+n} = \frac{a_n - ib_n}{2}\right]$

Parseval's theorem ⇒ the average power in $u(t)$ is equal to the sum of the average powers in each of its Fourier components.

Example: $u(t) = 2 + 2\cos 2\pi Ft + 4\sin 2\pi Ft - 2\sin 6\pi Ft$

$$\left\langle |u(t)|^2 \right\rangle = 4 + \frac{1}{2}\left(2^2 + 4^2 + (-2)^2\right) = 16$$





$$U_{0:3} = [2,\ 1-2i,\ 0,\ i] \quad \Rightarrow \quad |U_0|^2 + 2\sum_{n=1}^{\infty} |U_n|^2 = 16$$

The *spectrum* of a periodic signal is the values of $\{U_n\}$ versus nF.

The *magnitude spectrum* is the values of $\{|U_n|\} = \left\{\frac{1}{2}\sqrt{a_{|n|}^2 + b_{|n|}^2}\right\}$.

The *power spectrum* is the values of $\left\{|U_n|^2\right\} = \left\{\frac{1}{4}\left(a_{|n|}^2 + b_{|n|}^2\right)\right\}$.

Example:

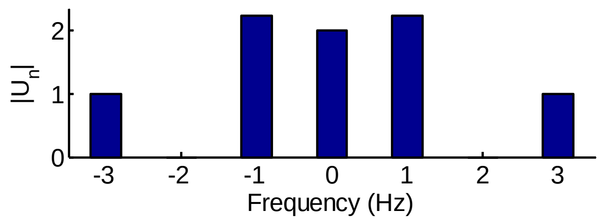
$$u(t) = 2 + 2\cos 2\pi Ft + 4\sin 2\pi Ft - 2\sin 6\pi Ft$$

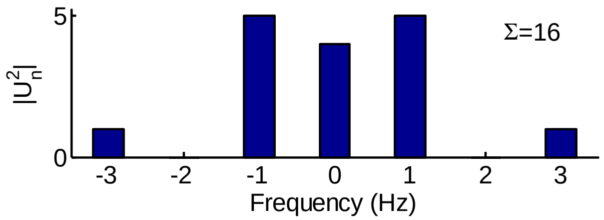
Fourier Coefficients: $a_{0:3} = [4,\ 2,\ 0,\ 0]$ $\quad$ $b_{1:3} = [4,\ 0,\ -2]$

Spectrum: $U_{-3:3} = [-i,\ 0,\ 1+2i,\ 2,\ 1-2i,\ 0,\ i]$

Magnitude Spectrum: $|U_{-3:3}| = \left[1,\ 0,\ \sqrt{5},\ 2,\ \sqrt{5},\ 0,\ 1\right]$

Power Spectrum: $\left|U_{-3:3}^2\right| = [1,\ 0,\ 5,\ 4,\ 5,\ 0,\ 1]$ $\quad$ $\left[\sum = \left\langle u^2(t)\right\rangle\right]$





The magnitude and power spectra of a real periodic signal are symmetrical.

A *one-sided power power spectrum* shows U_0 and then $2\left|U_n\right|^2$ for $n \geq 1$.

Product of Signals

Suppose we have two signals with the same period, $T = \frac{1}{F}$,

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$$
$$v(t) = \sum_{m=-\infty}^{\infty} V_n e^{i2\pi mFt}$$

If $w(t) = u(t)v(t)$ then $W_r = \sum_{m=-\infty}^{\infty} U_{r-m} V_m \triangleq U_r * V_r$

Proof:

$$w(t) = u(t)v(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} \sum_{m=-\infty}^{\infty} V_m e^{i2\pi mFt}$$
$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U_n V_m e^{i2\pi(m+n)Ft}$$

Now we change the summation variable to use r instead of n:

$$r = m + n \Rightarrow n = r - m$$

This is a one-to-one mapping: every pair (m, n) in the range $\pm\infty$ corresponds to exactly one pair (m, r) in the same range.

$$w(t) = \sum_{r=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U_{r-m} V_m e^{i2\pi rFt} = \sum_{r=-\infty}^{\infty} W_r e^{i2\pi rFt}$$

where $W_r = \sum_{m=-\infty}^{\infty} U_{r-m} V_m \triangleq U_r * V_r$.

W_r is the sum of all products $U_n V_m$ for which $m + n = r$.

The spectrum $W_r = U_r * V_r$ is called the convolution of U_r and V_r.

Convolution Properties

Convolution behaves algebraically like multiplication:

1) Commutative: $U_r * V_r = V_r * U_r$
2) Associative: $U_r * V_r * W_r = (U_r * V_r) * W_r = U_r * (V_r * W_r)$
3) Distributive over addition: $W_r * (U_r + V_r) = W_r * U_r + W_r * V_r$
4) Identity Element or "1": If $I_r = \begin{cases} 1 & r = 0 \\ 0 & r \neq 0 \end{cases}$, then $I_r * U_r = U_r$

Proofs: (all sums are over $\pm\infty$)

1) Substitute for m: $n = r - m \Leftrightarrow m = r - n$ [$1 \leftrightarrow 1$ for any r]
$\sum_m U_{r-m} V_m = \sum_n U_n V_{r-n}$

2) Substitute for n: $k = r + m - n \Leftrightarrow n = r + m - k$ [$1 \leftrightarrow 1$]
$$\sum_n \left(\left(\sum_m U_{n-m} V_m\right) W_{r-n}\right) = \sum_k \left(\left(\sum_m U_{r-k} V_m\right) W_{k-m}\right)$$
$$= \sum_k \sum_m U_{r-k} V_m W_{k-m} = \sum_k \left(U_{r-k} \left(\sum_m V_m W_{k-m}\right)\right)$$

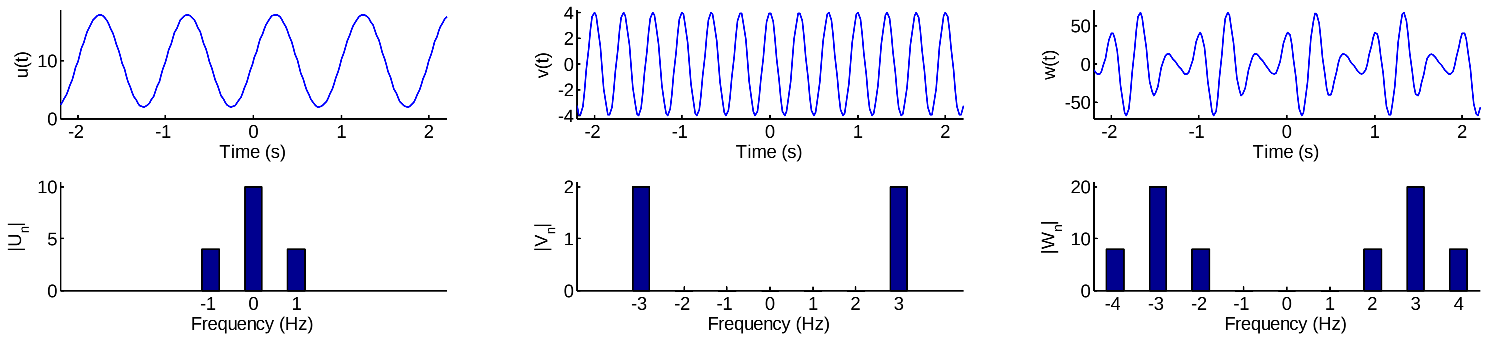
3) $\sum_m W_{r-m} (U_m + V_m) = \sum_m W_{r-m} U_m + \sum_m W_{r-m} V_m$

4) $I_{r-m} U_m = 0$ unless $m = r$. Hence $\sum_m I_{r-m} U_m = U_r$.

Convolution Example

$$u(t) = 10 + 8\sin 2\pi t \qquad v(t) = 4\cos 6\pi t$$

$$U_{-1:1} = [4i,\ \underline{10},\ -4i] \qquad V_{-3:3} = [2,\ 0,\ 0,\ \underline{0},\ 0,\ 0,\ 2] \qquad [\underline{0} = V_0]$$



$$\begin{aligned} w(t) = u(t)v(t) &= (10 + 8\sin 2\pi t)\, 4\cos 6\pi t \\ &= 40\cos 6\pi t + 32 \sin 2\pi t \cos 6\pi t \\ &= 40\cos 6\pi t + 16\sin 8\pi t - 16 \sin 4\pi t \end{aligned}$$

$$W_{-4:4} = [8i,\ 20,\ -8i,\ 0,\ \underline{0},\ 0,\ 8i,\ 20,\ -8i]$$

To convolve U_n and V_n:

Replace each harmonic in V_n by a scaled copy of the entire $\{U_n\}$ (or vice versa) and sum the complex-valued coefficients of any overlapping harmonics.

Two polynomials: $u(x) = U_3x^3 + U_2x^2 + U_1x + U_0$

$v(x) = V_2x^2 + V_1x + V_0$

Now multiply the two polynomials together:

$$w(x) = u(x)v(x)$$
$$= U_3V_2x^5 + (U_3V_1 + U_2V_2)\,x^4 + (U_3V_0 + U_2V_1 + U_1V_2)\,x^3$$
$$+ (U_2V_0 + U_1V_1 + U_0V_2)\,x^2 + (U_1V_0 + U_0V_1)\,x + U_0V_0$$

The coefficient of x^r consists of all the coefficient pair from U and V where the subscripts add up to r. For example, for $r = 3$:

$$W_3 = U_3V_0 + U_2V_1 + U_1V_2 = \sum_{m=0}^{2} U_{3-m}V_m$$

If all the missing coefficients are assumed to be zero, we can write

$$W_r = \sum_{m=-\infty}^{\infty} U_{r-m}V_m \triangleq U_r * V_r$$

So, to multiply two polynomials, you convolve their coefficient sequences.

Actually, the complex Fourier Series is iust a polynomial:

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} = \sum_{n=-\infty}^{\infty} U_n \left(e^{i2\pi Ft}\right)^n$$

- Parseval's Theorem: $\langle u^*(t)v(t)\rangle = \sum_{n=-\infty}^{\infty} U_n^* V_n$
  - Power Conservation: $\left\langle |u(t)|^2 \right\rangle = \sum_{n=-\infty}^{\infty} |U_n|^2$
  - or in terms of a_n and b_n:
    $$\left\langle |u(t)|^2 \right\rangle = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$
- Linearity: $w(t) = au(t) + bv(t) \Leftrightarrow W_n = aU_n + bV_n$
- Product of signals $\Leftrightarrow$ Convolution of complex Fourier coefficients:
  $w(t) = u(t)v(t) \Leftrightarrow W_n = U_n * V_n \triangleq \sum_{m=-\infty}^{\infty} U_{n-m}V_m$
- Convolution acts like multiplication:
  - Commutative: $U * V = V * U$
  - Associative: $U * V * W$ is unambiguous
  - Distributes over addition: $U * (V + W) = U * V + U * W$
  - Has an identity: $I_r = 1$ if $r = 0$ and $= 0$ otherwise
- Polynomial multiplication $\Leftrightarrow$ convolution of coefficients

For further details see RHB Chapter 12.8.

5: Gibbs Phenomenon

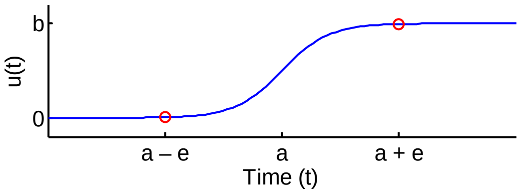
Discontinuities

A function, $v(t)$, has a discontinuity of amplitude b at $t = a$ if

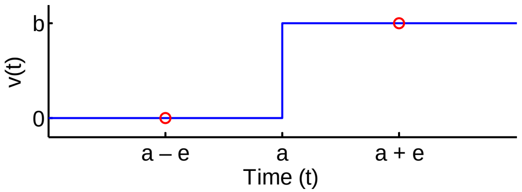
$$\lim_{e \to 0} \left(v(a+e) - v(a-e) \right) = b \neq 0$$

Conversely, $v(t)$, is continuous at $t = a$ if the limit, b, equals zero.

Examples:



Continuous



Discontinuous

We will see that if a periodic function, $v(t)$, is discontinuous, then its Fourier series behaves in a strange way.

Discontinuous Waveform

Pulse: $T = \frac{1}{F} = 20$, width$=\frac{1}{2}T$, height $A = 1$

$$U_m = \frac{1}{T}\int_0^{0.5T} Ae^{-i2\pi mFt}dt$$

$$= \frac{i}{2\pi mFT}\left[e^{-i2\pi mFt}\right]_0^{0.5T}$$

$$= \frac{i}{2\pi m}\left(e^{-i\pi m} - 1\right) = \frac{((-1)^m - 1)i}{2\pi m}$$

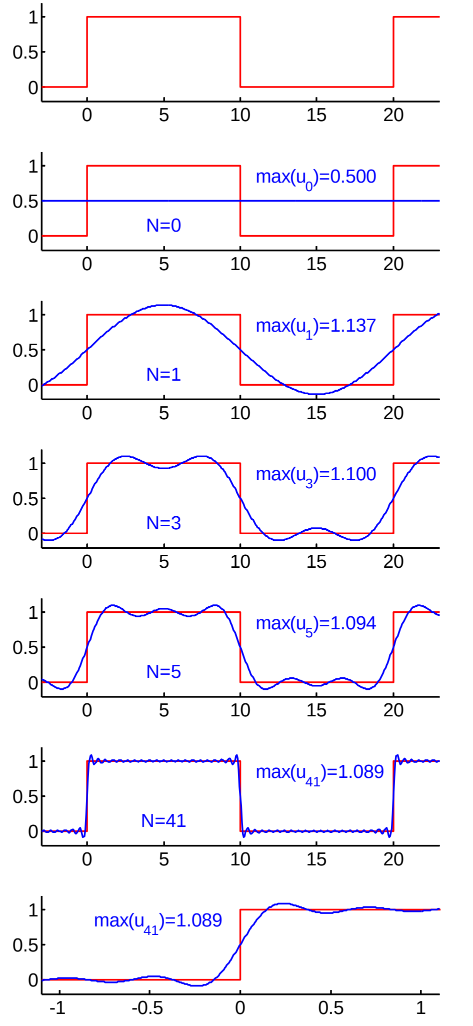
$$= \begin{cases} 0 & m \neq 0, \text{ even} \\ 0.5 & m = 0 \\ \frac{-i}{m\pi} & m \text{ odd} \end{cases}$$

So, $u(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin 2\pi Ft + \frac{1}{3}\sin 6\pi Ft + \frac{1}{5}\sin 10\pi Ft + \ldots\right)$

Define: $u_N(t) = \sum_{m=-N}^{N} U_m e^{i2\pi mFt}$

$$u_N(0) = 0.5 \;\forall N$$

$$\max_t u_N(t) \underset{N\to\infty}{\longrightarrow} \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt \approx 1.0895$$



[Enlarged View: $u_{41}(t)$]

[Powers of -1 and i]

Expressions involving $(-1)^m$ or, less commonly, i^m arise quite frequently and it is worth becoming familiar with them. They can arise in several guises:

$$e^{-i\pi m} = e^{i\pi m} = \left(e^{i\pi}\right)^m = \cos(\pi m) = (-1)^m$$
$$e^{i\frac{1}{2}\pi m} = \left(e^{i\frac{1}{2}\pi}\right)^m = i^m$$
$$e^{-i\frac{1}{2}\pi m} = \left(e^{-i\frac{1}{2}\pi}\right)^m = (-i)^m$$

As m increases these expressions repeat with periods of 2 or 4. Simple expressions involving these quantities make useful sequences.

m	-4	-3	-2	-1	0	1	2	3	4
$(-1)^m = \cos\pi m = e^{i\pi m}$	1	-1	1	-1	1	-1	1	-1	1
$i^m = e^{i0.5\pi m}$	1	i	-1	$-i$	1	i	-1	$-i$	1
$(-i)^m = e^{-i0.5\pi m}$	1	$-i$	-1	i	1	$-i$	-1	i	1
$\frac{1}{2}\left(1+(-1)^m\right)$	1	0	1	0	1	0	1	0	1
$\frac{1}{2}\left(1-(-1)^m\right)$	0	1	0	1	0	1	0	1	0
$\frac{1}{2}\left(i^m+(-i)^m\right) = \cos 0.5\pi m$	1	0	-1	0	1	0	-1	0	1
$\frac{1}{4}\left(1+(-1)^m+i^m+(-i)^m\right)$	1	0	0	0	1	0	0	0	1

Truncated Fourier Series: $u_N(t) = \sum_{m=-N}^{N} U_m e^{i2\pi mFt}$

If $u(t)$ has a discontinuity of height b at $t = a$ then:

(1) $u_N(a) \underset{N\to\infty}{\longrightarrow} \lim_{e\to 0} \frac{u(a-e)+u(a+e)}{2}$

(2) $u_N(t)$ has an overshoot of about 9% of b at the discontinuity. For large N the overshoot moves closer to the discontinuity but does not get smaller (Gibbs phenomenon). In the limit the overshoot equals $\left(-\frac{1}{2} + \frac{1}{\pi}\int_0^\pi \frac{\sin t}{t} dt\right) b \approx 0.0895b$.

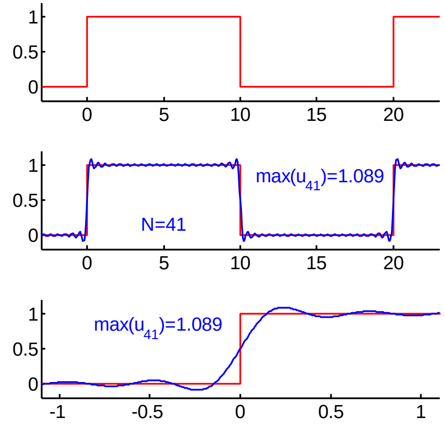
(3) For large m, the coefficients, U_m decrease no faster than $|m|^{-1}$.

Example:

$$u_N(0) \underset{N\to\infty}{\longrightarrow} 0.5$$

$$\max_t u_N(t) \underset{N\to\infty}{\longrightarrow} 1.0895\ldots$$

$$U_m = \begin{cases} 0 & m \neq 0,\text{ even} \\ 0.5 & m = 0 \\ \frac{-i}{m\pi} & m\text{ odd} \end{cases}$$



[Origin of Gibbs Phenomenon]

This topic is included for interest but is not examinable.

Our first goal is to express the partial Fourier series, $u_N(t)$, in terms of the original signal, $u(t)$. We begin by substituting the integral expression for U_n in the partial Fourier series

$$u_N(t) = \sum_{n=-N}^{+N} U_n e^{i2\pi nFt} = \sum_{n=-N}^{+N} \left(\frac{1}{T}\int_0^T u(\tau)e^{-i2\pi nF\tau}d\tau\right) e^{i2\pi nFt}$$

Now we swap the order of the integration and the finite summation (OK if the integral converges $\forall n$)

$$u_N(t) = \frac{1}{T}\int_0^T u(\tau)\left(\sum_{n=-N}^{+N} e^{i2\pi nF(t-\tau)}\right)d\tau$$

Now apply the formula for the sum of a geometric progression with $z = e^{i2\pi F(t-\tau)}$:

$$\sum_{n=-N}^{+N} z^n = \frac{z^{-N}-z^{N+1}}{1-z} = \frac{z^{-(N+0.5)}-z^{N+0.5}}{z^{-0.5}-z^{0.5}}$$

$$u_N(t) = \frac{1}{T}\int_0^T u(\tau)\frac{e^{i2\pi(N+0.5)F(\tau-t)}-e^{-i2\pi(N+0.5)F(\tau-t)}}{e^{i2\pi 0.5F(\tau-t)}-e^{-i2\pi 0.5F(\tau-t)}}d\tau$$

$$= \frac{1}{T}\int_0^T u(\tau)\frac{\sin\pi(2N+1)F(\tau-t)}{\sin\pi F(\tau-t)}d\tau$$

So if we define the Dirichlet Kernel to be $D_N(x) = \frac{\sin((N+0.5)x)}{\sin 0.5x}$, and set $x = 2\pi F(\tau - t)$, we obtain

$$u_N(t) = \frac{1}{T}\int_0^T u(\tau)D_N\left(2\pi F(\tau-t)\right)d\tau$$

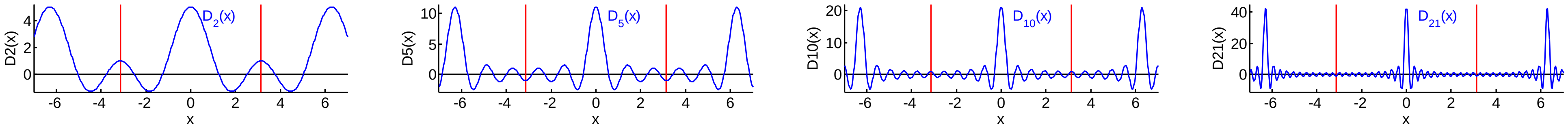
So what we have shown is that $u_N(t)$ can be obtained by multiplying $u(\tau)$ by a time-shifted Dirichlet Kernel and then integrating over one period. Next we will look at the properties of the Dirichlet Kernel.

[Dirichlet Kernel]

This topic is included for interest but is not examinable.

Dirichlet Kernel definition: $D_N(x) = \sum_{n=-N}^{+N} e^{inx} = 1 + 2\sum_{n=1}^{N} \cos nx = \frac{\sin((N+0.5)x)}{\sin 0.5x}$

$D_N(x)$ is plotted below for $N = \{2,\ 5,\ 10,\ 21\}$. The vertical red lines at $\pm\pi$ mark one period.



- Periodic: with period 2π
- Average value: $\langle D_N(x)\rangle = \frac{1}{2\pi}\int_{-\pi}^{+\pi} D_N(x)dx = 1$
- First Zeros: $D_N(x) = 0$ at $x = \pm\frac{\pi}{N+0.5}$ define the main lobe as $-\frac{\pi}{N+0.5} < x < \frac{\pi}{N+0.5}$
- Peak value: $2N+1$ at $x = 0$. The main lobe gets narrower but higher as N increases.
- Main Lobe semi-integral:

$$\int_{x=0}^{\frac{\pi}{N+0.5}} D_N(x)dx = \int_{x=0}^{\frac{\pi}{N+0.5}} \frac{\sin((N+0.5)x)}{\sin 0.5x}dx = \frac{1}{N+0.5}\int_{y=0}^{\pi} \frac{\sin y}{\sin\frac{y}{2N+1}}dy[y = (N+0.5)x]$$

where we substituted $y = (N+0.5)x$. Now, for large N, we can approximate $\sin\frac{y}{2N+1} \approx \frac{y}{2N+1}$:

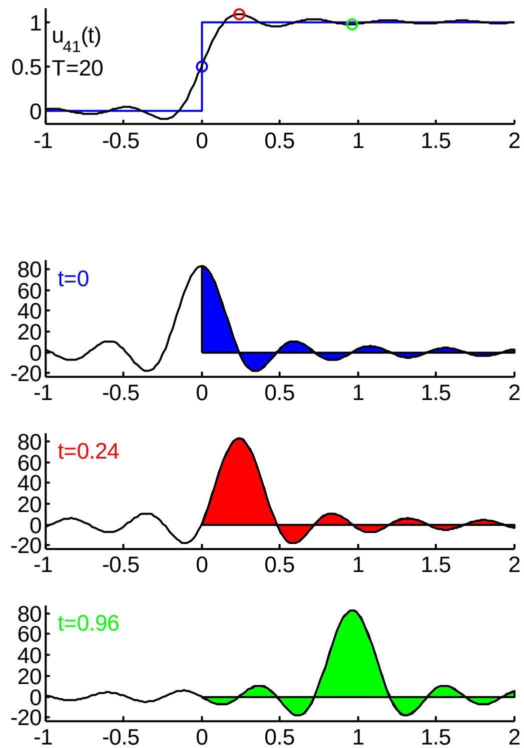
$$\int_{x=0}^{\frac{\pi}{N+0.5}} D_N(x)dx \approx \frac{1}{N+0.5}\int_{y=0}^{\pi} \frac{\sin y}{\frac{y}{2N+1}}dy = 2\int_{y=0}^{\pi}\frac{\sin y}{y}dy \approx 3.7038741 \approx 2\pi \times 0.58949$$

We see that, for large enough N, the main lobe semi-integral is independent of N.

[In MATLAB $D_N(x) = (2N+1) \times \text{diric}(x, 2N+1)$]

[Gibbs Phenomenon Overshoot]

This topic is included for interest but is not examinable.



The partial Fourier Series, $u_N(t)$, can be obtained by multiplying $u(t)$ by a shifted Dirichlet Kernel and integrating over one period:

$$u_N(t) = \frac{1}{T}\int_0^T u(\tau) D_N\left(2\pi F(\tau - t)\right) d\tau$$

For the special case when $u(t)$ is a pulse of height 1 and width $0.5T$:

$$u_N(t) = \frac{1}{T}\int_0^{0.5T} D_N\left(2\pi F(\tau - t)\right) d\tau$$

Substitute $x = 2\pi F(\tau - t)$

$$u_N(t) = \frac{1}{2\pi FT}\int_{-2\pi Ft}^{\pi FT - 2\pi Ft} D_N(x)\, dx = \frac{1}{2\pi}\int_{-2\pi Ft}^{\pi - 2\pi Ft} D_N(x)\, dx$$

- For $t = 0$ (the blue integral and the blue circle on the upper graph):
  $$u_N(0) = \frac{1}{2\pi}\int_0^{\pi} D_N(x)\, dx = 0.5$$
- For $t = \frac{T}{2N+1}$ (the red integral and the red circle on the upper graph):
  $$u_N\left(\frac{T}{2N+1}\right) = \frac{1}{2\pi}\int_{-\frac{\pi}{N+0.5}}^{\pi - \frac{\pi}{N+0.5}} D_N(x)\, dx = \frac{1}{2\pi}\int_{-\frac{\pi}{N+0.5}}^{0} D_N(x)\, dx + \frac{1}{2\pi}\int_0^{\pi - \frac{\pi}{N+0.5}} D_N(x)\, dx$$
  For large N, we replace the first term by a constant (since it is the semi-integral of the main lobe) and replace the upper limit of the second term by π:
  $$\approx 0.58949 + \frac{1}{2\pi}\int_0^{\pi} D_N(x)\, dx = 1.08949$$
- For $0 \ll t \ll 0.5T$, $u_N(t) \approx 1$ (the green integral and the green circle on the upper graph).

Integration

Suppose $u(t) = \sum_{m=-\infty}^{\infty} U_m e^{i2\pi mFt}$

Define $v(t)$ to be the integral of $u(t)$ [boundedness requires $U_0 = 0$]

$$v(t) = \int^t u(\tau)d\tau = \int^t \sum_{m=-\infty}^{\infty} U_m e^{i2\pi mF\tau} d\tau$$

$$= \sum_{m=-\infty}^{\infty} U_m \int^t e^{i2\pi mF\tau} d\tau \quad \text{[assume OK to swap } \textstyle\int \text{ and } \sum\text{]}$$

$$= c + \sum_{m=-\infty}^{\infty} U_m \frac{1}{i2\pi mF} e^{i2\pi mFt}$$

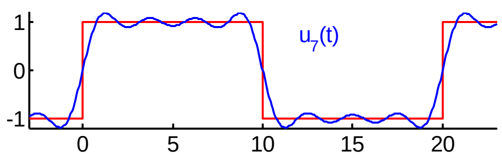
$$= c + \sum_{m=-\infty}^{\infty} V_m e^{i2\pi mFt} \text{ where } c \text{ is an integration constant}$$

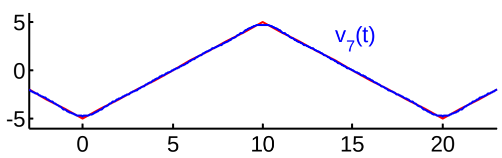
Hence $V_m = \frac{-i}{2\pi mF} U_m$ except for $V_0 = c$ (arbitrary constant)

Example:

Square wave: $U_m = \frac{-2i}{m\pi}$ for odd m (0 for even m)

Triangle wave: $V_m = \frac{-i}{2\pi mF} \times \frac{-2i}{m\pi} = \frac{-1}{\pi^2 m^2 F}$ for odd m (0 for even m)



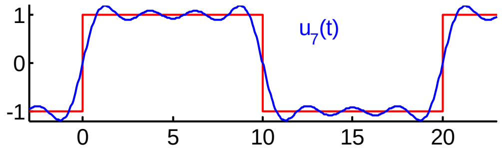


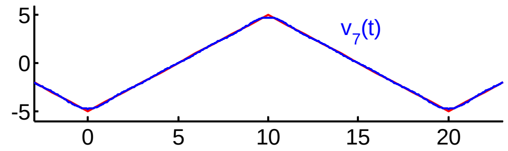
Convergence: $v(t)$ always converges if $u(t)$ does since $V_m \propto \frac{1}{m} U_m$

$v_N(t)$ is a good approximation even for small N

Rate at which coefficients decrease with m

Square wave: $U_m = \frac{-2i}{\pi} m^{-1}$ for odd m (0 for even m)
Triangle wave: $V_m = \frac{-1}{\pi^2 F} m^{-2}$ for odd m (0 for even m)





Integrating $u(t)$ multiplies the U_m by $\frac{-i}{2\pi F} \times m^{-1} \Rightarrow$ they decrease faster.

The rate at which the coefficients, U_m, decrease with m depends on the lowest derivative that has a discontinuity:

- Discontinuity in $u(t)$ itself (e.g. square wave)
  For large $|m|$, U_m decreases as $|m|^{-1}$
- Discontinuity in $u'(t)$ (e.g. triangle wave)
  For large $|m|$, U_m decreases as $|m|^{-2}$
- Discontinuity in $u^{(n)}(t)$
  For large $|m|$, U_m decreases as $|m|^{-(n+1)}$
- No discontinuous derivatives
  For large $|m|$, U_m decreases faster than any power (e.g. $e^{-|m|}$)

Differentiation

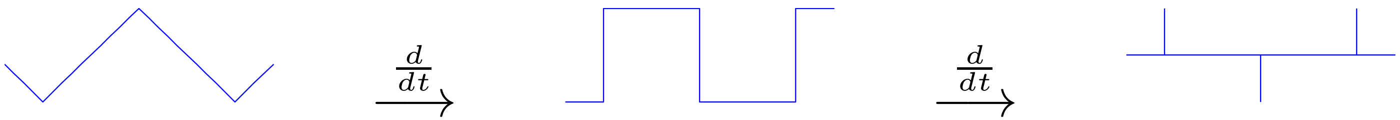
Integration multiplies U_m by $\frac{-i}{2\pi mF}$.

Hence differentiation multiplies U_m by $\frac{2\pi mF}{-i} = i2\pi mF$

If $u(t)$ is a continuous differentiable function and $w(t) = \frac{du(t)}{dt}$ then, provided that $w(t)$ satisfies the Dirichlet conditions, its Fourier coefficients are:

$$W_m = \begin{cases} 0 & m = 0 \\ i2\pi mFU_m & m \neq 0 \end{cases}.$$

Since we are multiplying U_m by m the coefficients W_m decrease more slowly with m and so the Fourier series for $w(t)$ may not converge (i.e. $w(t)$ may not satisfy the Dirichlet conditions).



$U_m \propto |m|^{-2}$ $\quad$ $U_m \propto |m|^{-1}$ $\quad$ $U_m \propto |m|^{-0}$

Differentiation makes waveforms spikier and less smooth.

Periodic Extension

Suppose $y(t)$ is only defined over a finite interval (a, b).

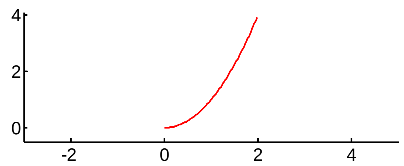
You have two reasonable choices to make a periodic version:

(a) $T = b - a, \qquad u(t) = y(t) \text{ for } a \le t < b$

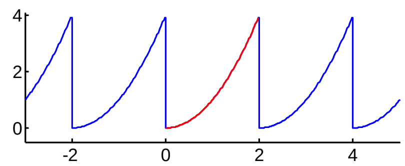
(b) $T = 2(b-a),\ u(t) = \begin{cases} y(t) & a \le t \le b \\ y(2b-t) & b \le t \le 2b-a \end{cases}$

Example:

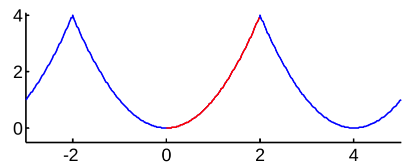
$y(t) = t^2$ for $0 \le t < 2$



$y(t)$



(a) $T = 2$



(b) $T = 4$

Option (b) has twice the period, no discontinuities, no Gibbs phenomenon overshoots and if $y(t)$ is continuous the coefficients decrease at least as fast as $|m|^{-2}$.

$y(t) = t^2$ for $0 \le t < 2$

Method (a): $T = \frac{1}{F} = 2$

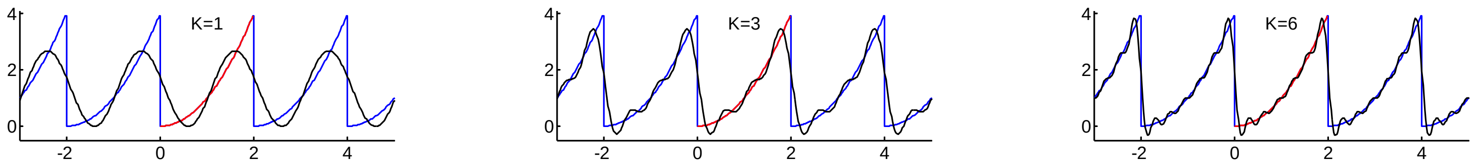
$$U_m = \frac{1}{T}\int_0^T t^2 e^{-i2\pi mFt}dt \qquad U_0 = \frac{1}{T}\int_0^T t^2 dt = \frac{4}{3}$$

$$= \frac{1}{T}\left[\frac{t^2e^{-i2\pi mFt}}{-i2\pi mF} - \frac{2te^{-i2\pi mFt}}{(-i2\pi mF)^2} + \frac{2e^{-i2\pi mFt}}{(-i2\pi mF)^3}\right]_0^T$$

Substitute $e^{-i2\pi mF0} = e^{-i2\pi mFT} = 1$ [for integer m]

$$= \frac{1}{T}\left[\frac{T^2}{-i2\pi mF} - \frac{2T}{(-i2\pi mF)^2}\right]$$

$$= \frac{2i}{\pi m} + \frac{2}{\pi^2 m^2}$$



$$U_{0:3} = [1.333,\ 0.203 + 0.637i,\ 0.051 + 0.318i,\ 0.023 + 0.212i]$$

$y(t) = t^2$ for $0 \le t < 2$

Method (b): $T = \frac{1}{F} = 4$

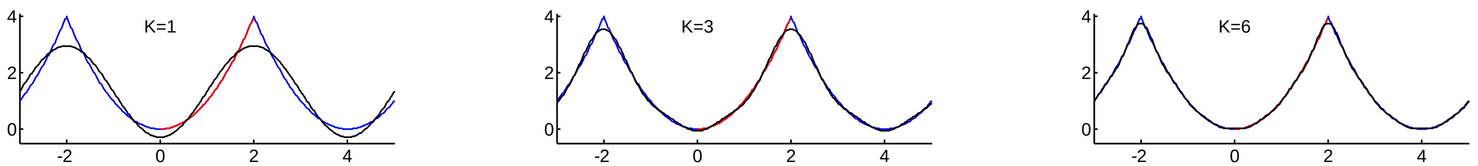
$$U_m = \frac{1}{T}\int_{-0.5T}^{0.5T} t^2 e^{-i2\pi mFt} dt \qquad U_0 = \frac{1}{T}\int_{-0.5T}^{0.5T} t^2 dt = \frac{4}{3}$$

$$= \frac{1}{T}\left[\frac{t^2 e^{-i2\pi mFt}}{-i2\pi mF} - \frac{2te^{-i2\pi mFt}}{(-i2\pi mF)^2} + \frac{2e^{-i2\pi mFt}}{(-i2\pi mF)^3}\right]_{-0.5T}^{0.5T}$$

Substitute $e^{\pm i\pi mFT} = e^{\pm i\pi m} = (-1)^m$ [for integer m]

$$= \frac{(-1)^m}{T}\left[\frac{-2T}{(-i2\pi mF)^2}\right]$$ [all even powers of t cancel out]

$$= \frac{(-1)^m T^2}{2\pi^2 m^2} = \frac{(-1)^m 8}{\pi^2 m^2}$$



$U_{0:3} = [1.333, -0.811, 0.203, -0.090]$ [$u(t)$ real+even $\Rightarrow U_m$ real]

Convergence is noticeably faster than for method (a)

Summary

- Discontinuity at $t = a$
  - Gibbs phenomenon: $u_N(t)$ overshoots by 9% of iump
  - $u_N(a) \to$ mid point of iump
- Integration: If $v(t) = \int^t u(\tau)d\tau$, then $V_m = \frac{-i}{2\pi mF} U_m$ and $V_0 = c$, an arbitrary constant. U_0 must be zero.
- Differentiation: If $w(t) = \frac{du(t)}{dt}$, then $W_m = i2\pi mFU_m$ provided $w(t)$ satisfies Dirichlet conditions (it might not)
- Rate of decay:
  - For large n, U_n decreases at a rate $|n|^{-(k+1)}$ where $\frac{d^k u(t)}{dt^k}$ is the first discontinuous derivative
  - Error power: $\left\langle (u(t) - u_N(t))^2 \right\rangle = \sum_{|n|>N} |U_n|^2$
- Periodic Extension of finite domain signal of length L
  - (a) Repeat indefinitely with period $T = L$
  - (b) Reflect alternate repetitions for period $T = 2L$
    no discontinuities or Gibbs phenomenon

For further details see RHB Chapter 12.4, 12.5, 12.6

6: Fourier Transform

Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

The harmonic frequencies are $nF \; \forall n$ and are spaced $F = \frac{1}{T}$ apart.

As T gets larger, the harmonic spacing becomes smaller.

e.g. $T = 1\,\text{s} \Rightarrow F = 1\,\text{Hz}$
$T = 1\,\text{day} \Rightarrow F = 11.57\,\mu\text{Hz}$

If $T \to \infty$ then the harmonic spacing becomes zero, the sum becomes an integral and we get the Fourier Transform:

$$u(t) = \int_{f=-\infty}^{+\infty} U(f) e^{i2\pi ft} df$$

Here, $U(f)$, is the *spectral density* of $u(t)$.

- $U(f)$ is a continuous function of f .
- $U(f)$ is complex-valued.
- $u(t)$ real $\Rightarrow U(f)$ is conjugate symmetric $\Leftrightarrow U(-f) = U(f)^*$.
- Units: if $u(t)$ is in volts, then $U(f)df$ must also be in volts
  $\Rightarrow U(f)$ is in volts/Hz (hence "spectral density").

Fourier Series: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

The summation is over a set of equally spaced frequencies $f_n = nF$ where the spacing between them is $\Delta f = F = \frac{1}{T}$.

$$U_n = \left\langle u(t)e^{-i2\pi nFt} \right\rangle = \Delta f \int_{t=-0.5T}^{0.5T} u(t)e^{-i2\pi nFt} dt$$

Spectral Density: If $u(t)$ has finite energy, $U_n \to 0$ as $\Delta f \to 0$. So we define a spectral density, $U(f_n) = \frac{U_n}{\Delta f}$, on the set of frequencies $\{f_n\}$:

$$U(f_n) = \frac{U_n}{\Delta f} = \int_{t=-0.5T}^{0.5T} u(t)e^{-i2\pi f_n t} dt$$

we can write [Substitute $U_n = U(f_n)\Delta f$]

$$u(t) = \sum_{n=-\infty}^{\infty} U(f_n) e^{i2\pi f_n t} \Delta f$$

Fourier Transform: Now if we take the limit as $\Delta f \to 0$, we get

$$u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df \quad \text{[Fourier Synthesis]}$$

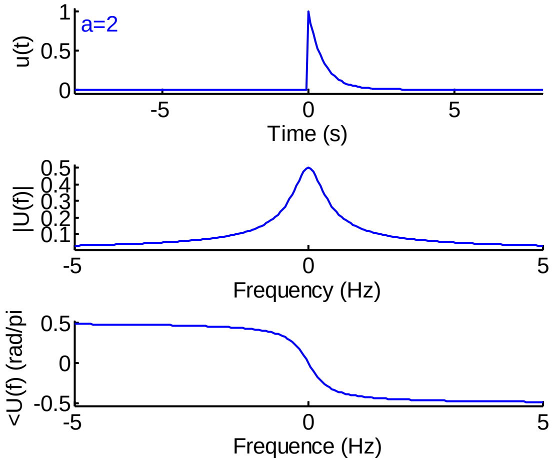
$$U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi ft} dt \quad \text{[Fourier Analysis]}$$

For non-periodic signals $U_n \to 0$ as $\Delta f \to 0$ and $U(f_n) = \frac{U_n}{\Delta f}$ remains finite. However, if $u(t)$ contains an exactly periodic component, then the corresponding $U(f_n)$ will become infinite as $\Delta f \to 0$. We will deal with it.

Fourier Transform Examples

Example 1:

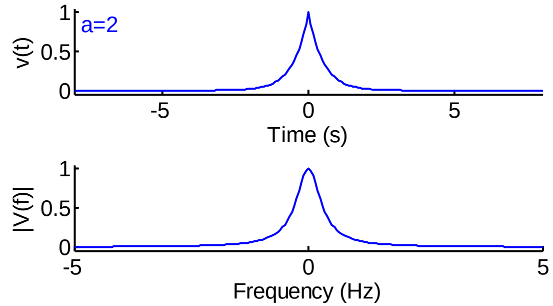
$$u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi f t} dt$$
$$= \int_{0}^{\infty} e^{-at} e^{-i2\pi f t} dt$$
$$= \int_{0}^{\infty} e^{(-a-i2\pi f)t} dt$$
$$= \frac{-1}{a+i2\pi f} \left[e^{(-a-i2\pi f)t} \right]_0^{\infty} = \frac{1}{a+i2\pi f}$$

Example 2:

$$v(t) = e^{-a|t|}$$

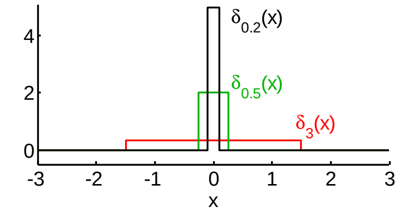


$$V(f) = \int_{-\infty}^{\infty} v(t) e^{-i2\pi f t} dt$$
$$= \int_{-\infty}^{0} e^{at} e^{-i2\pi f t} dt + \int_{0}^{\infty} e^{-at} e^{-i2\pi f t} dt$$
$$= \frac{1}{a-i2\pi f} \left[e^{(a-i2\pi f)t} \right]_{-\infty}^{0} + \frac{-1}{a+i2\pi f} \left[e^{(-a-i2\pi f)t} \right]_0^{\infty}$$
$$= \frac{1}{a-i2\pi f} + \frac{1}{a+i2\pi f} = \frac{2a}{a^2+4\pi^2 f^2}$$

[$v(t)$ real+symmetric
$\Rightarrow V(f)$ real+symmetric]

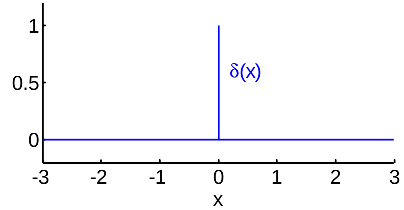
We define a unit area pulse of width w as

$$d_w(x) = \begin{cases} \frac{1}{w} & -0.5w \le x \le 0.5w \\ 0 & \text{otherwise} \end{cases}$$



This pulse has the property that its integral equals 1 for all values of w:

$$\int_{x=-\infty}^{\infty} d_w(x)dx = 1$$

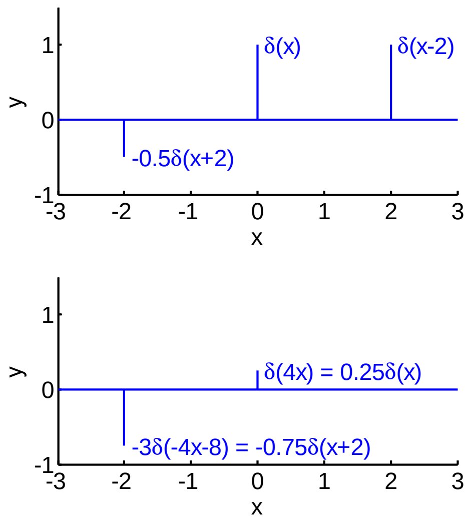


If we make w smaller, the pulse height increases to preserve unit area.

We define the Dirac delta function as $\delta(x) = \lim_{w \to 0} d_w(x)$

- $\delta(x)$ equals zero everywhere except at $x = 0$ where it is infinite.
- However its area still equals $1 \Rightarrow \int_{-\infty}^{\infty} \delta(x)dx = 1$
- We plot the height of $\delta(x)$ as its area rather than its true height of ∞.

$\delta(x)$ is not quite a proper function: it is called a generalized function.



Translation: $\delta(x-a)$

$\delta(x)$ is a pulse at $x=0$
$\delta(x-a)$ is a pulse at $x=a$

Amplitude Scaling: $b\delta(x)$

$\delta(x)$ has an area of $1 \Leftrightarrow \int_{-\infty}^{\infty} \delta(x)dx = 1$

$b\delta(x)$ has an area of b since

$$\int_{-\infty}^{\infty} (b\delta(x))\, dx = b\int_{-\infty}^{\infty} \delta(x)dx = b$$

b can be a complex number (on a graph, we then plot only its magnitude)

Time Scaling: $\delta(cx)$

$c>0$: $\int_{x=-\infty}^{\infty} \delta(cx)dx = \int_{y=-\infty}^{\infty} \delta(y)\frac{dy}{c}$ [sub $y=cx$]

$$= \frac{1}{c}\int_{y=-\infty}^{\infty} \delta(y)dy = \frac{1}{c} = \frac{1}{|c|}$$

$c<0$: $\int_{x=-\infty}^{\infty} \delta(cx)dx = \int_{y=+\infty}^{-\infty} \delta(y)\frac{dy}{c}$ [sub $y=cx$]

$$= \frac{-1}{c}\int_{y=-\infty}^{+\infty} \delta(y)dy = \frac{-1}{c} = \frac{1}{|c|}$$

In general, $\delta(cx) = \frac{1}{|c|}\delta(x)$ for $c \neq 0$

Dirac Delta Function: Products and Integrals

If we multiply $\delta(x-a)$ by a function of x:

$$y = x^2 \times \delta(x-2)$$

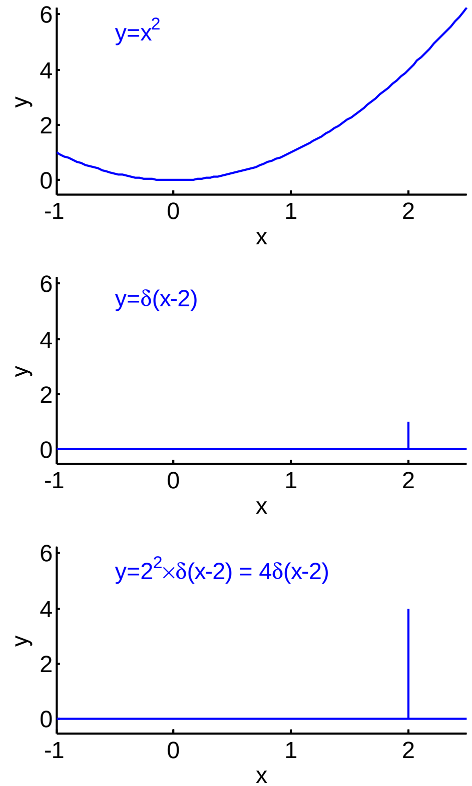
The product is 0 everywhere except at $x = 2$.

So $\delta(x-2)$ is multiplied by the value taken by x^2 at $x = 2$:

$$x^2 \times \delta(x-2) = \left[x^2\right]_{x=2} \times \delta(x-2)$$
$$= 4 \times \delta(x-2)$$

In general for any function, $f(x)$, that is continuous at $x = a$,

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$



Integrals:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(a)\delta(x-a)dx$$
$$= f(a)\int_{-\infty}^{\infty} \delta(x-a)dx$$
$$= f(a) \qquad \text{[if } f(x) \text{ continuous at } a\text{]}$$

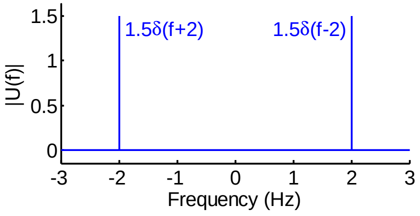
Example: $\int_{-\infty}^{\infty} \left(3x^2 - 2x\right)\delta(x-2)dx = \left[3x^2 - 2x\right]_{x=2} = 8$

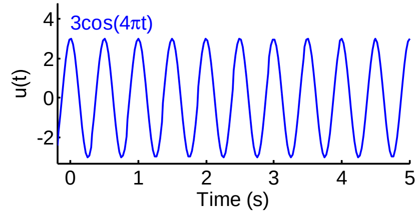
Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df$ [Fourier Synthesis]

$U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi ft} dt$ [Fourier Analysis]

Example: $U(f) = 1.5\delta(f+2) + 1.5\delta(f-2)$

$$\begin{aligned} u(t) &= \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df \\ &= \int_{-\infty}^{\infty} 1.5\delta(f+2) e^{i2\pi ft} df \\ &\quad + \int_{-\infty}^{\infty} 1.5\delta(f-2) e^{i2\pi ft} df \\ &= 1.5 \left[e^{i2\pi ft}\right]_{f=-2} + 1.5 \left[e^{i2\pi ft}\right]_{f=+2} \\ &= 1.5 \left(e^{i4\pi t} + e^{-i4\pi t}\right) = 3\cos 4\pi t \end{aligned}$$





If $u(t)$ is periodic then $U(f)$ is a sum of Dirac delta functions:

$$u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} \quad \Rightarrow \quad U(f) = \sum_{n=-\infty}^{\infty} U_n \delta(f - nF)$$

Proof: $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} df$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} U_n \delta(f - nF) e^{i2\pi ft} df \\ &= \sum_{n=-\infty}^{\infty} U_n \int_{-\infty}^{\infty} \delta(f - nF) e^{i2\pi ft} df \\ &= \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt} \end{aligned}$$

Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft}df$ [Fourier Synthesis]

$U(f) = \int_{t=-\infty}^{\infty} u(t)e^{-i2\pi ft}dt$ [Fourier Analysis]

Dual transform:

Suppose $v(t) = U(t)$, then

$$V(f) = \int_{t=-\infty}^{\infty} v(t)e^{-i2\pi ft}d\tau$$

$$V(g) = \int_{t=-\infty}^{\infty} U(t)e^{-i2\pi gt}dt$$ [substitute $f = g$, $v(t) = U(t)$]

$$= \int_{f=-\infty}^{\infty} U(f)e^{-i2\pi gf}df$$ [substitute $t = f$]

$$= u(-g)$$

So: $v(t) = U(t) \quad \Rightarrow \quad V(f) = u(-f)$

Example:

$u(t) = e^{-|t|} \quad \Rightarrow \quad U(f) = \frac{2}{1+4\pi^2 f^2}$ [from earlier]

$v(t) = \frac{2}{1+4\pi^2 t^2} \quad \Rightarrow \quad V(f) = e^{-|-f|} = e^{-|f|}$

Fourier Transform: $u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft}df$ [Fourier Synthesis]

$U(f) = \int_{t=-\infty}^{\infty} u(t)e^{-i2\pi ft}dt$ [Fourier Analysis]

Time Shifting and Scaling:

Suppose $v(t) = u(at+b)$, then

$$V(f) = \int_{t=-\infty}^{\infty} u(at+b)e^{-i2\pi ft}dt$$ [now sub $\tau = at+b$]

$$= \text{sgn}(a)\int_{\tau=-\infty}^{\infty} u(\tau)e^{-i2\pi f\left(\frac{\tau-b}{a}\right)}\frac{1}{a}d\tau$$

note that $\pm\infty$ limits swap if $a<0$ hence $\text{sgn}(a) = \begin{cases}1 & a>0\\ -1 & a<0\end{cases}$

$$= \frac{1}{|a|}e^{i\frac{2\pi fb}{a}}\int_{\tau=-\infty}^{\infty} u(\tau)e^{-i2\pi\frac{f}{a}\tau}d\tau$$

$$= \frac{1}{|a|}e^{i\frac{2\pi fb}{a}}U\left(\frac{f}{a}\right)$$

So: $v(t) = u(at+b) \quad\Rightarrow\quad V(f) = \frac{1}{|a|}e^{i\frac{2\pi fb}{a}}U\left(\frac{f}{a}\right)$

Gaussian Pulse: $u(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$

This is a Normal (or Gaussian) probability distribution, so $\int_{-\infty}^{\infty} u(t)dt = 1$.

$$U(f) = \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-i2\pi ft}dt$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t^2+i4\pi\sigma^2 ft\right)}dt$$

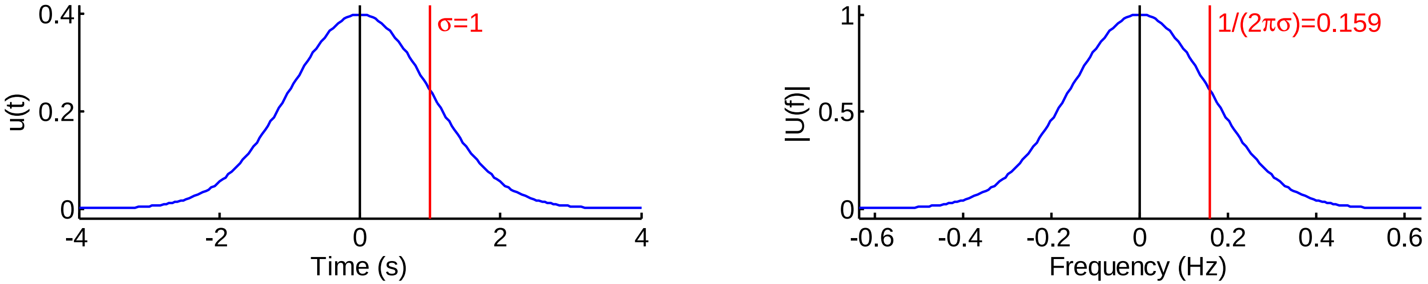
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t^2+i4\pi\sigma^2 ft+\left(i2\pi\sigma^2 f\right)^2-\left(i2\pi\sigma^2 f\right)^2\right)}dt$$

$$= e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t+i2\pi\sigma^2 f\right)^2}dt$$

$$\stackrel{(i)}{=} e^{\frac{1}{2\sigma^2}\left(i2\pi\sigma^2 f\right)^2} = e^{-\frac{1}{2}(2\pi\sigma f)^2}$$

[(i) uses a result from complex analysis theory that:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(t+i2\pi\sigma^2 f\right)^2}dt = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}t^2}dt = 1]$$



Uniquely, the Fourier Transform of a Gaussian pulse is a Gaussian pulse.

Summary

- Fourier Transform:
  - Inverse transform (synthesis): $u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi f t} df$
  - Forward transform (analysis): $U(f) = \int_{t=-\infty}^{\infty} u(t) e^{-i2\pi f t} dt$
    - ▷ $U(f)$ is the spectral density function (e.g. Volts/Hz)
- Dirac Delta Function:
  - $\delta(t)$ is a zero-width infinite-height pulse with $\int_{-\infty}^{\infty} \delta(t) dt = 1$
  - Integral: $\int_{-\infty}^{\infty} f(t)\delta(t-a) = f(a)$
  - Scaling: $\delta(ct) = \frac{1}{|c|}\delta(t)$
- Periodic Signals: $u(t) = \sum_{n=-\infty}^{\infty} U_n e^{i2\pi nFt}$

  $$\Rightarrow U(f) = \sum_{n=-\infty}^{\infty} U_n \delta(f - nF)$$
- Fourier Transform Properties:
  - $v(t) = U(t) \quad \Rightarrow \quad V(f) = u(-f)$
  - $v(t) = u(at+b) \quad \Rightarrow \quad V(f) = \frac{1}{|a|} e^{i\frac{2\pi f b}{a}} U\left(\frac{f}{a}\right)$
  - $v(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2} \quad \Rightarrow \quad V(f) = e^{-\frac{1}{2}(2\pi\sigma f)^2}$

For further details see RHB Chapter 13.1 (uses ω instead of f)

7: Fourier Transforms: Convolution and Parseval's Theorem

Question: What is the Fourier transform of $w(t) = u(t)v(t)$?

Let $u(t) = \int_{h=-\infty}^{+\infty} U(h)e^{i2\pi ht}dh$ and $v(t) = \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt}dg$

[Note use of different dummy variables]

$$w(t) = u(t)v(t)$$
$$= \int_{h=-\infty}^{+\infty} U(h)e^{i2\pi ht}dh \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt}dg$$
$$= \int_{h=-\infty}^{+\infty} U(h) \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi(h+g)t}dg\, dh \qquad \text{[merge } e^{(\cdots)}\text{]}$$

Now we make a change of variable in the second integral: $g = f - h$

$$= \int_{h=-\infty}^{+\infty} U(h) \int_{f=-\infty}^{+\infty} V(f-h)e^{i2\pi ft}df\, dh$$
$$= \int_{f=-\infty}^{\infty} \int_{h=-\infty}^{+\infty} U(h)V(f-h)e^{i2\pi ft}dh\, df \qquad \text{[swap } \textstyle\int\text{]}$$
$$= \int_{f=-\infty}^{+\infty} W(f)e^{i2\pi ft}df$$

where $W(f) = \int_{h=-\infty}^{+\infty} U(h)V(f-h)dh$ $\int_{h=-\infty}^{+\infty} U(h)V(f-h)dh \triangleq U(f) * V(f)$

This is the *convolution* of the two spectra $U(f)$ and $V(f)$.

$$w(t) = u(t)v(t) \quad \Leftrightarrow \quad W(f) = U(f) * V(f)$$

Multiplication Example

$$u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$U(f) = \frac{1}{a+i2\pi f}$$ [from before]

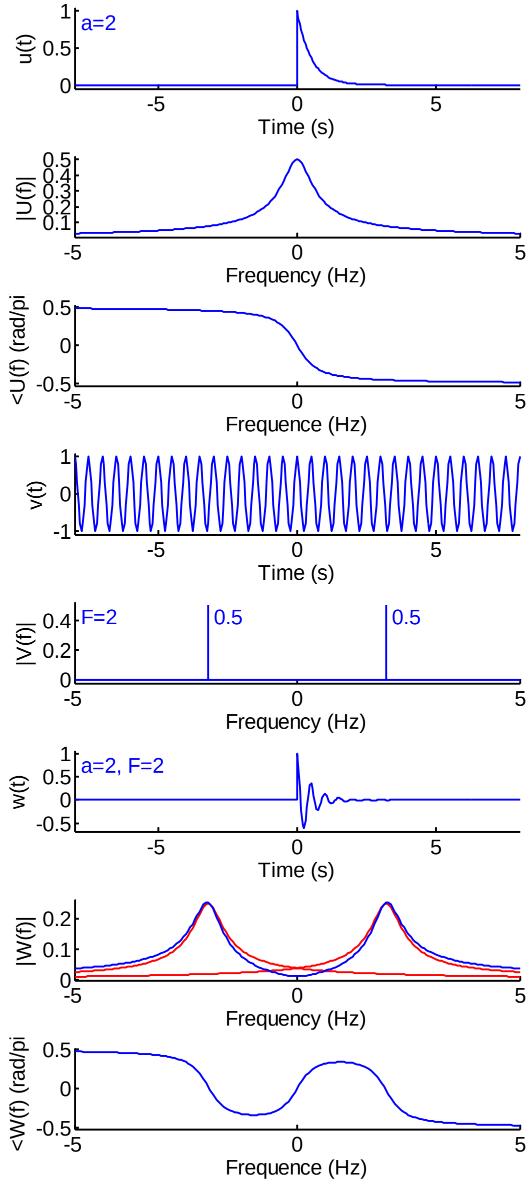
$$v(t) = \cos 2\pi F t$$

$$V(f) = 0.5\left(\delta(f+F) + \delta(f-F)\right)$$

$$w(t) = u(t)v(t)$$

$$W(f) = U(f) * V(f) = \frac{0.5}{a+i2\pi(f+F)} + \frac{0.5}{a+i2\pi(f-F)}$$

If $V(f)$ consists entirely of Dirac impulses then $U(f) * V(f)$ just replaces each impulse with a complete copy of $U(f)$ scaled by the area of the impulse and shifted so that $0\,\text{Hz}$ lies on the impulse. Then add the overlapping complex spectra.



Convolution Theorem:

$$w(t) = u(t)v(t) \quad \Leftrightarrow \quad W(f) = U(f) * V(f)$$
$$w(t) = u(t) * v(t) \quad \Leftrightarrow \quad W(f) = U(f)V(f)$$

Convolution in the time domain is equivalent to multiplication in the frequency domain and vice versa.

Proof of second line:

Given $u(t)$, $v(t)$ and $w(t)$ satisfying

$$w(t) = u(t)v(t) \quad \Leftrightarrow \quad W(f) = U(f) * V(f)$$

define dual waveforms $x(t)$, $y(t)$ and $z(t)$ as follows:

$$x(t) = U(t) \quad \Leftrightarrow \quad X(f) = u(-f) \qquad \text{[duality]}$$
$$y(t) = V(t) \quad \Leftrightarrow \quad Y(f) = v(-f)$$
$$z(t) = W(t) \quad \Leftrightarrow \quad Z(f) = w(-f)$$

Now the convolution property becomes:

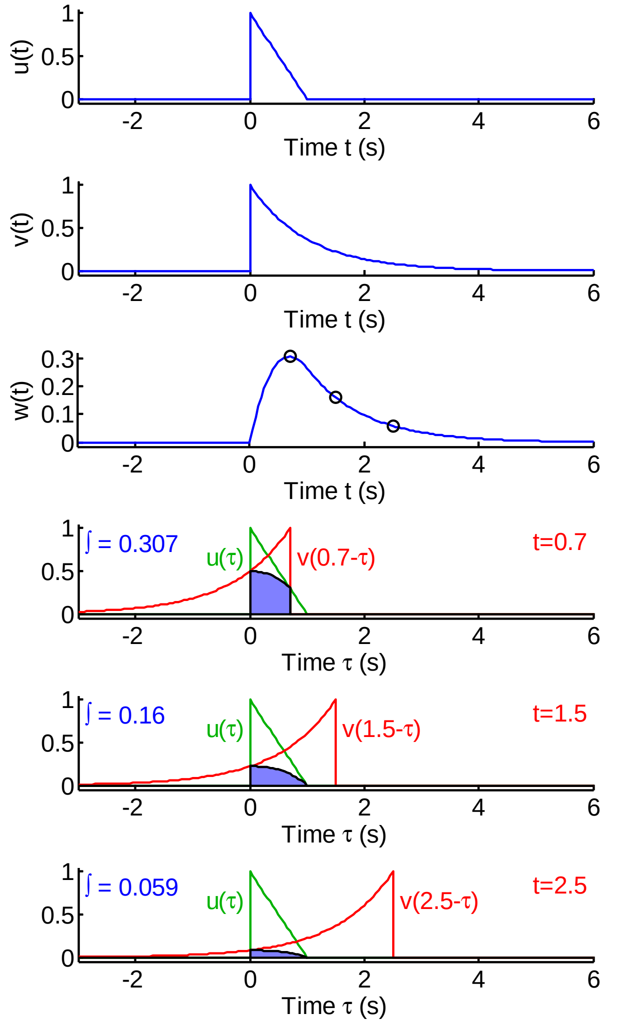
$$w(-f) = u(-f)v(-f) \quad \Leftrightarrow \quad W(t) = U(t) * V(t) \qquad [\text{sub } t \leftrightarrow \pm f]$$
$$Z(f) = X(f)Y(f) \quad \Leftrightarrow \quad z(t) = x(t) * y(t) \qquad \text{[duality]}$$

Convolution Example

$$u(t) = \begin{cases} 1-t & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$v(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$\begin{aligned} w(t) &= u(t) * v(t) \\ &= \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau \\ &= \int_0^{\min(t,1)} (1-\tau)e^{\tau-t}d\tau \\ &= \left[(2-\tau)\, e^{\tau-t}\right]_{\tau=0}^{\min(t,1)} \\ &= \begin{cases} 0 & t<0 \\ 2-t-2e^{-t} & 0 \le t < 1 \\ (e-2)\, e^{-t} & t \ge 1 \end{cases} \end{aligned}$$



Note how $v(t-\tau)$ is time-reversed (because of the $-\tau$) and time-shifted to put the time origin at $\tau = t$.

Convolution Properties

Convloution: $w(t) = u(t) * v(t) \triangleq \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau$

Convolution behaves algebraically like multiplication:

1) Commutative: $u(t) * v(t) = v(t) * u(t)$

2) Associative:

$$u(t) * v(t) * w(t) = (u(t) * v(t)) * w(t) = u(t) * (v(t) * w(t))$$

3) Distributive over addition:

$$w(t) * (u(t) + v(t)) = w(t) * u(t) + w(t) * v(t)$$

4) Identity Element or "1": $u(t) * \delta(t) = \delta(t) * u(t) = u(t)$

5) Bilinear: $(au(t)) * (bv(t)) = ab\,(u(t) * v(t))$

Proof: In the frequency domain, convolution is multiplication.

Also, if $u(t) * v(t) = w(t)$, then

6) Time Shifting: $u(t+a) * v(t+b) = w(t+a+b)$

7) Time Scaling: $u(at) * v(at) = \frac{1}{|a|} w(at)$

How to recognise a convolution integral:

the arguments of $u(\cdots)$ and $v(\cdots)$ sum to a constant.

Lemma:

$$X(f) = \delta(f-g) \quad \Rightarrow \quad x(t) = \int \delta(f-g) e^{i2\pi ft} df = e^{i2\pi gt}$$
$$\Rightarrow \quad X(f) = \int e^{i2\pi gt} e^{-i2\pi ft} dt = \int e^{i2\pi (g-f)t} dt = \delta(g-f)$$

Parseval's Theorem: $\int_{t=-\infty}^{\infty} u^*(t)v(t)dt = \int_{f=-\infty}^{+\infty} U^*(f)V(f)df$

Proof:

Let $u(t) = \int_{f=-\infty}^{+\infty} U(f)e^{i2\pi ft}df$ and $v(t) = \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt}dg$

[Note use of different dummy variables]

Now multiply $u^*(t) = u(t)$ and $v(t)$ together and integrate over time:

$$\begin{aligned}
&\int_{t=-\infty}^{\infty} u^*(t)v(t)dt \\
&= \int_{t=-\infty}^{\infty} \int_{f=-\infty}^{+\infty} U^*(f)e^{-i2\pi ft}df \int_{g=-\infty}^{+\infty} V(g)e^{i2\pi gt}dgdt \\
&= \int_{f=-\infty}^{+\infty} U^*(f) \int_{g=-\infty}^{+\infty} V(g) \int_{t=-\infty}^{\infty} e^{i2\pi (g-f)t}dtdgdf \\
&= \int_{f=-\infty}^{+\infty} U^*(f) \int_{g=-\infty}^{+\infty} V(g)\delta(g-f)dgdf \qquad \text{[lemma]} \\
&= \int_{f=-\infty}^{+\infty} U^*(f)V(f)df
\end{aligned}$$

Parseval's Theorem: $\int_{t=-\infty}^{\infty} u^*(t)v(t)dt = \int_{f=-\infty}^{+\infty} U^*(f)V(f)df$

For the special case $v(t) = u(t)$, Parseval's theorem becomes:

$$\int_{t=-\infty}^{\infty} u^*(t)u(t)dt = \int_{f=-\infty}^{+\infty} U^*(f)U(f)df$$

$$\Rightarrow \quad E_u = \int_{t=-\infty}^{\infty} |u(t)|^2\, dt = \int_{f=-\infty}^{+\infty} |U(f)|^2\, df$$

Energy Conservation: The energy in $u(t)$ equals the energy in $U(f)$.

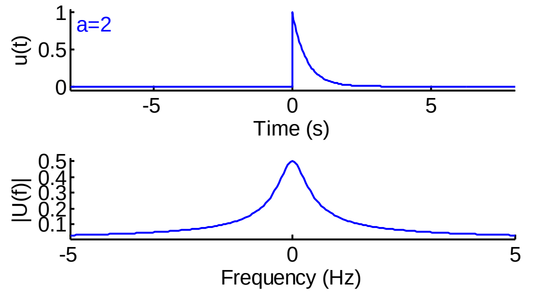
Example:

$$u(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \Rightarrow \quad E_u = \int |u(t)|^2\, dt = \left[\frac{-e^{-2at}}{2a}\right]_0^{\infty} = \frac{1}{2a}$$

$U(f) = \frac{1}{a+i2\pi f}$ [from before]

$$\Rightarrow \quad \int |U(f)|^2\, df = \int \frac{df}{a^2+4\pi^2 f^2}$$

$$= \left[\frac{\tan^{-1}\left(\frac{2\pi f}{a}\right)}{2\pi a}\right]_{-\infty}^{\infty} = \frac{\pi}{2\pi a} = \frac{1}{2a}$$



Energy Spectrum

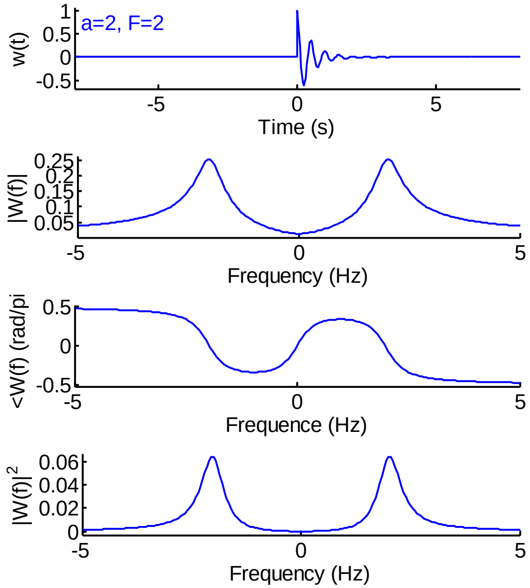
Example from before:

$$w(t) = \begin{cases} e^{-at}\cos 2\pi F t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$W(f) = \frac{0.5}{a+i2\pi(f+F)} + \frac{0.5}{a+i2\pi(f-F)}$$

$$= \frac{a+i2\pi f}{a^2+i4\pi a f-4\pi^2(f^2-F^2)}$$

$$|W(f)|^2 = \frac{a^2+4\pi^2 f^2}{(a^2-4\pi^2(f^2-F^2))^2+16\pi^2 a^2 f^2}$$



Energy Spectrum

- The units of $|W(f)|^2$ are "*energy per Hz*" so that its integral, $E_w = \int_{-\infty}^{\infty} |W(f)|^2 df$, has units of energy.
- The quantity $|W(f)|^2$ is called the *energy spectral density* of $w(t)$ at frequency f and its graph is the *energy spectrum* of $w(t)$. It shows how the energy of $w(t)$ is distributed over frequencies.
- If you divide $|W(f)|^2$ by the total energy, E_w, the result is non-negative and integrates to unity like a probability distribution.

Summary

- Convolution:
  - $u(t) * v(t) \triangleq \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau$
    - ▷ Arguments of $u(\cdots)$ and $v(\cdots)$ sum to t
  - Acts like multiplication + time scaling/shifting formulae
- Convolution Theorem: multiplication ↔ convolution
  - $w(t) = u(t)v(t) \quad \Leftrightarrow \quad W(f) = U(f) * V(f)$
  - $w(t) = u(t) * v(t) \quad \Leftrightarrow \quad W(f) = U(f)V(f)$
- Parseval's Theorem: $\int_{t=-\infty}^{\infty} u^*(t)v(t)dt = \int_{f=-\infty}^{+\infty} U^*(f)V(f)df$
- Energy Spectrum:
  - Energy spectral density: $|U(f)|^2$ (energy/Hz)
  - Parseval: $E_u = \int |u(t)|^2 dt = \int |U(f)|^2 df$

For further details see RHB Chapter 13.1

8: Correlation

The *cross-correlation* between two signals $u(t)$ and $v(t)$ is

$$w(t) = u(t) \otimes v(t) \triangleq \int_{-\infty}^{\infty} u^*(\tau)v(\tau+t)d\tau$$
$$= \int_{-\infty}^{\infty} u^*(\tau-t)v(\tau)d\tau \qquad \text{[sub: } \tau \to \tau - t\text{]}$$

The complex conjugate, $u^*(\tau)$ makes no difference if $u(t)$ is real-valued but makes the definition work even if $u(t)$ is complex-valued.

Correlation versus Convolution:

$$u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau)v(\tau+t)d\tau \qquad \text{[correlation]}$$
$$u(t) * v(t) = \int_{-\infty}^{\infty} u(\tau)v(t-\tau)d\tau \qquad \text{[convolution]}$$

Unlike convolution, the integration variable, τ, has the same sign in the arguments of $u(\cdots)$ and $v(\cdots)$ so the arguments have a constant difference instead of a constant sum (i.e. $v(t)$ is not time-flipped).

Notes: (a) The argument of $w(t)$ is called the "lag" (= delay of u versus v).
(b) Some people write $u(t) \star v(t)$ instead of $u(t) \otimes v(t)$.
(c) Some swap u and v and/or negate t in the integral.

It is all rather inconsistent ☹.

Signal Matching

Cross correlation is used to find where two signals match: $u(t)$ is the test waveform.

Example 1:

$v(t)$ contains $u(t)$ with an unknown delay and added noise.

$w(t) = u(t) \otimes v(t)$
$= \int u^*(\tau - t)v(\tau)dt$ gives a peak at the time lag where $u(\tau - t)$ best matches $v(\tau)$; in this case at $t = 450$

Example 2:

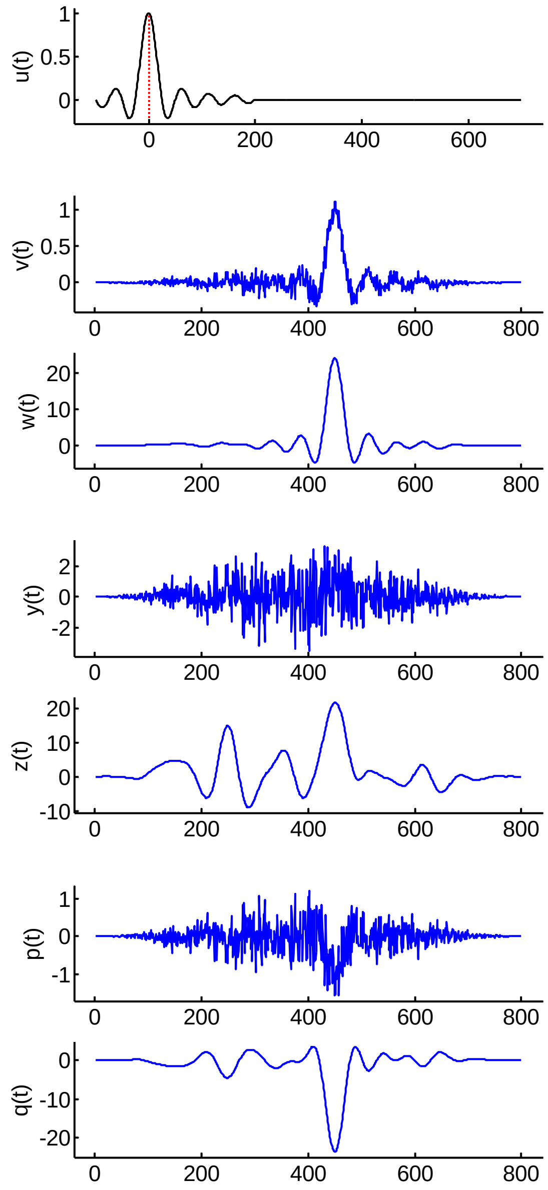
$y(t)$ is the same as $v(t)$ with more noise

$z(t) = u(t) \otimes y(t)$ can still detect the correct time delay (hard for humans)

Example 3:

$p(t)$ contains $-u(t)$ so that

$q(t) = u(t) \otimes p(t)$ has a negative peak



Correlation: $w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau$

If we define $x(t) = u^*(-t)$ then

$$x(t) * v(t) \triangleq \int_{-\infty}^{\infty} x(t-\tau)v(\tau)d\tau = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau$$
$$= u(t) \otimes v(t)$$

Fourier Transform of $x(t)$:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} u^*(-t)e^{-i2\pi ft}dt$$
$$= \int_{-\infty}^{\infty} u^*(t)e^{i2\pi ft}dt = \left(\int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt\right)^*$$
$$= U^*(f)$$

So $w(t) = x(t) * v(t) \quad \Rightarrow \quad W(f) = X(f)V(f) = U^*(f)V(f)$

Hence the Cross-correlation theorem:

$$w(t) = u(t) \otimes v(t) \quad \Leftrightarrow \quad W(f) = U^*(f)V(f)$$
$$= u^*(-t) * v(t)$$

Note that, unlike convolution, correlation is not associative or commutative:

$$v(t) \otimes u(t) = v^*(-t) * u(t) = u(t) * v^*(-t) = w^*(-t)$$

Normalized Cross-correlation

Correlation: $w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau$

If we define $y(t) = u(t - t_0)$ for some fixed t_0, then $E_y = E_u$:

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 \, dt = \int_{-\infty}^{\infty} |u(t - t_0)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} |u(\tau)|^2 \, d\tau = E_u \qquad [t \to \tau + t_0]$$

Cauchy-Schwarz inequality: $\left|\int_{-\infty}^{\infty} y^*(\tau)v(\tau)d\tau\right|^2 \le E_y E_v$

$$\Rightarrow |w(t_0)|^2 = \left|\int_{-\infty}^{\infty} u^*(\tau - t_0)v(\tau)d\tau\right|^2 \le E_y E_v = E_u E_v$$

but t_0 was arbitrary, so we must have $|w(t)| \le \sqrt{E_u E_v}$ for all t

We can define the *normalized cross-correlation*

$$z(t) = \frac{u(t) \otimes v(t)}{\sqrt{E_u E_v}}$$

with properties: (1) $|z(t)| \le 1$ for all t

(2) $|z(t_0)| = 1 \Leftrightarrow v(\tau) = \alpha u(\tau - t_0)$ with α constant

[Cauchy-Schwarz Inequality Proof]

You do not need to memorize this proof

We want to prove the Cauchy-Schwarz Inequality: $\left|\int_{-\infty}^{\infty} u^*(t)v(t)dt\right|^2 \le E_u E_v$

where $E_u \triangleq \int_{-\infty}^{\infty} |u(t)|^2\, dt$.

Suppose we define $w \triangleq \int_{-\infty}^{\infty} u^*(t)v(t)dt$. Then,

$$
\begin{aligned}
0 &\le \int |E_v u(t) - w^* v(t)|^2\, dt && [|\cdots|^2 \text{ always } \ge 0]\\
&= \int \left(E_v u^*(t) - w v^*(t)\right)\left(E_v u(t) - w^* v(t)\right) dt && [|z|^2 = z^* z]\\
&= E_v^2 \int u^*(t)u(t)dt + |w|^2 \int v^*(t)v(t)dt - w^* E_v \int u^*(t)v(t)dt - w E_v \int u(t)v^*(t)dt\\
&= E_v^2 \int |u(t)|^2\, dt + |w|^2 \int |v(t)|^2\, dt - E_v w^* w - E_v w w^* && [\text{definition of } w]\\
&= E_v^2 E_u + |w|^2 E_v - 2|w|^2 E_v = E_v\left(E_u E_v - |w|^2\right) && [|z|^2 = z^* z]
\end{aligned}
$$

Unless $E_v = 0$ (in which case, $v(t) \equiv 0$ and the C-S inequality is true), we must have $|w|^2 \le E_u E_v$ which proves the C-S inequality.

Also, $E_u E_v = |w|^2$ only if we have equality in the first line,

that is, $\int |E_v u(t) - w^* v(t)|^2\, dt = 0$ which implies that the integrand is zero for all t.

This implies that $u(t) = \frac{w^*}{E_v} v(t)$.

So we have shown that $E_u E_v = |w|^2$ if and only if $u(t)$ and $v(t)$ are proportional to each other.

The correlation of a signal with itself is its *autocorrelation*:

$$w(t) = u(t) \otimes u(t) = \int_{-\infty}^{\infty} u^*(\tau - t)u(\tau)d\tau$$

The autocorrelation at zero lag:

$$\begin{aligned} w(0) &= \int_{-\infty}^{\infty} u^*(\tau - 0)u(\tau)d\tau \\ &= \int_{-\infty}^{\infty} u^*(\tau)u(\tau)d\tau \\ &= \int_{-\infty}^{\infty} |u(\tau)|^2 \, d\tau = E_u \end{aligned}$$

The autocorrelation at zero lag, $w(0)$, is the energy of the signal.

The normalized autocorrelation: $z(t) = \frac{u(t) \otimes u(t)}{E_u}$

satisfies $z(0) = 1$ and $|z(t)| \le 1$ for any t.

Wiener-Khinchin Theorem: [Cross-correlation theorem when $v(t) = u(t)$]

$$w(t) = u(t) \otimes u(t) \quad \Leftrightarrow \quad W(f) = U^*(f)U(f) = |U(f)|^2$$

The Fourier transform of the autocorrelation is the energy spectrum.

Cross-correlation is used to find when two different signals are similar.
Autocorrelation is used to find when a signal is similar to itself delayed.

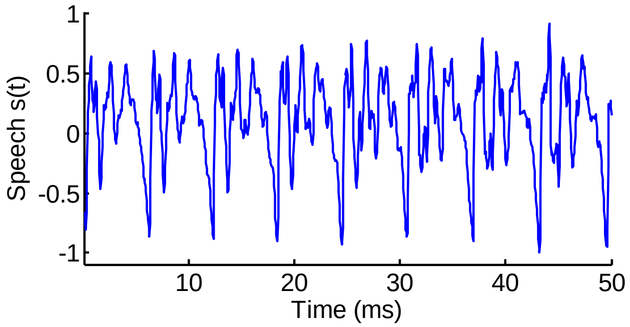
First graph shows $s(t)$ a segment of the microphone signal from the initial vowel of "early" spoken by me. The waveform is "quasi-periodic" = "almost periodic but not quite".

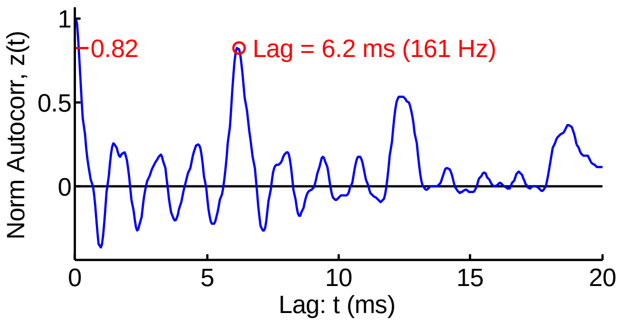
Second graph shows normalized autocorrelation, $z(t) = \frac{s(t) \otimes s(t)}{E_s}$.

$z(0) = 1$ for $t = 0$ since a signal always matches itself exactly.

$z(t) = 0.82$ for $t = 6.2\,\text{ms}$ = one period lag (not an exact match).

$z(t) = 0.53$ for $t = 12.4\,\text{ms}$ = two periods lag (even worse match).





There are three different versions of the Fourier Transform in current use.

(1) Frequency version (we have used this in lectures)

$$U(f) = \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}dt \qquad u(t) = \int_{-\infty}^{\infty} U(f)e^{i2\pi ft}df$$

- Used in the communications/broadcasting industry and textbooks.
- The formulae do not need scale factors of 2π anywhere. ☺☺☺

(2) Angular frequency version

$$\widetilde{U}(\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt \qquad u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{U}(\omega)e^{i\omega t}d\omega$$

Continuous spectra are unchanged: $\widetilde{U}(\omega) = U(f) = U(\frac{\omega}{2\pi})$

However δ-function spectral components are multiplied by 2π so that

$$U(f) = \delta(f - f_0) \quad \Rightarrow \quad \widetilde{U}(\omega) = 2\pi \times \delta(\omega - 2\pi f_0)$$

- Used in most signal processing and control theory textbooks.

(3) Angular frequency + symmetrical scale factor

$$\widehat{U}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(t)e^{-i\omega t}dt \qquad u(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{U}(\omega)e^{i\omega t}d\omega$$

In all cases $\widehat{U}(\omega) = \frac{1}{\sqrt{2\pi}}\widetilde{U}(\omega)$

- Used in many Maths textbooks (mathematicians like symmetry)

Fourier Transform using Angular Frequency:

$$\widetilde{U}(\omega) = \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt \qquad u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{U}(\omega) e^{i\omega t} d\omega$$

Any formula involving $\int df$ will change to $\frac{1}{2\pi} \int d\omega$ [since $d\omega = 2\pi df$]

Parseval's Theorem:

$$\int u^*(t) v(t) dt = \frac{1}{2\pi} \int \widetilde{U}^*(\omega) \widetilde{V}(\omega) d\omega$$

$$E_u = \int |u(t)|^2 dt = \frac{1}{2\pi} \int \left| \widetilde{U}(\omega) \right|^2 d\omega$$

Waveform Multiplication: (convolution implicitly involves integration)

$$w(t) = u(t)v(t) \quad \Rightarrow \quad \widetilde{W}(\omega) = \frac{1}{2\pi} \widetilde{U}(\omega) * \widetilde{V}(\omega)$$

Spectrum Multiplication: (multiplication $\nRightarrow$ integration)

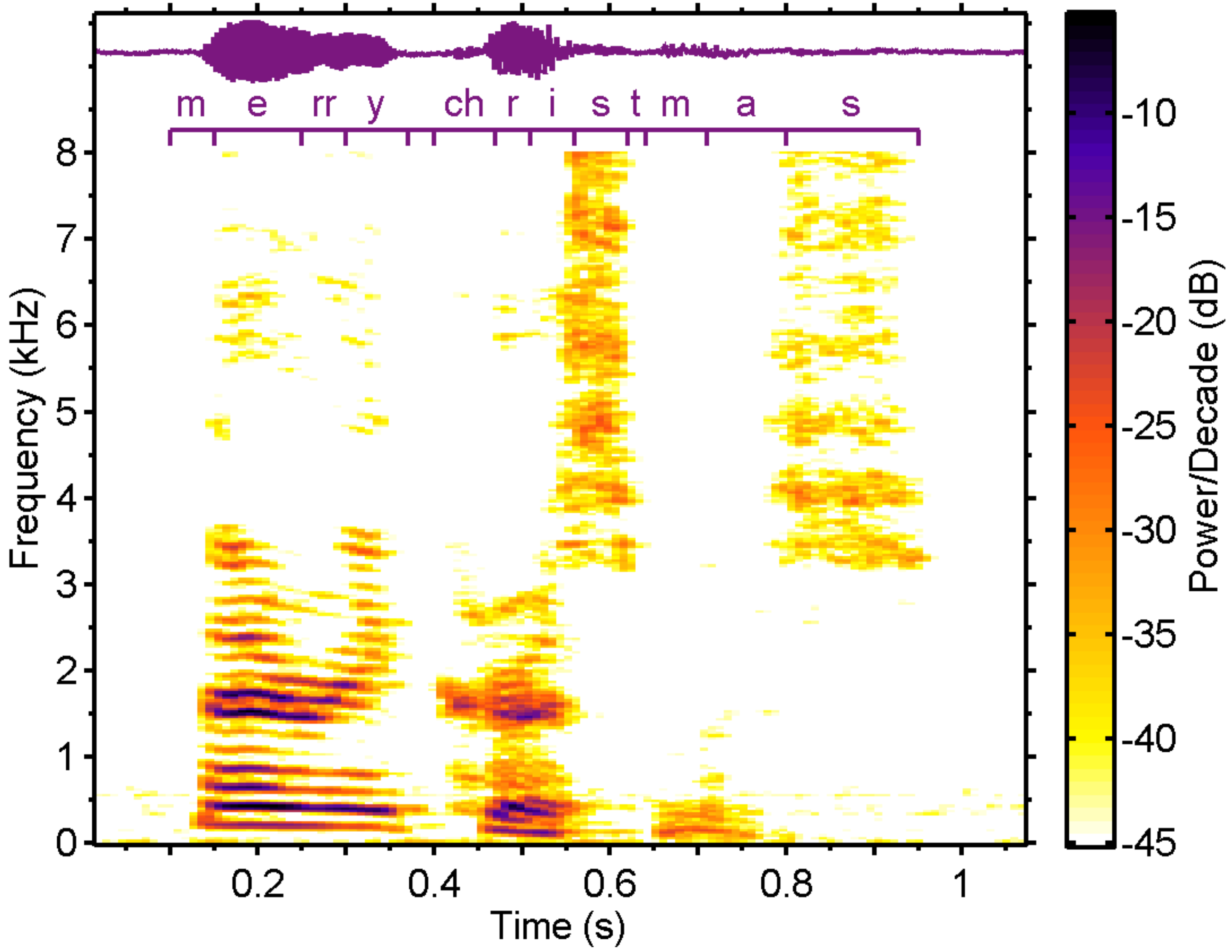
$$w(t) = u(t) * v(t) \quad \Rightarrow \quad \widetilde{W}(\omega) = \widetilde{U}(\omega) \widetilde{V}(\omega)$$

To obtain formulae for version (3) of the Fourier Transform, $\widehat{U}(\omega)$, substitute into the above formulae: $\widetilde{U}(\omega) = \sqrt{2\pi} \widehat{U}(\omega)$.

- Cross-Correlation: $w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau$
  - Used to find similarities between $v(t)$ and a delayed $u(t)$
  - Cross-correlation theorem: $W(f) = U^*(f)V(f)$
  - Cauchy-Schwarz Inequality: $|u(t) \otimes v(t)| \leq \sqrt{E_u E_v}$
    - Normalized cross-correlation: $\left|\frac{u(t)\otimes v(t)}{\sqrt{E_u E_v}}\right| \leq 1$
- Autocorrelation: $x(t) = u(t) \otimes u(t) = \int_{-\infty}^{\infty} u^*(\tau - t)u(\tau)d\tau \leq E_u$
  - Wiener-Khinchin: $X(f)$ = energy spectral density, $|U(f)|^2$
  - Used to find periodicity in $u(t)$
- Fourier Transform using ω:
  - Continuous spectra unchanged; spectral impulses multiplied by 2π
  - In formulae: $\int df \to \frac{1}{2\pi}\int d\omega$; ω-convolution involves an integral

For further details see RHB Chapter 13.1

Spectrogram of “Merry Christmas” spoken by Mike Brookes



[Complex Fourier Series]

All waveforms have period $T = 1$. $\delta_{condition}$ is 1 whenever "condition" is true and otherwise 0.

Waveform	$x(t)$ for $\|t\| < 0.5$	X_n
Square wave	$2\delta_{\|t\|<0.25} - 1$	$\frac{2\sin 0.5\pi n}{\pi n} \times \delta_{n\neq 0}$
Pulse of width d	$\delta_{\|t\|<0.5d}$	$\frac{\sin \pi d n}{\pi n}$
Sawtooth wave	$2t$	$\frac{i(-1)^n}{\pi n} \times \delta_{n\neq 0}$
Triangle wave	$1 - 4\|t\|$	$\frac{2(1-(-1)^n)}{\pi^2 n^2}$

[Fourier Transform Properties A]

You need not memorize these properties. All integrals are $\int_{-\infty}^{\infty}$

Property	$x(t)$	$Xf)$
Forward	$x(t)$	$\int x(t)e^{-i2\pi ft}dt$
Inverse	$\int X(f)e^{i2\pi ft}df$	$X(f)$
Spectral Zero	$\int x(t)dt$	$= X(0)$
Temporal Zero	$x(0)$	$= \int X(f)df$
Duality	$X(t)$	$x(-f)$
Reversal	$x(-t)$	$X(-f)$
conjugate	$x^*(t)$	$X^*(-f)$
Temporal Derivative	$\frac{d^n}{dt^n}x(t)$	$(i2\pi f)^n X(f)$
Spectral Derivative	$(-i2\pi t)^n x(t)$	$\frac{d^n}{df^n}X(f)$
Integral	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{i2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$
Scaling	$x(\alpha t + \beta)$	$\frac{1}{\lvert\alpha\rvert}e^{\frac{2i\pi f\beta}{\alpha}}X(\frac{f}{\alpha})$
Time Shift	$x(t-T)$	$X(f)e^{-i2\pi fT}$
Frequency Shift	$x(t)e^{i2\pi Ft}$	$X(f-F)$

[Fourier Transform Properties B]

You need not memorize these properties. All integrals are $\int_{-\infty}^{\infty}$

Property	$x(t)$	$X f)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(f) + \beta Y(f)$
Multiplication	$x(t)y(t)$	$X(f) * Y(f)$
Convolution	$x(t) * y(t)$	$X(f)Y(f)$
Correlation	$x(t) \otimes y(t)$	$X^*(f)Y(f)$
Autocorrelation	$x(t) \otimes x(t)$	$\|X(f)\|^2$
Parseval or Plancherel	$\int x^*(t)y(t)dt$ $E_x = \int \|x(t)\|^2 dt$	$= \int X^*(f)Y(f)df$ $= \int \|X(f)\|^2 df$
Repetition	$\sum_n x(t - nT)$	$\left\|\frac{1}{T}\right\| \sum_k X\left(\frac{k}{T}\right) \delta\left(f - \frac{k}{T}\right)$
Sampling	$\sum_n x(nT)\delta(t - nT)$	$\left\|\frac{1}{T}\right\| \sum_k X\left(f - \frac{k}{T}\right)$
Modulation	$x(t)\cos(2\pi F t)$	$\frac{1}{2}X(f - F) + \frac{1}{2}X(f + F)$

Convolution: $x(t) * y(t) = \int x(\tau)y(t - \tau)d\tau$

Cross-correlation: $x(t) \otimes y(t) = \int x^*(\tau)y(\tau + t)d\tau = \int x^*(\tau - t)y(\tau)d\tau$

[Fourier Transform Pairs]

You need not memorize these pairs.

$x(t)$	$X(f)$	$x(t)$	$X(f)$
$\delta(t)$	1	1	$\delta(f)$
$\text{rect}(t)$	$\frac{\sin(\pi f)}{\pi f}$	$\frac{\sin(t)}{t}$	$\pi\text{rect}(\pi f)$
$\text{tri}(t)$	$\frac{\sin^2(\pi f)}{\pi^2 f^2}$	$\frac{\sin^2(t)}{t^2}$	$\pi\text{tri}(\pi f)$
$\cos(2\pi\alpha t)$	$\frac{1}{2}\delta\left(f+\alpha\right)+\frac{1}{2}\delta\left(f-\alpha\right)$	$\sin(2\pi\alpha t)$	$\frac{i}{2}\delta\left(f+\alpha\right)-\frac{i}{2}\delta\left(f-\alpha\right)$
$e^{-\alpha t}u(t)$	$\frac{1}{\alpha+2\pi i f}$	$te^{-\alpha t}u(t)$	$\frac{1}{(\alpha+2\pi i f)^2}$
$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2+4\pi^2 f^2}$	$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\text{sgn}(t)$	$\frac{1}{i\pi f}$	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{2\pi i f}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\left\|\frac{1}{T}\right\|\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$		

Elementary Functions:

$$\text{rect}(t)=\begin{cases}1, & |t|<0.5\\ 0, & \text{elsewhere}\end{cases} \qquad \text{tri}(t)=\begin{cases}1-|t|, & |t|<1\\ 0, & \text{elsewhere}\end{cases}$$

$$\text{sgn}(t)=\begin{cases}-1, & t<0\\ 0, & t=0\\ 1, & t>0\end{cases} \qquad u(t)=\frac{1}{2}\left(1+\text{sgn}(t)\right)=\begin{cases}0, & x<0\\ 0.5, & x=0\\ 1, & x>0\end{cases}$$