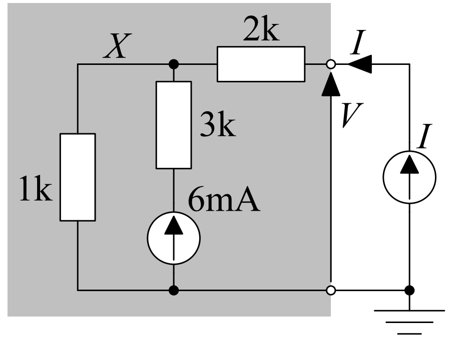
5: Thévenin and Norton Equivalents

Equivalent Networks

From linearity theorem: $V = aI + b$.



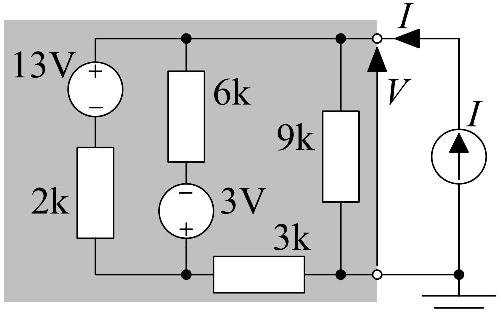
Use nodal analysis:

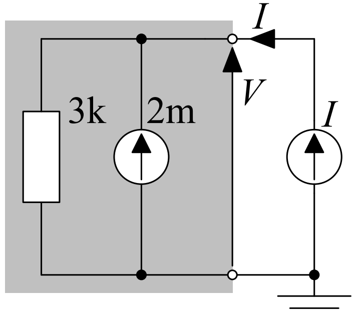
KCL@X: $\frac{X}{1} - 6 + \frac{X-V}{2} = 0$

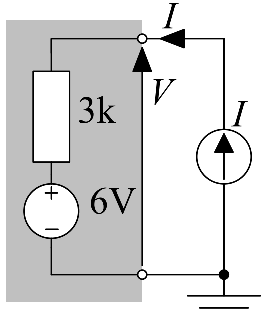
KCL@V: $\frac{V-X}{2} - I = 0$

Eliminating X gives: $V = 3I + 6$.

There are infinitely many networks with the same values of a and b:





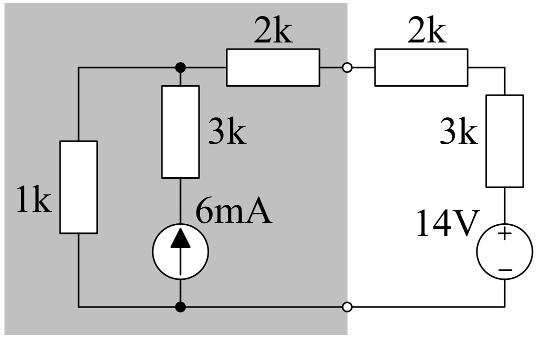


These four shaded networks are *equivalent* because the relationship between V and I is *exactly* the same in each case.

The last two are particularly simple and are respectively called the *Norton* and *Thévenin* equivalent networks.

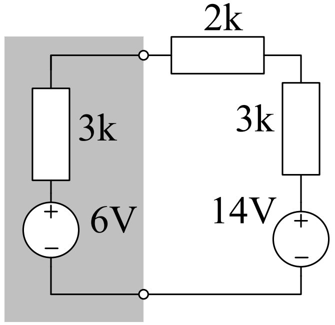
Thévenin Theorem: Any two-terminal network consisting of resistors, fixed voltage/current sources and linear dependent sources is externally equivalent to a circuit consisting of a resistor in series with a fixed voltage source.

We can replace the shaded part of the circuit with its Thévenin equivalent network.



The voltages and currents in the unshaded part of the circuit will be identical in both circuits.

The new components are called the *Thévenin equivalent resistance*, R_{Th}, and the *Thévenin equivalent voltage*, V_{Th}, of the original network.



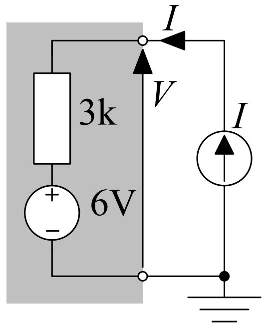
This is often a useful way to simplify a complicated circuit (provided that you do not want to know the voltages and currents in the shaded part).

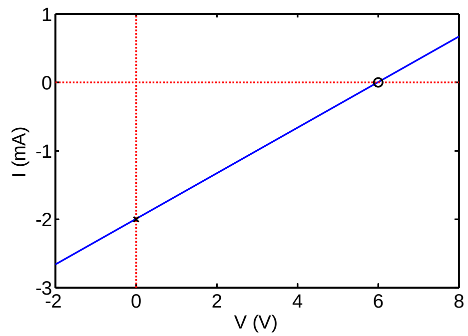
Thévenin Circuit Properties

A Thévenin equivalent circuit has a straight line characteristic with the equation:

$$V = R_{Th}I + V_{Th}$$

$$\Leftrightarrow I = \frac{1}{R_{Th}}V - \frac{V_{Th}}{R_{Th}}$$





Three important quantities are:

Open Circuit Voltage: If $I = 0$ then $V_{OC} = V_{Th}$. (X-intercept: o)

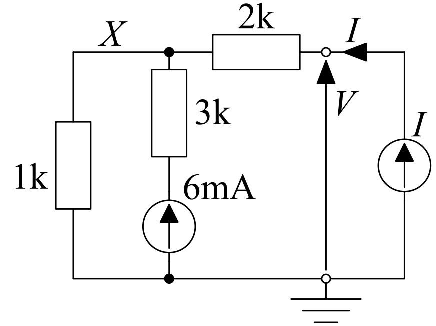
Short Circuit Current: If $V = 0$ then $I_{SC} = -\frac{V_{Th}}{R_{Th}}$ (Y-intercept: x)

Thévenin Resistance: The slope of the characteristic is $\frac{dI}{dV} = \frac{1}{R_{Th}}$.

If we know the value of any two of these three quantities, we can work out V_{Th} and R_{Th}.

In any two-terminal circuit with the same characteristic, the three quantities will have the same values. So if we can determine two of them, we can work out the Thévenin equivalent.

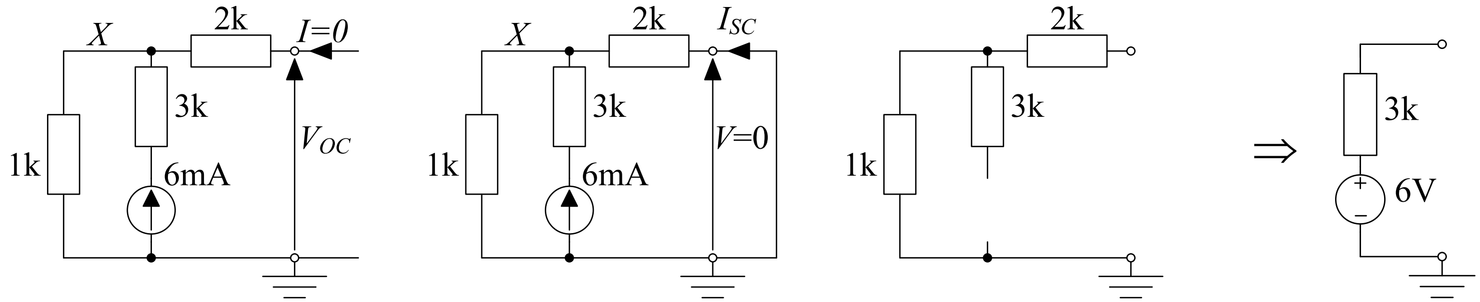
We need any two of the following:



Open Circuit Voltage: $V_{OC} = V_{Th} = 6\,\mathrm{V}$

Short Circuit Current: $I_{SC} = -\frac{V_{Th}}{R_{Th}} = -2\,\mathrm{mA}$

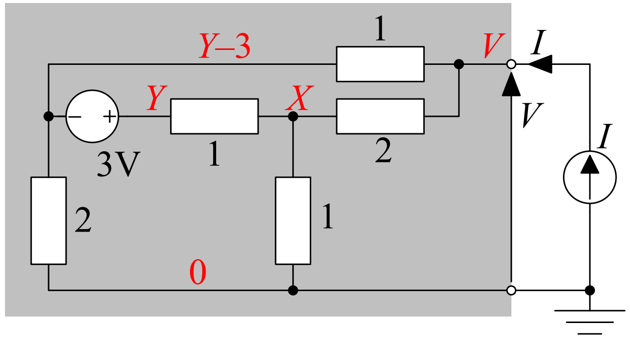
Thévenin Resistance: $\mathrm{R}_{Th} = 2\,\mathrm{k} + 1\,\mathrm{k} = 3\,\mathrm{k}\Omega$



Thévenin Resistance:
We set all the independent sources to zero (voltage sources → short circuit, current sources→ open circuit). Then we find the equivalent resistance between the two terminals.
The $3\,\mathrm{k}$ resistor has no effect so $R_{Th} = 2\,\mathrm{k} + 1\,\mathrm{k} = 3\,\mathrm{k}$.
Any measurement gives the same result on an equivalent circuit.

For a complicated circuit, you can use nodal analysis to find the Thévenin equivalent directly in the form:

$$V = V_{Th} + IR_{Th}.$$



Step 1: Label ground as an output terminal + label other nodes.

Step 2: Write down the equations (Y is a supernode)

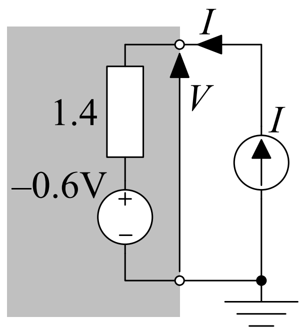
$$\frac{X-V}{2} + \frac{X}{1} + \frac{X-Y}{1} = 0$$

$$\frac{Y-3-V}{1} + \frac{Y-X}{1} + \frac{Y-3}{2} = 0$$

$$\frac{V-Y+3}{1} + \frac{V-X}{2} - I = 0$$

Step 3: Eliminate X and Y and solve for V in terms of I:

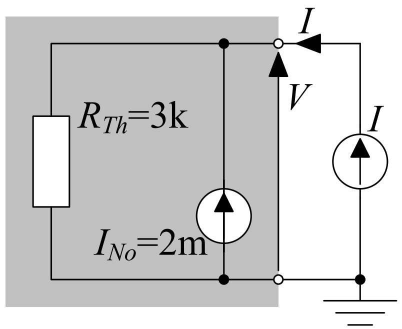
$$V = \frac{7}{5}I - \frac{3}{5} = R_{Th}I + V_{Th}$$

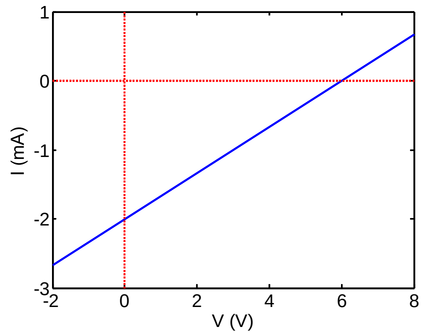


Norton Theorem: Any two-terminal network consisting of resistors, fixed voltage/current sources and linear dependent sources is externally equivalent to a circuit consisting of a resistor in parallel with a fixed current source.

KCL: $-I - I_{No} + \frac{V}{R_{Th}} = 0$
$\Leftrightarrow I = \frac{1}{R_{Th}} V - I_{No}$

c.f. Thévenin (slide 5-4):
Same R and $I_{No} = \frac{V_{Th}}{R_{Th}}$





Open Circuit Voltage: If $I = 0$ then $V_{OC} = I_{No} R_{Th}$.

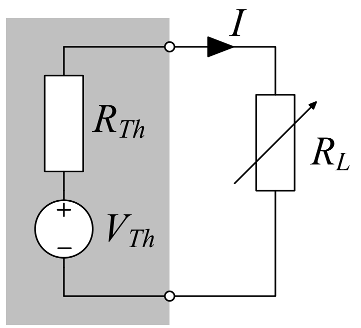
Short Circuit Current: If $V = 0$ then $I_{SC} = -I_{No}$

Thévenin Resistance: The slope of the characteristic is $\frac{1}{R_{Th}}$.

Easy to change between Norton and Thévenin: $V_{Th} = I_{No} R_{Th}$.
Usually best to use Thévenin for small R_{Th} and Norton for large R_{Th} compared to the other impedances in the circuit.

Suppose we connect a variable resistor, R_L, across a two-terminal network. From Thévenin's theorem, even a complicated network is equivalent to a voltage source and a resistor.

We know $I = \frac{V_{Th}}{R_{Th}+R_L}$



$\Rightarrow$ power in R_L is $P_L = I^2 R_L = \frac{V_{Th}^2 R_L}{(R_{Th}+R_L)^2}$

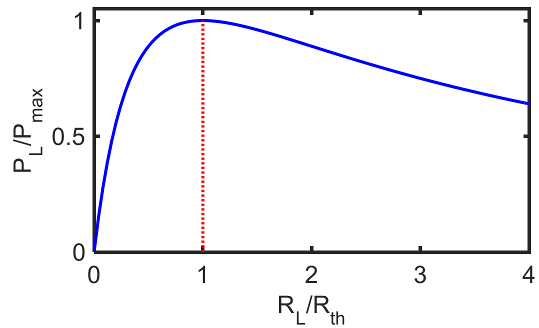
To find the R_L that maximizes P_L:

$$0 = \frac{dP_L}{dR_L} = \frac{(R_{Th}+R_L)^2 V_{Th}^2 - 2V_{Th}^2 R_L (R_{Th}+R_L)}{(R_{Th}+R_L)^4}$$

$$= \frac{V_{Th}^2 (R_{Th}+R_L) - 2V_{Th}^2 R_L}{(R_{Th}+R_L)^3}$$

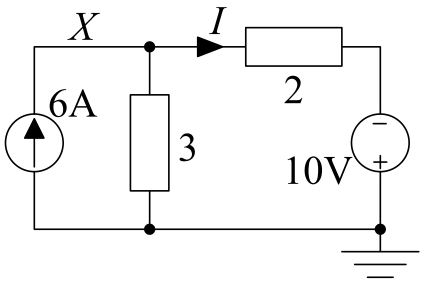
$$\Rightarrow V_{Th}^2 \left((R_{Th}+R_L) - 2R_L\right) = 0$$

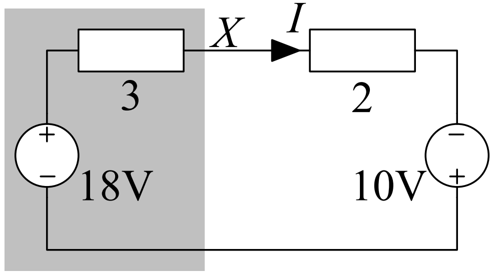
$$\Rightarrow R_L = R_{Th} \;\Rightarrow\; P_{(max)} = \frac{V_{Th}^2}{4R_{Th}}$$



For fixed R_{Th}, the maximum power transfer is when $R_L = R_{Th}$ ("*matched load*").

Sometimes changing between Thévenin and Norton can simplify a circuit. Suppose we want to calculate I.





Norton → Thévenin on current source: $I = \frac{18-(-10)}{5} = 5.6\text{ A}$

If you can't spot any clever tricks, you can always find out everything with nodal analysis.

$$-6 + \frac{X}{3} + \frac{X-(-10)}{2} = 0$$

$$\Rightarrow \quad 5X = 36 - 30 = 6$$

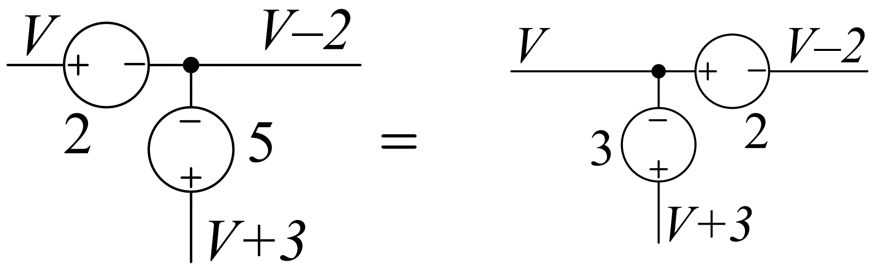
$$\Rightarrow \quad X = 1.2$$

$$\Rightarrow \quad I = \frac{X-(-10)}{2} = 5.6$$

If all but one branches connecting to a node are voltage sources or are current sources, you can choose any of the branches to be the sourceless one.

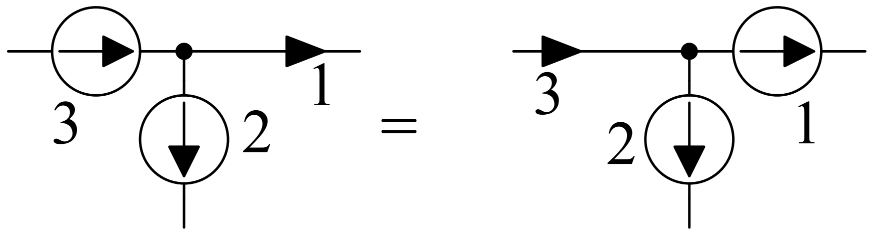
Voltage Sources:

We can use the left node as the reference



Current Sources:

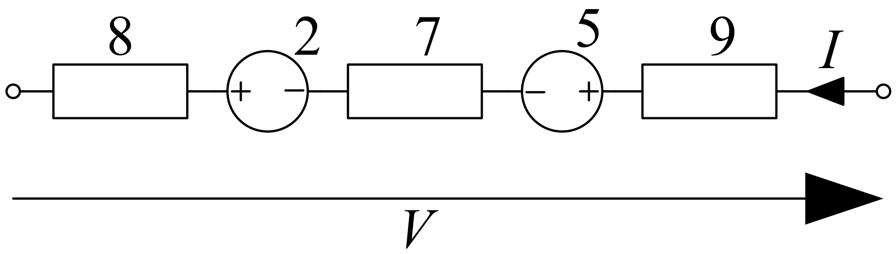
KCL gives current into rightmost node



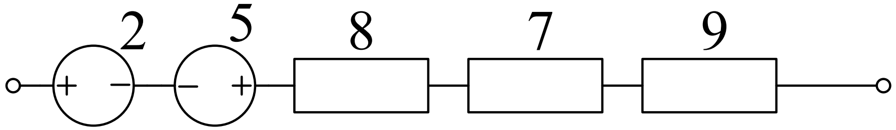
Series Rearrangement

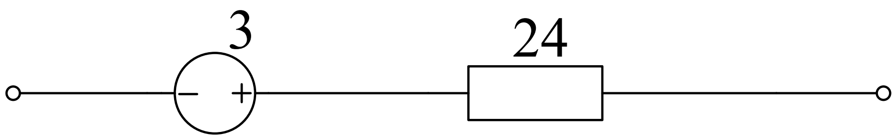
If we have any number of voltage sources and resistors in series we can calculate the total voltage across the chain as:

$$V = 8I - 2 + 7I + 5 + 9I = (-2+5) + (8+7+9)I$$
$$= 3 + 24I$$



We can arbitrarily rearrange the order of the components without affecting $V = 3 + 24I$.





If we move all the voltage sources together and all the resistors together we can merge them and then we get the Thévenin equivalent.

Summary

- Thévenin and Norton Equivalent Circuits
  - A network has Thévenin and Norton equivalents if:
    - only 2 terminals connect it to the outside world
    - it is made of resistors + sources + linear dependent sources
  - How to determine V_{Th}, I_{No} and R_{Th}
    - Method 1: Connect current source → Nodal analysis
    - Method 2: Find any two of:
      (a) $V_{OC} = V_{Th}$, the open-circuit voltage
      (b) $I_{SC} = -I_{No}$, the short-circuit current
      (c) R_{Th}, equivalent resistance with all sources set to zero
    - Related by Ohm's law: $V_{Th} = I_{No}R_{Th}$
- Load resistor for maximum power transfer $= R_{Th}$
- Source Transformation and Rearrangement

For further details see Hayt Ch 5 & A3 or Irwin Ch 5.