E1.1 Circuit Analysis

Problem Sheet 5 - Solutions

1. Each of the circuits may be viewed as a potential divider, so we can write down the transfer function without doing any nodal analysis. In two cases, one element of the potential divider consists of a parallel $R||C$ combination. This parallel combination has the impedance $\frac{1}{1/R+j\omega C} = \frac{R}{1+j\omega RC}$. Graphs of the magnitude responses are shown in Fig. 1(i)-(vi) together with their straight-line approximations.

   (i) $\frac{Y}{X} = \frac{R}{R+1/j\omega C} = \frac{j\omega RC}{1+j\omega RC} = \frac{j\omega/500}{1+j\omega/500}$ where $RC = 2\,\text{ms}$. LF asymptote is $0.002j\omega$; HF asymptote is 1. Denominator corner frequency is $\frac{1}{RC} = 500\,\text{rad/s}$. At the corner, the gain is $1 = 0\,\text{dB}$.

   (ii) $\frac{Y}{X} = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{1+j\omega RC} = \frac{1}{1+j\omega/500}$ where $RC = 2\,\text{ms}$. LF asymptote is 1; HF asymptote is $500\,(j\omega)^{-1}$. Denominator corner frequency is $\frac{1}{RC} = 500\,\text{rad/s}$. At the corner, the gain is $1 = 0\,\text{dB}$.

   (iii) $\frac{Y}{X} = \frac{j\omega L}{R+j\omega L} = \frac{j\omega L/R}{1+j\omega L/R}$ where $\frac{L}{R} = 100\,\mu\text{s}$. LF asymptote is $10^{-4}j\omega$; HF asymptote is 1. Denominator corner frequency is $\frac{R}{L} = 10^4\,\text{rad/s}$. At the corner, the gain is $1 = 0\,\text{dB}$.

   (iv) For convenience, we define $R = 1\,\text{k}$. Then $\frac{Y}{X} = \frac{R+j\omega L}{5R+j\omega L} = 0.2\frac{1+j\omega\frac{L}{R}}{1+j\omega\frac{L}{5R}}$ where $\frac{L}{R} = 1\,\mu\text{s}$. LF asymptote is 0.2; HF asymptote is 1. Numerator corner frequency is $\frac{R}{L} = 10\,\text{krad/s}$ with a gain at the corner of $0.2 = -14\,\text{dB}$. Denominator corner frequency is $\frac{5R}{L} = 50\,\text{krad/s}$ with a gain at the corner of $1 = 0\,\text{dB}$. As can be seen in Fig. 1(iv), the magnitude response turns up at $10\,\text{krad/s}$ and then flattens out again at $50\,\text{krad/s}$. In between these two frequencies the slope $\frac{\log|H|}{\log\omega} = +1$ or, equivalently, $+6\,\text{dB/octave}$ or $+20\,\text{dB/decade}$; all these are the same as saying that $|H| \propto \omega$; thus from $\omega = 10\,\text{krad/s}$ to $50\,\text{krad/s}$, the frequency increases by a factor of 5 and the gain also increases by a factor of 5.

   (v) For convenience, we define $R = 1\,\text{k}$. Then $\frac{Y}{X} = \frac{R}{2R+\frac{8R}{1+8j\omega RC}} = \frac{1+8j\omega RC}{10+16j\omega RC} = 0.1\frac{1+8j\omega RC}{1+1.6j\omega RC}$ where $RC = 100\,\mu\text{s}$. LF asymptote is 0.1; HF asymptote is 0.5. Numerator corner frequency is $\frac{1}{8RC} = 1250\,\text{rad/s}$ with a gain at the corner of $0.1 = -20\,\text{dB}$. Denominator corner frequency is $\frac{1}{1.6RC} = 6.25\,\text{krad/s}$ with a gain at the corner of $0.5 = -6\,\text{dB}$. As in the previous part, both the frequency and the gain change by a factor of 5 between the corner frequencies.

   (vi) For convenience, we define $R = 10\,\text{k}$. Then $\frac{Y}{X} = \frac{\frac{R}{1+j\omega RC}}{2R+\frac{1}{j\omega C}+\frac{R}{1+j\omega RC}} = \frac{j\omega RC}{2j\omega RC(1+j\omega RC)+(1+j\omega RC)+j\omega RC} = \frac{j\omega RC}{1+4j\omega RC+2(j\omega RC)^2}$ where $RC = 1\,\text{ms}$. LF asymptote is $0.001j\omega$; HF asymptote is $500\,(j\omega)^{-1}$. We can factorize the denominator to give $1+4j\omega RC+2\,(j\omega RC)^2 = \left(1+\frac{j\omega}{a}\right)\left(1+\frac{j\omega}{b}\right)$ where a and b are -1 times the roots of the quadratic equation $2R^2C^2x^2+4RCx+1$ or $\frac{1\pm\sqrt{0.5}}{RC}$. This gives denominator corner frequencies $a = 293$ and $b = 1707\,\text{rad/s}$. The gain in between these two frequencies can be obtained by substituting $\omega = a$ into the LF asymptote expression to give a value of $aRC = 1-\sqrt{0.5} = 0.293 = -10.7\,\text{dB}$.

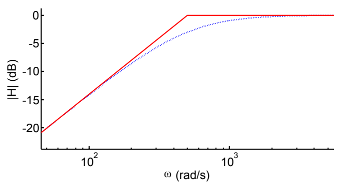

Fig. 1(i)

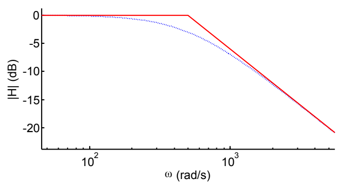

Fig. 1(ii)

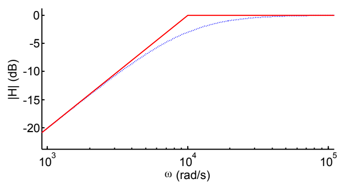

Fig. 1(iii)

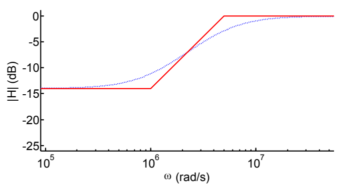

Fig. 1(iv)

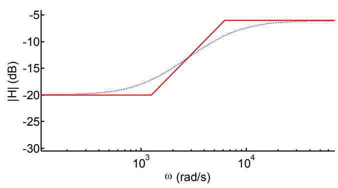

Fig. 1(v)

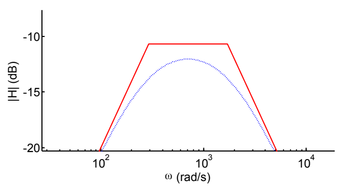

Fig. 1(vi)

2. For convenience, we define $R = 1\,\text{k}$. Then $\frac{Y}{X} = \frac{R}{2R+\frac{8R}{1+8j\omega RC}} = \frac{1+8j\omega RC}{10+16j\omega RC} = 0.1\frac{1+8j\omega RC}{1+1.6j\omega RC}$ where $RC = 100\,\mu\text{s}$. LF asymptote is 0.1; HF asymptote is 0.5; both of these are real and so have zero phase shift. The magnitude plot has a numerator corner frequency of $\frac{1}{8RC} = 1250\,\text{rad/s}$ and denominator corner frequency of $\frac{1}{1.6RC} = 6.25\,\text{krad/s}$. Each of these generates a pair of corner frequencies on the phase plot at $0.1\times$ and $10\times$ the frequency. Thus we have corners at $\omega = 125\,(+)$, $625\,(-)$, $12.5\,\text{k}\,(-)$, $62.5\,\text{k}\,(+)\,\text{rad/s}$ where the sign in parentheses indicates the gradient change $\pm\frac{\pi}{4}$ rad/decade. Between 125 and 625 rad/s the gradient is $\frac{\pi}{4}$ rad/decade so the phase will change by $\frac{\pi}{4} \times \log_{10}\frac{625}{125} = \frac{\pi}{4} \times 0.7 = +0.55\,\text{rad}$. This is therefore the phase shift for the flat part of the phase response. (see Fig. 2).

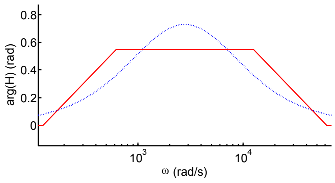

Fig. 2

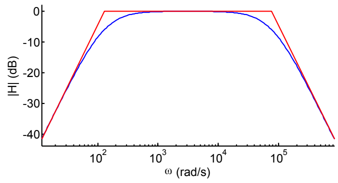

Fig. 3

3. At $\omega = 2000\pi = 6283.2$ we know $|H(j\omega)| = 1$. Hence $k = \frac{|j\omega+a|^2|j\omega+b|^2}{|j\omega|^2} = \frac{(\omega^2+a^2)(\omega^2+b^2)}{\omega^2} = \frac{3.9495\times10^7\times5.9155\times10^9}{3.9478\times10^7} = 5.918\times10^9$.

4. (i) This is the same low-pass filter as Fig. 1(ii) but with a corner frequency of 50 rad/s. So we want $RC = \frac{1}{50} = 20\,\text{ms}$. One possible choice is shown in Fig. 4(i).

(ii) This is the same high-pass filter as Fig. 1(i) but with a corner frequency of 1000 rad/s and a high frequency gain of $0.5 = -6$ dB. So we want $RC = \frac{1}{1000} = 1\,\text{ms}$. One possible choice is shown in Fig. 4(ii); the two resistors give the correct high frequency gain.

(iii) We want a circuit whose gain decreases from $\frac{1}{2}$ at low frequencies to $\frac{1}{8}$ at high frequencies. We can do this by using a capacitor to short out part of the vertical limb of the potential divider at high frequencies as shown in Fig. 4(iii). This design has a gain of $\frac{1}{8}$ when the capacitor is a short circuit; with the capacitor open circuit (low frequencies), we add in an additional $6R$ which gives a gain of $\frac{1}{2}$. The impedance of $6R||C$ is $\frac{R+\frac{6R}{1+6j\omega RC}}{8R+\frac{6R}{1+6j\omega RC}} = \frac{7+6j\omega RC}{14+48j\omega RC}$ which, as a check, we see has the correct LF and HF asymptotes. The numerator corner frequency is at $\omega = \frac{7}{6RC}$ which needs to be at 1000 rad/s. From this, $RC = 1.17\,\text{ms}$ so one possible set of value is $C = 100\,\text{nF}$ and $R = 12\,\text{k}\Omega$.

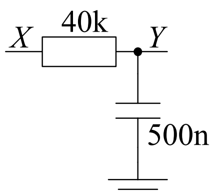

Fig. 4(i)

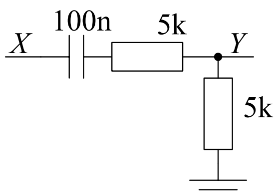

Fig. 4(ii)

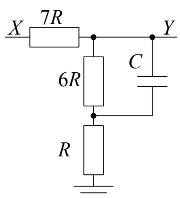

Fig. 4

5. (a) $\frac{-2\omega^2-2j\omega^3}{1-2\omega^2+\omega^4} = \frac{2(j\omega)^2(j\omega+1)}{((j\omega)^2+1)((j\omega)^2+1)}$.

(b) $\frac{-2(1+\omega^2)}{(1-\omega^2)+2j\omega} = \frac{2(j\omega+1)(j\omega-1)}{(j\omega+1)(j\omega+1)} = \frac{2(j\omega-1)}{(j\omega+1)}$

(c) $\frac{10(j\omega)^2+2j\omega+10}{(j\omega)^2+2j\omega+1} = \frac{10((j\omega)^2+0.2j\omega+1)}{(j\omega+1)(j\omega+1)}$

(d) $\frac{1}{j\omega+6(j\omega)^{-1}+5} = \frac{j\omega}{(j\omega)^2+5j\omega+6} = \frac{j\omega}{(j\omega+2)(j\omega+3)}$

6. To find the low frequency asymptote, you take the lowest power of $j\omega$ in each of the numerator and denominator factors and multiply them together. Likewise, for the high frequency asymptote, you take the highest power of $j\omega$ in each of the factors. There is no need (or indeed advantage) to do any factorization or to multiply out existing factors.

   (a) $H_{\text{LF}} = \frac{-2\omega^2}{1} = 2(j\omega)^2$, $H_{\text{HF}} = \frac{-2j\omega^3}{\omega^4} = 2(j\omega)^{-1}$

   (b) $H_{\text{LF}} = \frac{3}{1} = 3$, $H_{\text{HF}} = \frac{2(j\omega)^3}{4(j\omega)^4} = 0.5(j\omega)^{-1}$

   (c) $H_{\text{LF}} = \frac{j\omega\times3\times3}{2\times1\times5} = 0.9j\omega$, $H_{\text{HF}} = \frac{j\omega\times2(j\omega)^6\times5(j\omega)^3}{2\times(j\omega)^5\times(j\omega)^5} = 5$

   (d) $H_{\text{LF}} = \frac{12}{6(j\omega)^{-1}} = 2j\omega$, $H_{\text{HF}} = \frac{12}{j\omega} = 12(j\omega)^{-1}$

7. We must have $\alpha \geq 0$ because, if α were negative, $(j\omega)^{\alpha}$ would increase without limit as $\omega \to 0$ and the transfer function would exceed G at some point. Similarly, we must have $\beta \leq 0$ because otherwise $(j\omega)^{\beta}$would increase without limit as $\omega \to \infty$. A consequence of this is that the order of the numerator can never exceed that of the denominator in a transfer function whose magnitude is bounded.

8. Graphs of the transfer functions are shown in Fig. 8(a)-(c).

   (a) We have a LF asymptote of $5 = 14\,\text{dB}$. We have corner frequencies at $\omega = 100\,(-)$, $500\,(+)$, $2000\,(-)$ where the sign in parentheses indicates the polarity of gradient change. To estimate the gain at $\omega = 1000$, we assume that a factor $\left|1 + \frac{j\omega}{a}\right|$ is equal to 1 if $\omega < a$ or else $\frac{\omega}{a}$ if $\omega > a$. This gives $|H(1000j)| \simeq \left|\frac{5(\omega/500)}{(\omega/100)(1)}\right| = 1 = 0\,\text{dB}$.

   (b) We have a LF asymptote of $2 = 6\,\text{dB}$. We have corner frequencies at $\omega = 100\,(-)$, $5000\,(+)$. Using the same technique as in part (a), $|H(1000j)| \simeq \left|\frac{2(1)}{(\omega/100)}\right| = \left|\frac{200}{1000}\right| = 0.2 = -14\,\text{dB}$.

   (c) We have corner frequencies at $\omega = 100\,(-)$, $500\,(+)$, $2000\,(-)$, $5000\,(-)$. We have a LF asymptote of $3j\omega = 6\,\text{dB}$ which at the first corner ($\omega = 100$) is $300j = 50\,\text{dB}$. Using the same technique as in part (a), $|H(1000j)| \simeq \left|\frac{3\times\omega(\omega/500)}{(\omega/100)(1)(1)}\right| = \left|\frac{3000}{5}\right| = 600 = 55.6\,\text{dB}$. To obtain this expression from the transfer function, any term whose corner frequency is $> \omega$ has been replaced by (1).

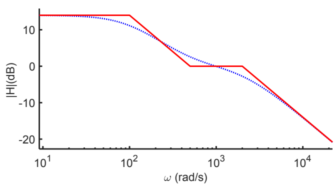

Fig. 8(a)

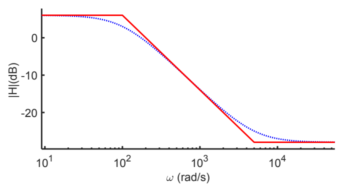

Fig. 8(b)

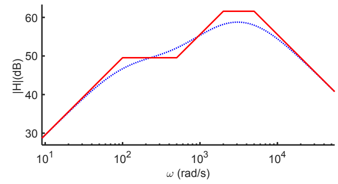

Fig. 8(c)

9. The corner frequency is $p = \frac{1}{\zeta RC} = 2\pi \times 1000$. Rearranging this gives $R = \frac{1}{2000\pi\zeta C} = 22508\,\Omega$. The upper resistor therefore has a value $\zeta^2 R = 0.5R = 11254\,\Omega$. The complete circuit is shown in Fig. 9. At $\omega = 100\,\text{Hz}$, 1 kHz and 10 kHz the value of $\frac{j\omega}{p} = 0.1j$, j and $10j$ respectively. From the question, the transfer function is given by $\frac{\left(\frac{j\omega}{p}\right)^2}{\left(\frac{j\omega}{p}\right)^2 + 2\zeta\left(\frac{j\omega}{p}\right) + 1}$. Substituting for $\frac{j\omega}{p}$ and $\zeta = \sqrt{0.5}$, this equals $\frac{-0.01}{-0.01+0.1414j+1} = -0.0099 + 0.0014j = 0.01\angle 172°$, $\frac{-1}{-1+1.414j+1} = 0.707j = 0.707\angle 90°$ and $\frac{-100}{-100+14.14j+1} = 0.9899 + 0.1414j = 1\angle 8°$ respectively.

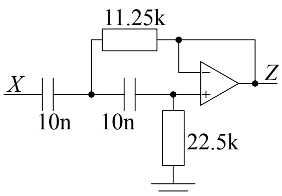

Fig. 9

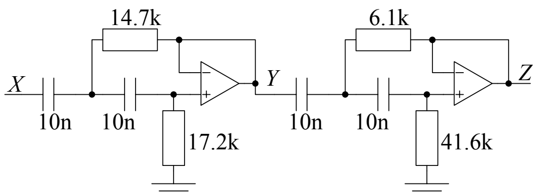

Fig. 10(a)

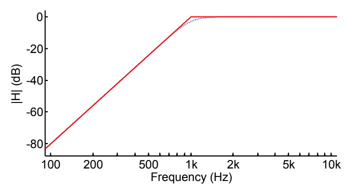

Fig. 10(b)

10. For a 4th order filter, we need $N = 2$ and from the formula given in the question, we use $\zeta_1 = \cos\left(\frac{\pi}{8}\right) = 0.924$ and $\zeta_2 = \cos\left(\frac{3\pi}{8}\right) = 0.383$. If we stick to $C = 10$ nF as in Q9, we obtain $R_1 = \frac{1}{2000\pi\zeta_1 C} = 17.2\,\text{k}\Omega$ and $R_2 = \frac{1}{2000\pi\zeta_2 C} = 41.6\,\text{k}\Omega$ with $\zeta_1^2 R_1 = 14.7\,\text{k}\Omega$ and $\zeta_2^2 R_2 = 6.1\,\text{k}\Omega$. This gives the circuit shown in Fig. 10(a). The transfer function is $H(j\omega) = \frac{\left(\frac{j\omega}{p}\right)^4}{\left(\left(\frac{j\omega}{P}\right)^2+2\zeta_1\left(\frac{j\omega}{p}\right)+1\right)\left(\left(\frac{j\omega}{P}\right)^2+2\zeta_2\left(\frac{j\omega}{p}\right)+1\right)}$; this is plotted in Fig. 10(b).

11. To shift the frequency response up by a factor of 5, we need to divide the value of each C or L component by 5. This gives the circuit of Fig. 11(a). We could also, if we wanted, multiply all the capacitor values by k and divide all the resistor values by k for any scale factor k without changing the transfer function. For this particular circuit, it would be a bad idea to use a value of $k > 1$ because, at $3\,\text{k}\Omega$ the feedback resistor is already a little on the low side for many op-amps (which have a limited current output capability).

    To reflect the magnitude response in the line $\omega_m = 10000$, we need to convert resistors into capacitors and vice-versa. From the notes, the formulae are: $R' = \frac{k}{\omega_m C}$, $C' = \frac{1}{\omega_m k R}$. For the circuit of Fig. 11(b), I have chosen $k = 3.33$ in order to get reasonable component values but other choices are also possible. A full analysis of this low-pass filter circuit is the subject of question 15.

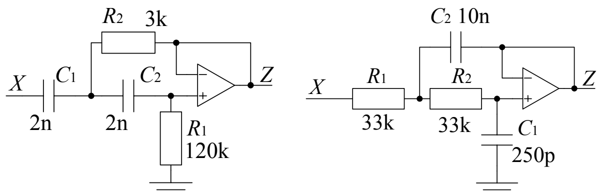

Fig. 11(a) Fig. 11(b)

12. (a) The parallel combination of $C||(R+L)$ has an impedance $Z = \frac{\frac{1}{j\omega C}(R+j\omega L)}{\frac{1}{j\omega C}+R+j\omega L} = \frac{R+j\omega L}{1+j\omega RC+(j\omega)^2 LC}$. We want to find the value of ω that makes this real. The easiest way to do this is to insist that the ratio of imaginary to real part is the same for the numerator and denominator (this implies that they have the same argument). Thus $\frac{\omega_r L}{R} = \frac{\omega_r RC}{1-\omega_r^2 LC}$ from which cross multiplying (after dividing both numerators by ω_r) gives $L - \omega_r^2 L^2 C = R^2 C$ from which $\omega_r = \sqrt{\frac{L-R^2C}{L^2C}} = 9\,950$ rad/s. Note that this is close, but not exactly equal to, $\omega_0 = 10\,000$ where the capacitor and inductor impedances have the same magnitude. The value of Z at resonance can now be found as the ratio between the real (or equivalently the imaginary) parts of the numerator and denominator of the previous expression. Thus $Z = \frac{R+j\omega L}{1+j\omega RC+(j\omega)^2 LC} = \frac{R}{1-\omega^2 LC} = \frac{j\omega L}{j\omega RC} = 1000$.

    (b) By definition Q equals ω_r times the average stored energy divided by the average power loss. If the input voltage phasor is V, then the peak energy stored in the capacitor is $\frac{1}{2}C|V|^2$ and its average stored energy is half this, namely $\frac{1}{4}C|V|^2$. The current through the resistor is $I_R = \frac{V}{R+j\omega_r L}$. The peak energy stored in the inductor is $\frac{1}{2}L|I_R|^2 = \frac{1}{2}L\frac{|V|^2}{R^2+\omega_r^2 L^2} = \frac{1}{2}L\frac{|V|^2}{R^2+\frac{L-R^2C}{L^2C}L^2} = \frac{1}{2}L\frac{C|V|^2}{R^2C+L-R^2C} = \frac{1}{2}C|V|^2$ which is the same as the peak capacitor energy; likewise, the average energy stored in the inductor is $\frac{1}{4}C|V|^2$. The average power loss in the resistor is $\frac{1}{2}R|I_R|^2 = \frac{RC}{2L}|V|^2$. Calculating Q from its definition gives $Q = \omega_r \frac{\frac{1}{4}C|V|^2+\frac{1}{4}C|V|^2}{\frac{RC}{2L}|V|^2} = \frac{\omega_r L}{R} = 9.95$. Since the capacitor and inductor store the same amount of energy on average, the Q can be determined more simply as $Q = \omega_r \frac{\frac{1}{2}L|I_R|^2}{\frac{1}{2}R|I_R|^2} = \frac{\omega_r L}{R} = 9.95$.

    (c) Note that the capacitor is unchanged in the two networks, so we can ignore it when matching their impedances. When choosing components to make two networks have the same impedance, your have a choice: you can either match their impedances or their admittances. You get the same answer in either case, but the algebra can sometimes be much simpler in one case than the other. In this question, it is easiest to use admittances because the components whose values are unknown are in parallel and so their admittances add: the total admittance of R_P and L_P in parallel is $\frac{1}{R_P} - \frac{j}{\omega_r L_P}$ and R_P and L_P remain unentangled in this expression. The admittance of $R_S + L_S$ is

$\frac{1}{R_S+j\omega_r L_S} = \frac{R_S-j\omega_r L_S}{R_S^2+\omega_r^2 L_S^2} = \frac{1}{R_P} - \frac{j}{\omega_r L_P}$. Equating the real and imaginary parts of this equation gives, $R_P = \frac{R_S^2+\omega_r^2 L_S^2}{R_S} = 1\,\text{k}\Omega$ and $L_P = \frac{R_S^2+\omega_r^2 L_S^2}{\omega_r^2 L_S} = L_S + \frac{R^2}{\omega_r^2 L_S} = 10.1\,\text{mH}$.

13. (a) At $\omega = 10\,000$, $Z_L = 100j$ and $Z_C = -100j$. Therefore the currents in L and C are equal and opposite. So the peak power supplied by V is the peak power absorbed by the resistor which equals $\frac{|V|^2}{R} = 100\,\text{mW}$.

    (b) The energy stored in the capacitor at time t is $W_C = \frac{1}{2}Cv(t)^2$. So the power absorbed by the capacitor is $\frac{dW_C}{dt} = Cv\frac{dv}{dt}$. Since you are told that the phasor $V = 10$, you know that the waveform $v(t) = 10\cos(\omega t)$ and, differentiating gives $\frac{dv}{dt} = -10\omega\sin(\omega t)$. Multiplying everything out gives $Cv\frac{dv}{dt} = -100\omega C\cos(\omega t)\sin(\omega t) = -50\omega C\sin(2\omega t)$. This has a peak value of $50\omega C = 500\,\text{mW}$. As is common in resonant circuits, this is 5 times greater than the answer to part (a).

14. (i) This is an inverting amplifier: $\frac{Y}{X} = -\frac{Z_F}{R} = -\frac{1}{R} \times \frac{2R}{1+2j\omega RC} = -\frac{2}{1+2j\omega RC}$.

(ii) This is a non-inverting amplifier: $\frac{Y}{X} = 1 + \frac{Z_F}{R} = 1 + \frac{2}{1+2j\omega RC} = \frac{3+2j\omega RC}{1+2j\omega RC}$.

(iii) This is the same as the previous circuit, but with an additional CR circuit at the input. $\frac{Y}{X} = \frac{4j\omega RC}{1+4j\omega RC} \times \frac{3+2j\omega RC}{1+2j\omega RC}$. This has corner frequencies at $\omega RC = \frac{1}{4}\,(-),\ \frac{1}{2}\,(-),\ \frac{3}{2}\,(+)$.

(iv) The circuit has negative feedback so we can assume $V_+ = V_- = 0$. KCL @ V_- gives: $\frac{0-X}{R} + \frac{0-Y}{R} + \frac{0-Z}{R} = 0$ from which $-Z = X + Y$. Now KCL @ Z gives: $(Z-0)j\omega C + \frac{Z-Y}{R} + \frac{Z}{R} = 0$ from which $Y - Z\,(2 + j\omega RC) = 0$. Substituting $-Z = X + Y$ gives $Y + (X+Y)\,(2 + j\omega RC) = 0$ from which $\frac{Y}{X} = -\frac{2+j\omega RC}{3+j\omega RC}$.

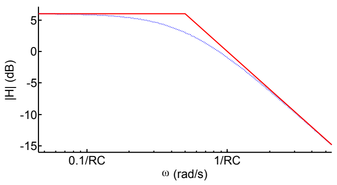

Fig. 14(i)

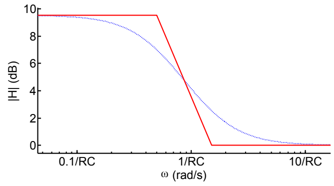

Fig. 14(ii)

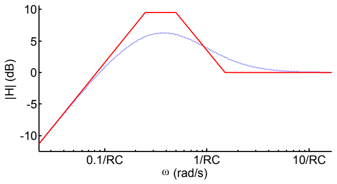

Fig. 14(iii)

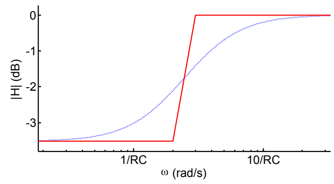

Fig. 14(iv)

15. In this circuit, the output, Y, is fed back to both V_+ and V_- so it is not immediately obvious that the overall feedback is negative. However, we see that $V_- = Y$ whereas $|V_+|$ will be attenuated by the network and will be $< Y$, so all is well. We can therefore assume that $Z = V_+ = V_- = Y$.

(a) $\frac{Z}{W}$ is just a potential divider, so $\frac{Z}{W} = \frac{Y}{W} = \frac{\frac{1}{j\omega C_1}}{R_2 + \frac{1}{j\omega C_1}} = \frac{1}{1+j\omega R_2 C_1}$. $Y = Z$ as noted above. From this we get $W = Y\,(1 + j\omega R_2 C_1)$.

(b) KCL @ W gives: $\frac{W-X}{R_1} + \frac{W-Z}{R_2} + (W-Y)\,j\omega C_2 = 0$ from which (substituting $Z = Y$),

$W\,(R_1 + R_2 + j\omega R_1 R_2 C_2) - Y\,(R_1 + j\omega R_1 R_2 C_2) - X R_2 = 0$.

Substituting the expression for W above gives

$Y\,(1 + j\omega R_2 C_1)\,(R_1 + R_2 + j\omega R_1 R_2 C_2) - Y\,(R_1 + j\omega R_1 R_2 C_2) = X R_2$

from which $Y\left(R_2 + j\omega R_2\,(R_1 + R_2)\,C_1 + (j\omega)^2\,R_1 R_2^2 C_1 C_2\right) = X R_2$.

Hence $\frac{Y}{X}\,(j\omega) = \frac{1}{R_1 R_2 C_1 C_2 (j\omega)^2 + (R_1+R_2) C_1 j\omega + 1}$.

(c) Squaring the expression for ζ gives $\zeta^2 = \frac{p^2 (R_1+R_2)^2 C_1^2}{4} = \frac{(R_1+R_2)^2 C_1^2}{4 R_1 R_2 C_1 C_2}$ which gives $\frac{4 R_1 R_2}{(R_1+R_2)^2} = \frac{C_1}{\zeta^2 C_2}$. Using quite a common algebraic trick, we can write the numerator as the difference of two squares:

$$\frac{4R_1R_2}{(R_1+R_2)^2} = \frac{(R_1+R_2)^2 - (R_1-R_2)^2}{(R_1+R_2)^2} = 1 - \left(\frac{R_1-R_2}{R_1+R_2}\right)^2 = 1 - \left(\frac{2R_1}{R_1+R_2} - 1\right)^2 = 1 - \left(\frac{2}{1+\frac{R_2}{R_1}} - 1\right)^2.$$

Rearranging $\frac{C_1}{\zeta^2 C_2} = 1 - \left(\frac{2}{1+\frac{R_2}{R_1}} - 1\right)^2$ gives $\frac{2}{1+\frac{R_2}{R_1}} = 1 + \sqrt{1 - \frac{C_1}{\zeta^2 C_2}}$ from which

$\left(1 + \frac{R_2}{R_1}\right)\left(1 + \sqrt{1 - \frac{C_1}{\zeta^2 C_2}}\right) = 2$.

The usefulness of this relationship is that it allows you to determine the resistor ratio, $\frac{R_2}{R_1}$, if you know the capacitor ratio $\frac{C_2}{C_1}$. For the square root to be a real number, we must have $1 - \frac{C_1}{\zeta^2 C_2} \ge 0$ which implies $C_1 \le \zeta^2 C_2$.

(d) We must have $\frac{C_2}{C_1} \ge \frac{1}{\zeta^2} = 4$. Given our restricted choice of capacitor value, we must therefore choose $C_2 = 47$ nF and $C_1 = 10$ nF. So, substituting $\frac{C_1}{\zeta^2 C_2} = 0.851$ into the expression from the previous part, we find $\left(1+\frac{R_2}{R_1}\right)\times 1.386 = 2$ from which $\frac{R_2}{R_1} = 0.443$. From the expression for p^2, we can write $0.443R_1^2 = R_1R_2 = \frac{1}{p^2C_1C_2} = 53.9\times 10^6$. Hence $R_1 = \sqrt{\frac{53.9\times 10^6}{0.443}} = 11\,\text{k}\Omega$ and $R_2 = 0.443R_1 = 4.9\,\text{k}\Omega$.

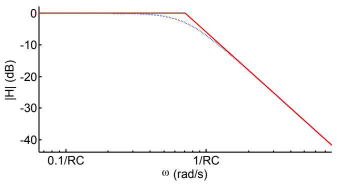

Fig. 15(mag)

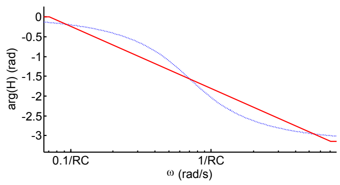

Fig. 15(phase)

16. (a) This circuit is a potential divider, so (setting $R = 20$) we can write down the transfer function: $\frac{Y}{X} = \frac{4R}{5R+j\omega L+\frac{1}{j\omega C}} = \frac{4j\omega RC}{1+5j\omega RC+(j\omega)^2LC} = \frac{2\zeta\left(\frac{j\omega}{a}\right)}{1+2\zeta\left(\frac{j\omega}{a}\right)+\left(\frac{j\omega}{a}\right)^2}$ where $a = \sqrt{\frac{1}{LC}} = 5000$ and $\zeta = 2.5aRC = 0.1$.

(b) To find the maximum of $\left|\frac{Y}{X}\right|$ it is easiest to find instead the maximum of $\left|\frac{Y}{X}\right|^2 = \frac{Y\times Y^*}{X\times X^*}$ where the * denotes the complex conjugate. Note that (i) a number multiplied by its complex conjugate is just the sum of the squares of its real and imaginary parts and that (ii) the magnitude of a complex fraction is the magnitude of the numerator divided by the magnitude of the denominator; very rarely is it necessary to multiply the top and bottom of a fraction by the complex conjugate of the denominator.

The difficult way to find the maximum is to differentiate the expression $\left|\frac{Y}{X}\right|^2 = \left|\frac{4j\omega RC}{1+5j\omega RC+(j\omega)^2LC}\right|^2 = \frac{(4\omega RC)^2}{(1-\omega^2LC)^2+(5\omega RC)^2}$ and set the derivative to zero. Much easier is to take the first expression above: $\left|\frac{Y}{X}\right|^2 = \left|\frac{4R}{5R+j\omega L+\frac{1}{j\omega C}}\right|^2 = \frac{16R^2}{25R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$. This is clearly maximized by making $\left(\omega L-\frac{1}{\omega C}\right) = 0$ which means $\omega_0 = \sqrt{\frac{1}{LC}}$. At this frequency $\frac{Y}{X} = 0.8 = -1.9\,\text{dB}$.

(c) The 3dB bandwidth is when $\left|\frac{Y}{X}\right|^2$ has fallen by a factor of 2. This will happen when $\left(\omega L-\frac{1}{\omega C}\right)^2 = 25R^2$ or $\omega L-\frac{1}{\omega C} = \pm 5R$. So we need to solve the quadratic equation $LC\omega^2 \pm 5RC\omega - 1 = 0$. The solution is $\omega = \frac{\pm 5RC\pm\sqrt{25R^2C^2+4LC}}{2LC}$ of which the positive solutions are $\omega = \frac{\pm 5RC+\sqrt{25R^2C^2+4LC}}{2LC}$. This gives $\omega_{3\text{dB}} = \{4525,\ 5525\}$. The bandwidth is the difference between these which is $\frac{10RC}{LC} = 1000\,\text{rad/s}$. Notice that ω_0 is the geometric mean of the two 3dB frequencies but is not the arithmetic mean which is 5025 rad/s. The Q (quality factor) of the resonance is $Q = \frac{1}{2\zeta} = 5$. This also equals the ratio of $\omega_0$to the bandwidth and the height of the peak above the intersection of the asymptotes.

The circles in Fig. 16 indicate ω_0 and the two 3dB frequencies.

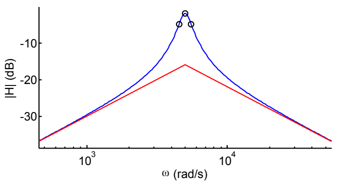

Fig. 16(mag)

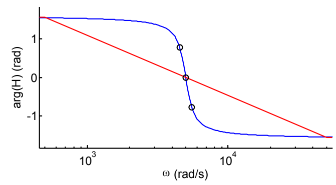

Fig. 16(phase)