E1.1 Analysis of Circuits

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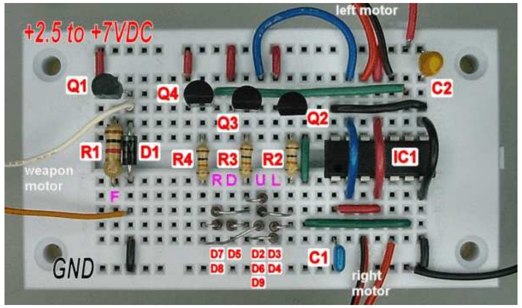
1: Introduction

Organization

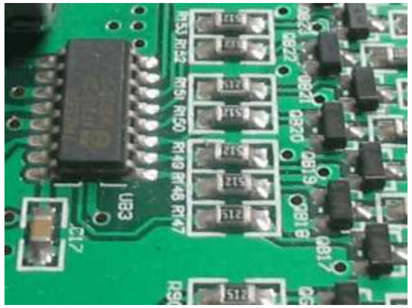
- 18 lectures: feel free to ask questions
- Buy the textbook: Hayt, Kemmerly & Durbin "Engineering Circuit Analysis" ISBN: 0071217066 (£44) or Irwin, Nelms & Patnaik "Engineering Circuit Analysis" ISBN: 1118960637 (£37)
- Weekly study group: Problem sheets - KEEP UP TO DATE
- Fortnightly tutorial: tutorial problems
- Lecture slides (including animations) and problem sheets + answers available via Blackboard or from my website: http://www.ee.ic.ac.uk/hp/staff/dmb/courses/ccts1/ccts1.htm
  - Quite dense: you should understand every word
- Email me with any errors or confusions in slides or problems/answers
- Christmas Test in January
- Exam in June (sample papers + solutions available via Blackboard)

What are circuits?

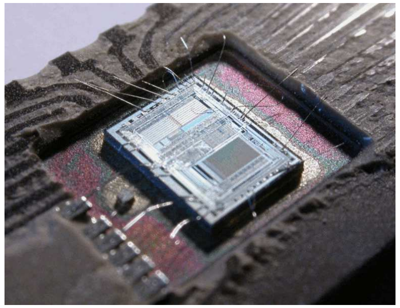
- A *circuit* consists of electrical or electronic components interconnected with metal wires
- Every electrical or electronic device is a circuit



Breadboard



Printed



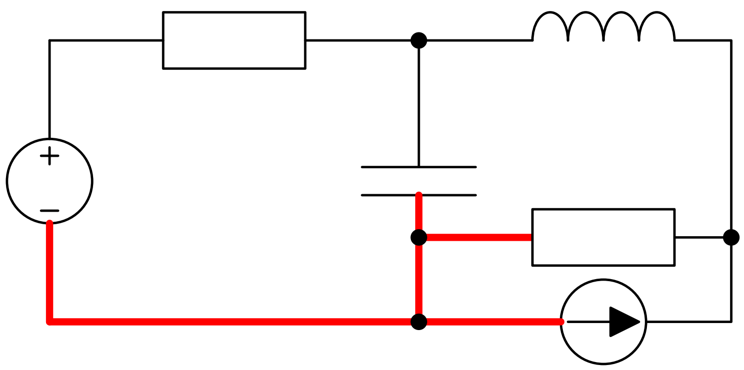
Integrated

- The function of the circuit is determined by which components are used and how they are interconnected: the physical positioning of the components usually has hardly any effect.

Circuit Diagrams

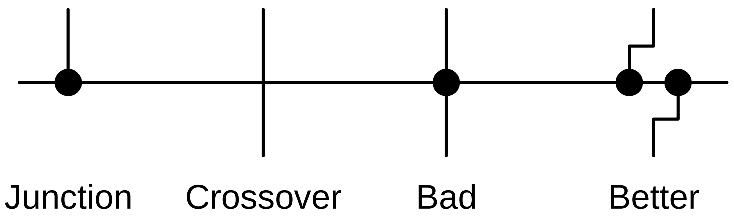
A *circuit diagram* shows the way in which the components are connected

- Each component has a special symbol
- The interconnecting wires are shown as lines



A *node* in a circuit is all the points that are connected together via the interconnecting wires. One of the four nodes in the diagram is coloured red.

Assumption: Interconnecting wires have zero resistance so everywhere along a node has the same voltage.



Indicate three meeting wires with a • and crossovers without one.

Avoid having four meeting wires in case the • disappears; stagger the wires instead.

Charge

Charge is an electrical property possessed by some atomic particles
Charge is measured in Colombs (abbreviated C)
An electron has a charge $-1.6 \times 10^{-19}\mathrm{C}$, a proton $+1.6 \times 10^{-19}\mathrm{C}$
Unlike charges attract, like charges repel: the force is fantastically huge

Two people $384,000\,\mathrm{km}$ apart
Each with 1% extra electrons

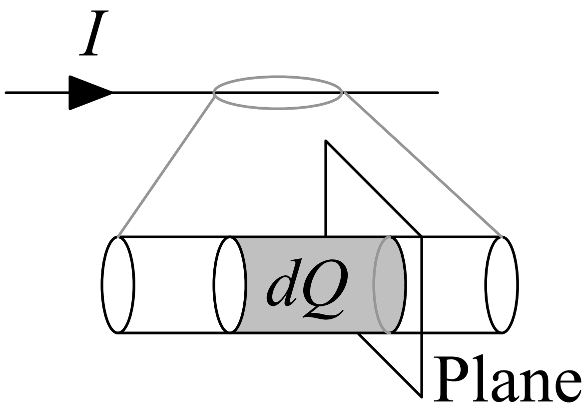
Force $= 2 \times 10^8\mathrm{N}$
$= 20,000\,\mathrm{tonne-force}$
$= 360,000\times$ their weight

Consequence: Charge never accumulates in a conductor: everywhere in a conducting path stays electrically neutral at all times.

Current

Current is the flow of charged particles past a measurement boundary
Using an ammeter, we measure current in Ampères (usually abbreviated to Amps or A): $1\text{ A} = 1\text{ C/s}$
Analogy: the flow of water in a pipe or river is measured in litres per second

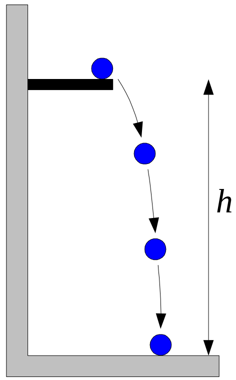
The arrow in a circuit diagram indicates the direction we choose to measure the current.
$I = +1\text{ A} \Rightarrow 1\text{ C}$ of $+$ve charge passes each point every second in the direction of the arrow (or else 1 C of –ve charge in the opposite direction)



$I = -1\text{ A} \Rightarrow 1\text{ C}$ of +ve charge in the direction opposite to the arrow

- Average electron velocity is surprisingly slow (e.g. 1 mm/s) but (like a water pipe) the signal travels much faster.
- In metals the charge carriers (electrons) are actually –ve: in this course you should ignore this always.

When a ball falls from a shelf, it loses potential energy of mgh or, equivalently, gh per kg.



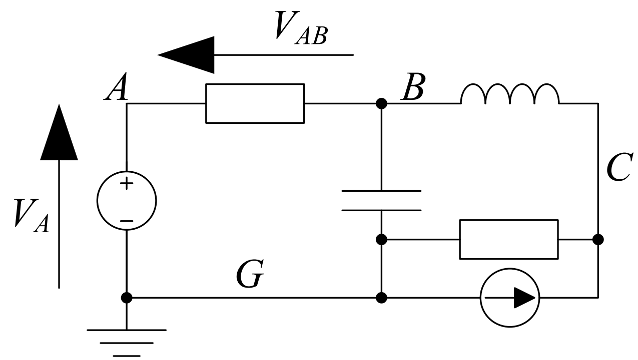
The potential energy per kg of any point on a mountain range is equal to gh where h is measured relative to an equipotential reference surface (e.g. the surface of a lake).

The potential energy difference between any two points is the energy needed to move 1 kg from one point to the other.
The potential energy difference does not depend on the route taken between the points.
The potential enegy difference does not depend on your choice of reference surface (e.g. lake surface or sea level).

The *electrical potential difference* (or *voltage difference*) between any two nodes in a circuit is the energy per coulomb needed to move a small +ve charge from one node to the the other.
We usually pick one of the nodes as a reference and define the *voltage at a node* to be the voltage difference between that node and the reference.

The four nodes are labelled A, B, C, G.

We have chosen G as the reference node; indicated by the "ground" symbol.

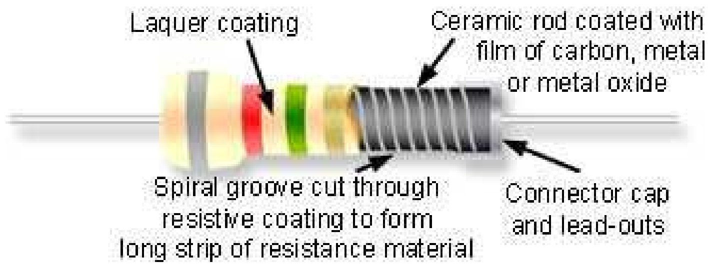


The potential difference between A and the ground reference, G, is written V_A and is also called "the voltage at A".
The potential difference between A and B is written as V_{AB} and shown as an arrow pointing towards A. This is the energy per coulomb in going from B to A and satisfies $V_{AB} = V_A - V_B$. (Different from vectors)

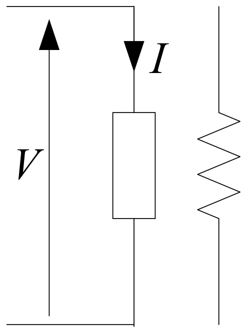
Easy algebra shows that $V_{AB} = -V_{BA}$ and that $V_{AC} = V_{AB} + V_{BC}$.

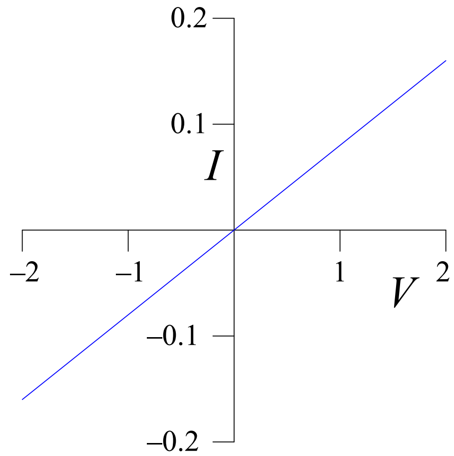
Resistors

A *resistor* is made from a thin strip of metal film deposited onto an insulating ceramic base.



The *characteristic* of a component is a graph showing how the voltage and current are related. We always choose the current and voltage arrows in opposite directions: this is *the passive sign convention*.



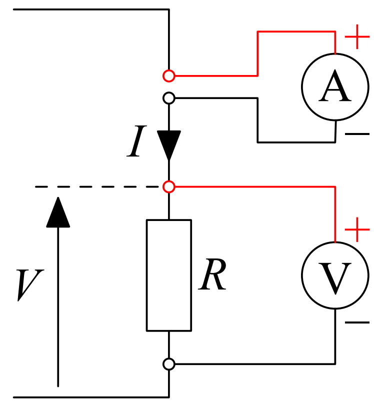


For a resistor, $I \propto V$ and $\frac{V}{I} = R$, its *resistance* which is measured in Ohms (Ω). This is Ohm's Law. Sometimes it is more convenient to work in terms of the *conductance*, $G = \frac{1}{R} = \frac{I}{V}$ measured in Siemens (S).

The graph shows the characteristic of a $12.5\,\Omega$ resistor. The gradient of the graph equals the conductance $G = 80\,\mathrm{mS}$. Alternative zigzag symbol.

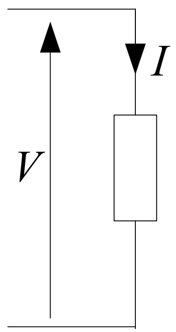
[Passive Sign Convention]

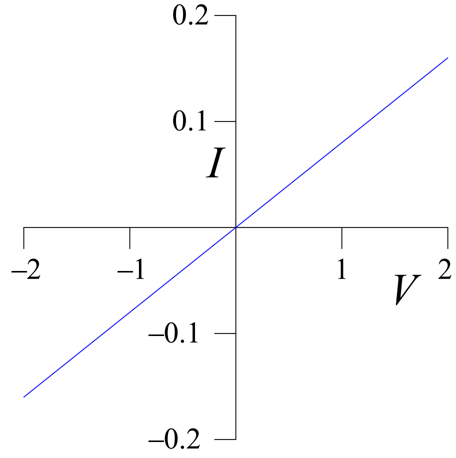
To measure the voltage in a physical circuit, you use a voltmeter (V in the figure) which has two test leads connected to it usually coloured red (marked +) and black (marked –) respectively. The reading on the voltmeter shows the voltage at the red lead relative to that at the black lead (or equivalently the red voltage minus the black voltage). To measure the voltage V in the figure, you would connect the red lead to the top end of the arrow (pointed end) and the black lead to the bottom (blunt end).



To measure current you use an ammeter (A in the figure) which also has two test leads coloured red and black respectively. The reading shows the current flowing through the ammeter into the red lead and out of the black lead. To measure the current I on the previous slide, you would need to break the wire carrying the current and insert the ammeter as shown in the figure.
With the connections shown in the figure, the readings on V and A will always have the same sign: either both positive or both negative and will satisfy Ohm's law: $V = IR$. However, if the connections are reversed on either V or A, then the two readings will have opposite signs and $V = -IR$ which does not satisfy Ohm's law.
So, if you want Ohm's law to be true you must be sure to connect the measuring devices the right way round according to the passive sign convention.

Ohm's law relates the voltage drop across a resistor to the current flowing in it.





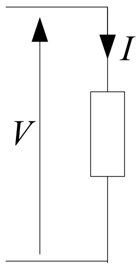
If the voltage, V, is fixed elsewhere in the circuit, it is convenient to think that V *causes* the current I to flow.

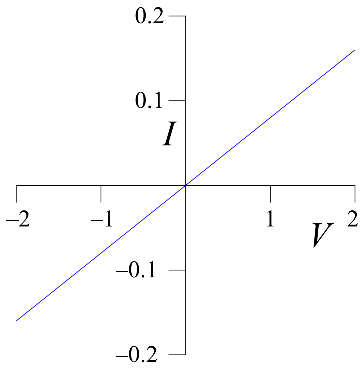
If the current, I, is fixed elsewhere in the circuit, it is more convenient to think that V is *caused by* the current I flowing through the resistor.

Neither statement is "more true" than the other. It is perhaps truer to say that I and V are *constrained to satisfy* $V = I \times R$.

Gravitational potential energy, mgh, lost by a falling object is transformed into kinetic energy or heat.

Current in a resistor always flows from a high voltage (more positive) to a low voltage (more negative).





When current flows through a resistor, the electrical potential energy that is lost is transformed into heat.

The power dissipated as heat in a resistor is equal to VI Watts (W). 1 Watt equals one Joule of energy per second. Since V and I always have the same sign (see graph) the power dissipation is always positive.

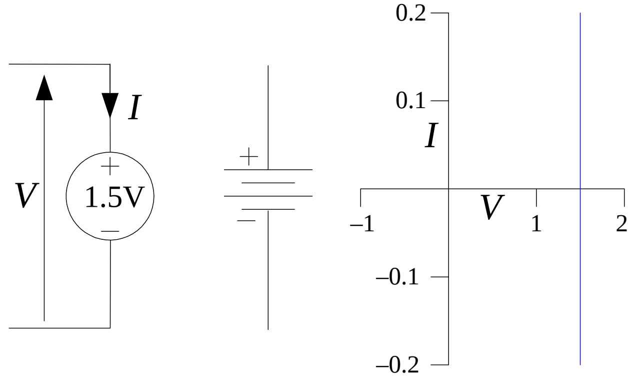
Any component: $P = VI$ gives the power absorbed by any component.

For a resistor only: $\frac{V}{I} = R \quad \Rightarrow \quad P = VI = \frac{V^2}{R} = I^2R.$

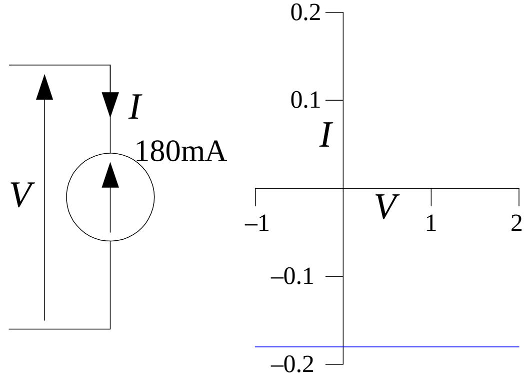
Voltage and Current Sources

Energy in an electrical circuit is supplied by voltage and current sources

An *ideal voltage source* maintains the same value of V for all currents. Its characteristic is a vertical line with infinite gradient. There are two common symbols.



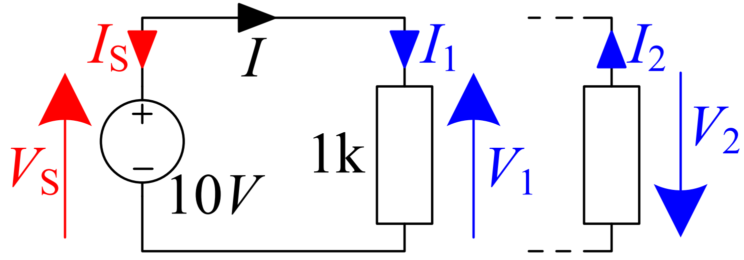
An *ideal current source* maintains the same value of I for all voltages. Its characteristic is a horizontal line with zero gradient. Notice that I is negative.



If the source is supplying electrical energy to a circuit, then $VI < 0$.
However, when a recharcheable battery is charging, $VI > 0$.

In any circuit some circuit elements will be supplying energy and others absorbing it. At all times, the power absorbed by all the elements will sum to zero.

The circuit has two nodes whose potential difference is 10 V.



Ohm's Law:

$I = \frac{V}{R} = 0.01\text{ A}$

Power absorbed by resistor:

$$P_R = V_1 \times I_1 = (+10) \times (+0.01) = +0.1\text{ W}$$

For Ohm's law or power dissipation, V and I can be measured either way round but must be in opposite directions (passive sign convention).

$$P_R = V_2 \times I_2 = (-10) \times (-0.01) = +0.1\text{ W}$$

Power absorbed by voltage source:

$$P_S = V_S \times I_S = (+10) \times (-0.01) = -0.1\text{ W}$$

Total power absorbed by circuit elements: $P_S + P_R = 0$

Units and Multipliers

Quantity	Letter	Unit	Symbol
Charge	Q	Coulomb	C
Conductance	G	Siemens	S
Current	I	Amp	A
Energy	W	Joule	J
Potential	V	Volt	V
Power	P	Watt	W
Resistance	R	Ohm	Ω

Value	Prefix	Symbol
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f

Value	Prefix	Symbol
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P

Summary

- Circuits and Nodes
- Charge, Current and Voltage
- Resistors, Voltage Source and Current Sources
- Power Dissipation and Power Conservation

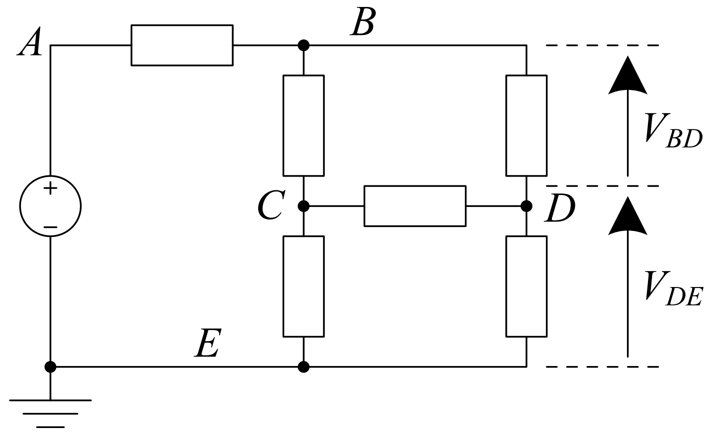
For further details see Hayt Ch 2 or Irwin Ch 1.

2: Resistor Circuits

Kirchoff's Voltage Law

The five nodes are labelled A, B, C, D, E where E is the reference node.

Each component that links a pair of nodes is called a *branch* of the network.



Kirchoff's Voltage Law (KVL) is a consequence of the fact that the work done in moving a charge from one node to another does not depend on the route you take; in particular the work done in going from one node back to the same node by any route is zero.

KVL: the sum of the voltage changes around any closed loop is zero.

Example: $V_{DE} + V_{BD} + V_{AB} + V_{EA} = 0$

Equivalent formulation:

$V_{XY} = V_{XE} - V_{YE} = V_X - V_Y$ for any nodes X and Y.

Wherever charges are free to move around, they will move to ensure charge neutrality everywhere at all times.

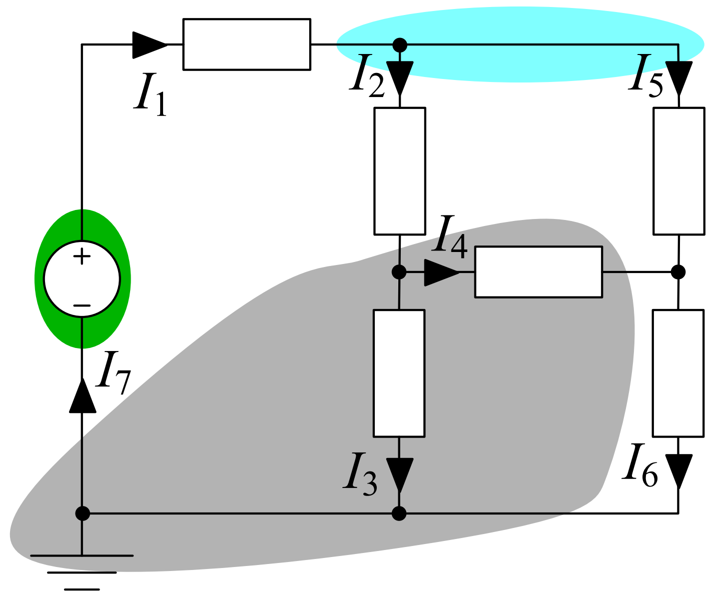
A consequence is Kirchoff's Current Law (KCL) which says that the current going into any closed region of a circuit must equal the current coming out.

KCL: The currents flowing out of any closed region of a circuit sum to zero.

Green: $I_1 = I_7$

Blue: $-I_1 + I_2 + I_5 = 0$

Gray: $-I_2 + I_4 - I_6 + I_7 = 0$

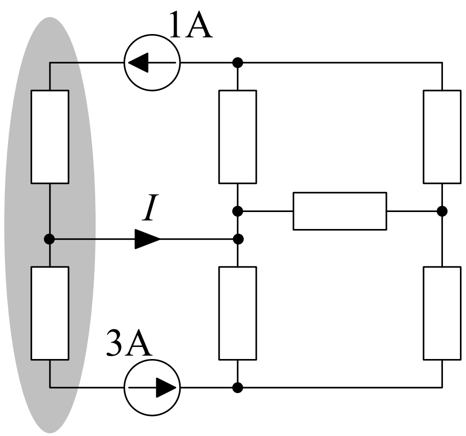


The currents and voltages in any linear circuit can be determined by using KCL, KVL and Ohm's law.

Sometimes KCL allows you to determine currents very easily without having to solve any simultaneous equations:

How do we calculate I ?

KCL: $-1 + I + 3 = 0$
$\Longrightarrow I = -2\,\mathrm{A}$



Note that here I ends up negative which means we chose the wrong arrow direction to label the circuit. This does not matter. You can choose the directions arbitrarily and let the algebra take care of reality.

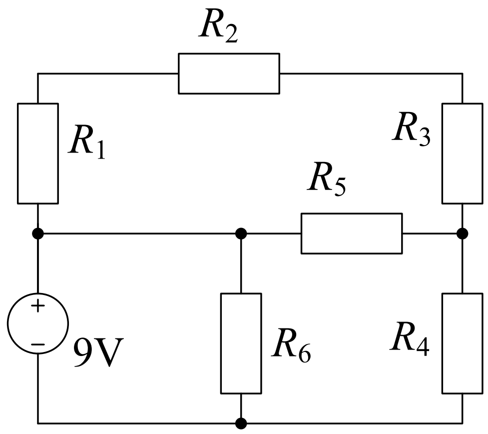
Series and Parallel

Series: Components that are connected in a chain so that the same current flows through each one are said to be *in series*.

R_1, R_2, R_3 are in series and the same current always flows through each.

Within the chain, each internal node connects to only two branches.

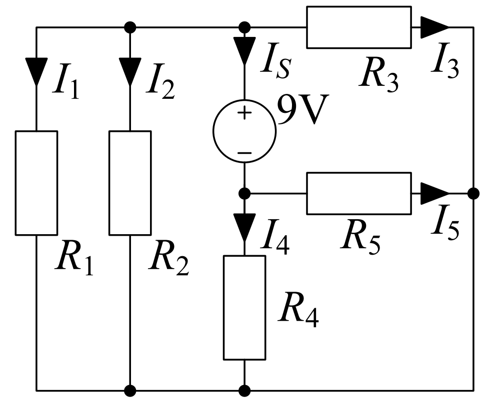
R_3 and R_4 are not in series and do not necessarily have the same current.



Parallel: Components that are connected to the same pair of nodes are said to be *in parallel*.

R_1, R_2, R_3 are in parallel and the same voltage is across each resistor (even though R_3 is not close to the others).

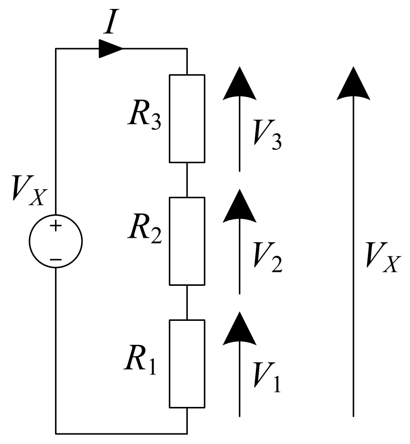
R_4 and R_5 are also in parallel.



$$V_X = V_1 + V_2 + V_3$$
$$= IR_1 + IR_2 + IR_3$$
$$= I(R_1 + R_2 + R_3)$$

$$\frac{V_1}{V_X} = \frac{IR_1}{I(R_1+R_2+R_3)}$$
$$= \frac{R_1}{R_1+R_2+R_3} = \frac{R_1}{R_T}$$

where $R_T = R_1 + R_2 + R_3$ is the total resistance of the chain.

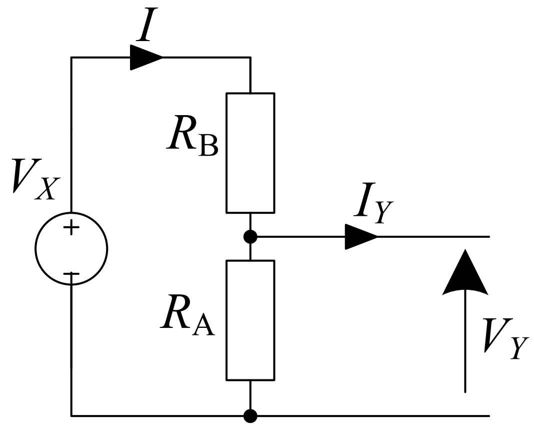


V_X is divided into $V_1 : V_2 : V_3$ in the proportions $R_1 : R_2 : R_3$.

Approximate Voltage Divider:

If $I_Y = 0$, then $V_Y = \frac{R_A}{R_A+R_B} V_X$.

If $I_Y \ll I$, then $V_Y \approx \frac{R_A}{R_A+R_B} V_X$.

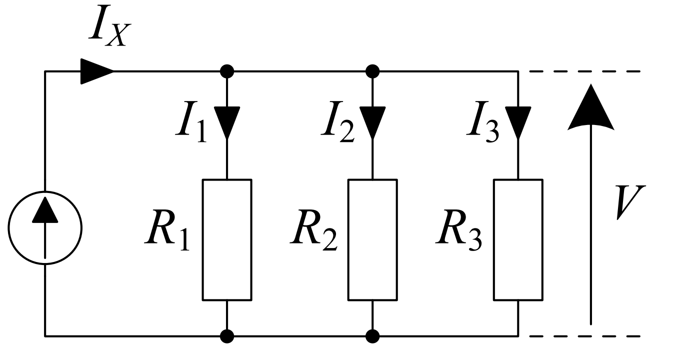


Parallel Resistors: Current Divider

Parallel resistors all share the same V.

$I_1 = \frac{V}{R_1} = VG_1$ where $G_1 = \frac{1}{R_1}$ is the *conductance* of R_1.

$$\begin{aligned} I_X &= I_1 + I_2 + I_3 \\ &= VG_1 + VG_2 + VG_3 \\ &= V(G_1 + G_2 + G_3) \end{aligned}$$



$$\frac{I_1}{I_X} = \frac{VG_1}{V(G_1+G_2+G_3)} = \frac{G_1}{G_1+G_2+G_3} = \frac{G_1}{G_P}$$

where $G_P = G_1 + G_2 + G_3$ is the total conductance of the resistors.

I_X is divided into $I_1 : I_2 : I_3$ in the proportions $G_1 : G_2 : G_3$.

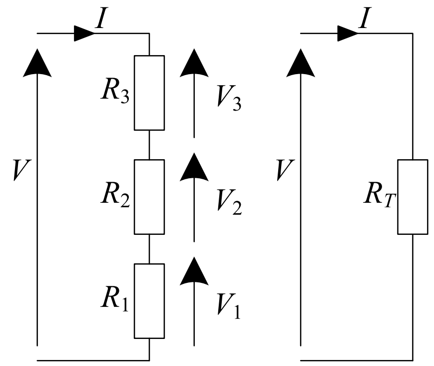
Special case for only two resistors:

$$I_1 : I_2 = G_1 : G_2 = R_2 : R_1 \Rightarrow I_1 = \frac{R_2}{R_1+R_2} I_X.$$

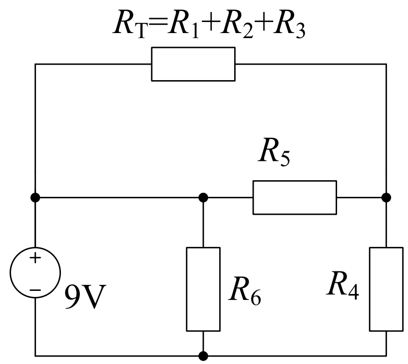
Equivalent Resistance: Series

We know that $V = V_1 + V_2 + V_3 = I(R_1 + R_2 + R_3) = IR_T$

So we can replace the three resistors by a single *equivalent resistor* of value R_T without affecting the relationship between V and I.



Replacing series resistors by their equivalent resistor will not affect any of the voltages or currents in the rest of the circuit.



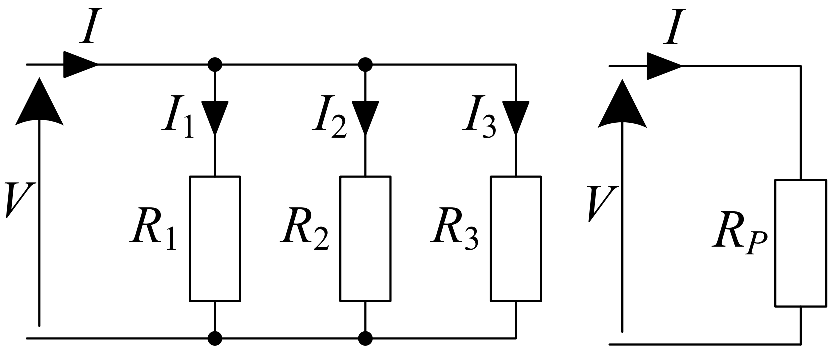
However the individual voltages V_1, V_2 and V_3 are no longer accessible.

Equivalent Resistance: Parallel

Similarly we known that $I = I_1 + I_2 + I_3 = V(G_1 + G_2 + G_3) = VG_P$.

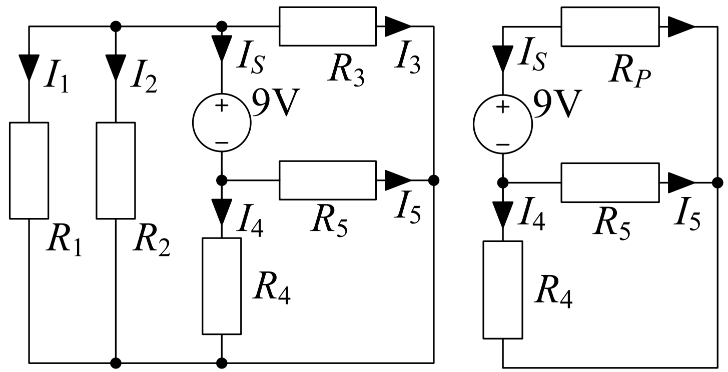
So $V = IR_P$ where $R_P = \frac{1}{G_P} = \frac{1}{G_1+G_2+G_3} = \frac{1}{1/R_1+1/R_2+1/R_3}$

We can use a single *equivalent resistor* of resistance R_P without affecting the relationship between V and I.



Replacing parallel resistors by their equivalent resistor will not affect any of the voltages or currents in the rest of the circuit.

$R_4$and R_5 are also in parallel.



Much simpler - although none of the original currents $I_1, \cdots, I_5$ are now accessible. Current I_S and the three node voltages are identical.

Equivalent Resistance: Parallel Formulae

For parallel resistors $G_P = G_1 + G_2 + G_3$

or equivalently $R_P = R_1||R_2||R_3 = \frac{1}{1/R_1 + 1/R_2 + 1/R_3}$.

These formulae work for any number of resistors.

- For the special case of two parallel resistors

$R_P = \frac{1}{1/R_1 + 1/R_2} = \frac{R_1 R_2}{R_1 + R_2}$ ("product over sum")

- If one resistor is a multiple of the other

Suppose $R_2 = kR_1$, then

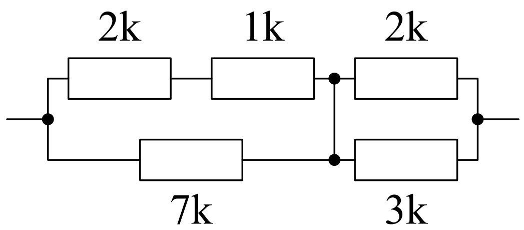
$R_P = \frac{R_1 R_2}{R_1 + R_2} = \frac{kR_1^2}{(k+1)R_1} = \frac{k}{k+1} R_1 = (1 - \frac{1}{k+1}) R_1$

Example: $1\ \mathrm{k\Omega}\ ||\ 99\ \mathrm{k\Omega} = \frac{99}{100}\ \mathrm{k\Omega} = \left(1 - \frac{1}{100}\right)\ \mathrm{k\Omega}$

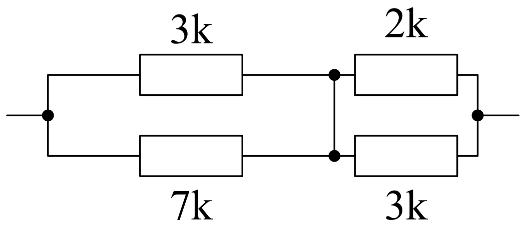
Important: The equivalent resistance of parallel resistors is always less than any of them.

Simplifying Resistor Networks

Many resistor circuits can be simplified by alternately combining series and parallel resistors.

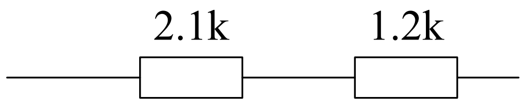


Series: $2\,\mathrm{k} + 1\,\mathrm{k} = 3\,\mathrm{k}$

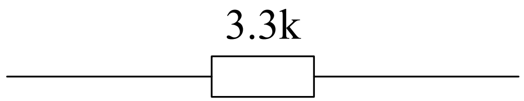


Parallel: $3\,\mathrm{k}\,||\,7\,\mathrm{k} = 2.1\,\mathrm{k}$

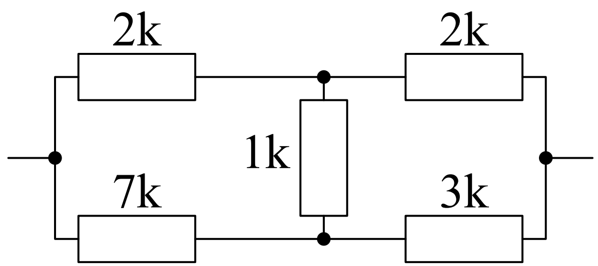
Parallel: $2\,\mathrm{k}\,||\,3\,\mathrm{k} = 1.2\,\mathrm{k}$



Series: $2.1\,\mathrm{k} + 1.2\,\mathrm{k} = 3.3\,\mathrm{k}$

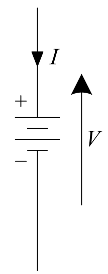


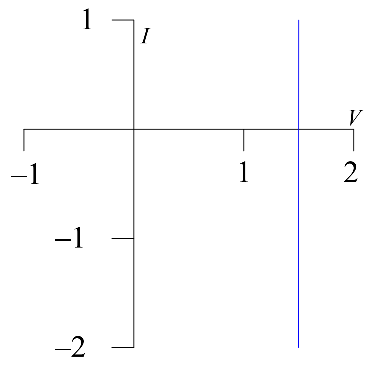
Sadly this method does not always work: there are no series or parallel resistors here.



Non-ideal Voltage Source

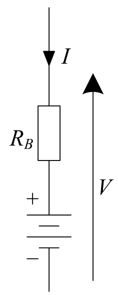
An ideal battery has a characteristic that is vertical: battery voltage does not vary with current.
Normally a battery is supplying energy so V and I have opposite signs, so $I \leq 0$.

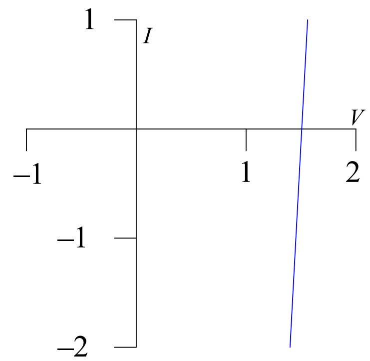




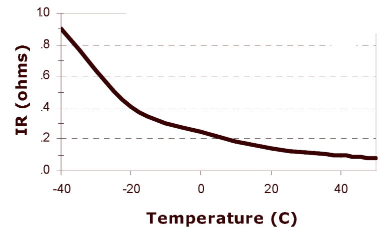
An real battery has a characteristic that has a slight positive slope: battery voltage decreases as the (negative) current increases.

Model this by including a small resistor in series. $V = V_B + IR_B$.





The equivalent resistance for a battery increases at low temperatures.



Summary

- ☐ Kichoff's Voltage and Current Laws
- ☐ Series and Parallel components
- ☐ Voltage and Current Dividers
- ☐ Simplifying Resistor Networks
- ☐ Battery Internal Resistance

For further details see Hayt Ch 3 or Irwin Ch 2.

3: Nodal Analysis

Aim of Nodal Analysis

The aim of nodal analysis is to determine the voltage at each node relative to the reference node (or ground). Once you have done this you can easily work out anything else you need.

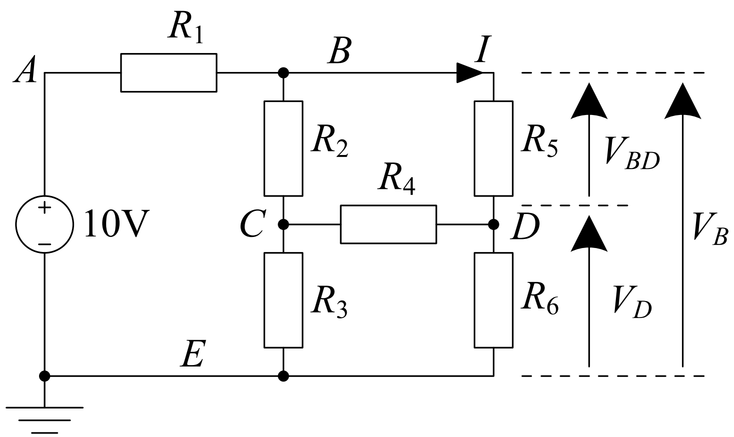
There are two ways to do this:

(1) Nodal Analysis - systematic; always works

(2) Circuit Manipulation - ad hoc; but can be less work and clearer

Reminders:

A node is all the points in a circuit that are directly interconnected. We assume the interconnections have zero resistance so all points within a node have the same voltage. Five nodes: $A, \cdots, E$.



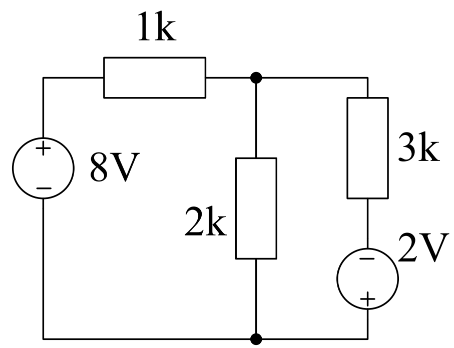
Ohm's Law: $V_{BD} = IR_5$

KVL: $V_{BD} = V_B - V_D$

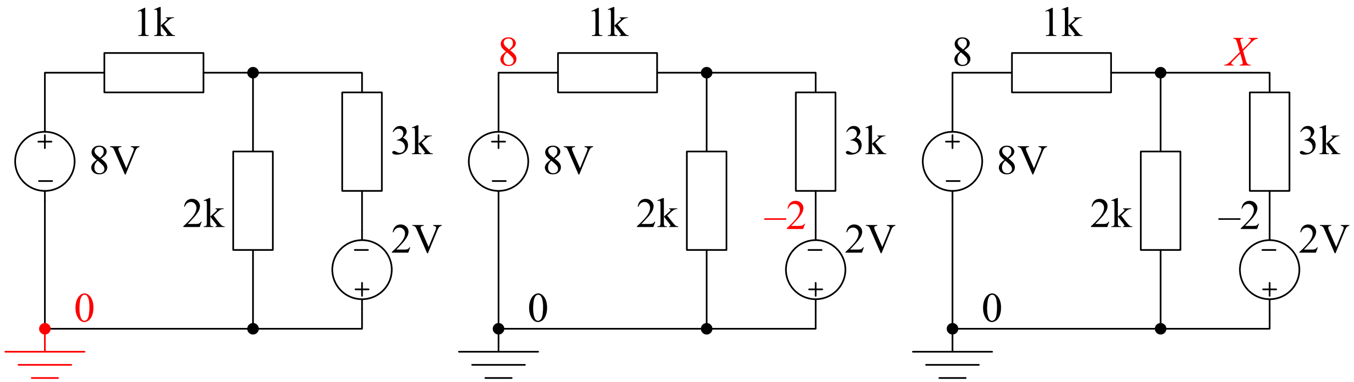
KCL: Total current exiting any closed region is zero.

Nodal Analysis Stage 1: Label Nodes

To find the voltage at each node, the first step is to label each node with its voltage as follows

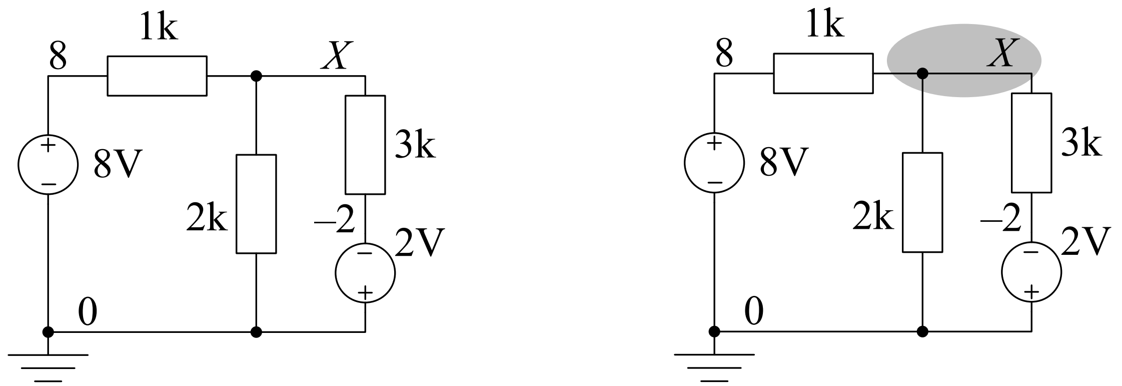


(1) Pick any node as the voltage reference. Label its voltage as 0 V.
(2) If any fixed voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end.
(3) Pick an unlabelled node and label it with X, Y, …, then go back to step (2) until all nodes are labelled.



Nodal Analysis Stage 2: KCL Equations

The second step is to write down a KCL equation for each node labelled with a variable by setting the total current flowing out of the node to zero. For a circuit with N nodes and S voltage sources you will have $N - S - 1$ simultaneous equations to solve.



We only have one variable:

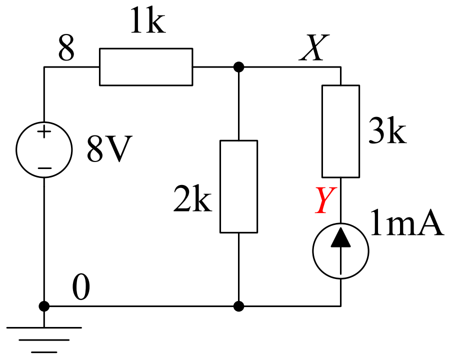
$$\frac{X-8}{1\text{ k}} + \frac{X-0}{2\text{ k}} + \frac{X-(-2)}{3\text{ k}} = 0 \quad \Rightarrow \quad (6X - 48) + 3X + (2X + 4) = 0$$

$$11X = 44 \quad \Rightarrow \quad X = 4$$

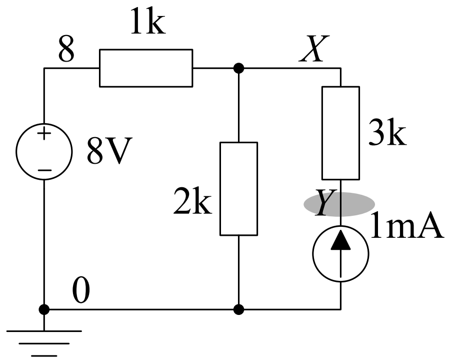
Numerator for a resistor is always of the form $X - V_N$ where V_N is the voltage on the other side of the resistor.

Current sources cause no problems.

(1) Pick reference node.
(2) Label nodes: 8, X and Y.



(3) Write equations



$$\frac{X-8}{1} + \frac{X}{2} + \frac{X-Y}{3} = 0$$

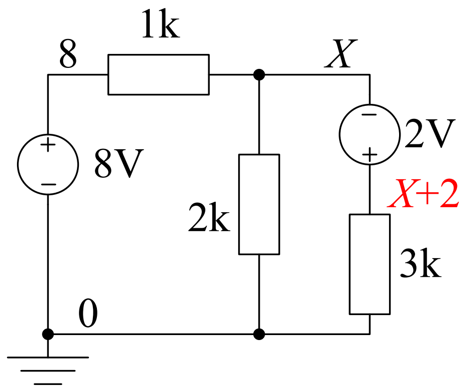
$$\frac{Y-X}{3} + (-1) = 0$$

Ohm's law works OK if all resistors are in $\mathrm{k}\Omega$ and all currents in mA.
(4) Solve the equations: $X = 6$, $Y = 9$

Floating Voltage Sources

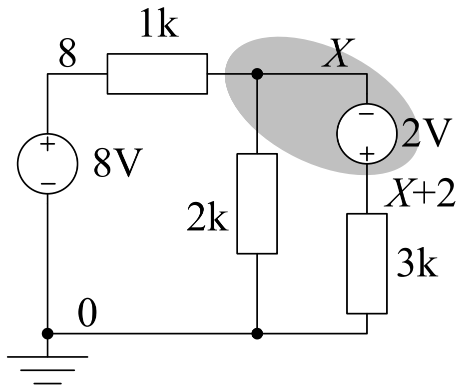
Floating voltage sources have neither end connected to a known fixed voltage. We have to change how we form the KCL equations slightly.

(1) Pick reference node.



(2) Label nodes: 8, X and $X+2$ since it is joined to X via a voltage source.

(3) Write KCL equations but count all the nodes connected via floating voltage sources as a single "super-node" giving one equation



$$\frac{X-8}{1}+\frac{X}{2}+\frac{(X+2)-0}{3}=0$$

(4) Solve the equations: $X=4$

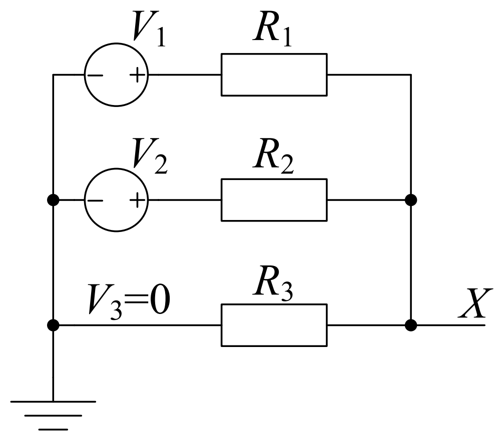
Ohm's law always involves the difference between the voltages at either end of a resistor. (Obvious but easily forgotten)

A very useful sub-circuit that calculates the weighted average of any number of voltages.

KCL equation for node X:

$$\frac{X-V_1}{R_1}+\frac{X-V_2}{R_2}+\frac{X-V_3}{R_3}=0$$

Still works if $V_3=0$.



Or using conductances:

$$(X-V_1)G_1+(X-V_2)G_2+(X-V_3)G_3=0$$

$$X(G_1+G_2+G_3)=V_1G_1+V_2G_2+V_3G_3$$

$$X=\frac{V_1G_1+V_2G_2+V_3G_3}{G_1+G_2+G_3}=\frac{\sum_{i=1}^{3}V_iG_i}{\sum_{i=1}^{3}G_i}$$

Voltage X is the average of V_1, V_2, V_3 weighted by the conductances.

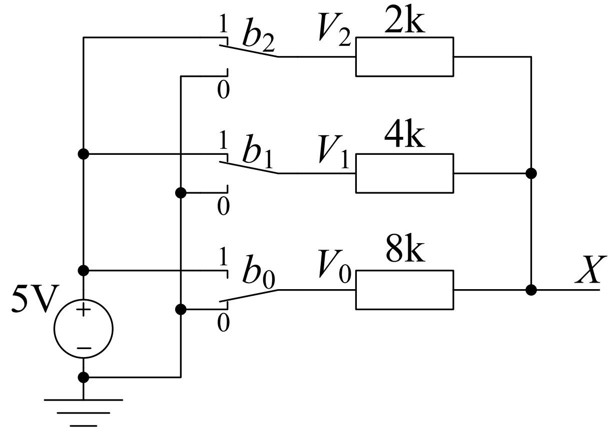
Digital-to-Analog Converter

A 3-bit binary number, b, has bit-weights of 4, 2 and 1. Thus 110 has a value 6 in decimal. If we label the bits $b_2b_1b_0$, then $b = 4b_2 + 2b_1 + b_0$.

We use $b_2b_1b_0$ to control the switches which determine whether $V_i = 5\,\mathrm{V}$ or $V_i = 0\,\mathrm{V}$. Thus $V_i = 5b_i$. Switches shown for $b = 6$.

$$X = \frac{\frac{1}{2}V_2 + \frac{1}{4}V_1 + \frac{1}{8}V_0}{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}$$

$$= \frac{1}{7}\left(4V_2 + 2V_1 + V_0\right)$$



but $V_i = 5 \times b_i$ since it connects to either $0\,\mathrm{V}$ or $5\,\mathrm{V}$

$$= \frac{5}{7}\left(4b_2 + 2b_1 + b_0\right) = \frac{5}{7}b$$

$$G_2 = \frac{1}{R_2} = \frac{1}{2}\,\mathrm{mS},\ \ldots$$

So we have made a circuit in which X is proportional to a binary number b.

A *dependent* voltage or current source is one whose value is determined by voltages or currents elsewhere in the circuit. These are most commonly used when modelling the behaviour of transistors or op-amps. Each dependent source has a defining equation.

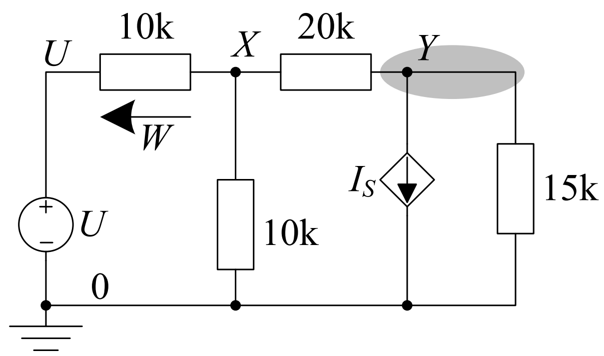
In this circuit: $I_S = 0.2W$ mA where W is in volts.

(1) Pick reference node.

(2) Label nodes: 0, U, X and Y .

(3) Write equation for the dependent source, I_S, in terms of node voltages:

$$I_S = 0.2\,(U - X)$$



(4) Write KCL equations:

$$\frac{X-U}{10} + \frac{X}{10} + \frac{X-Y}{20} = 0 \qquad \frac{Y-X}{20} + I_S + \frac{Y}{15} = 0$$

(5) Solve all three equations to find X, Y and I_S in terms of U:

$$X = 0.1U,\; Y = -1.5U,\; I_S = 0.18U$$

Note that the value of U is assumed to be known.

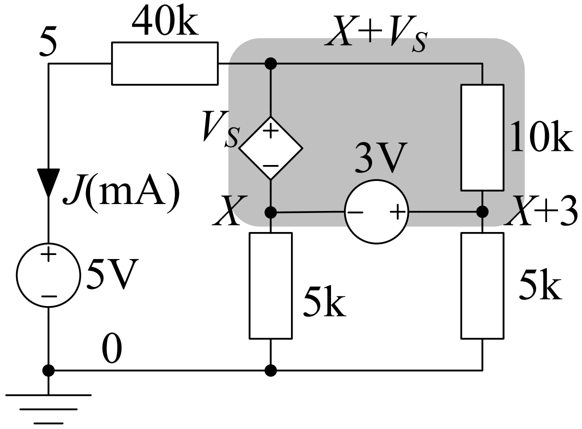
Dependent Voltage Sources

The value of the highlighted dependent voltage source is $V_S = 10J$ Volts where J is the indicated current in mA.

(1) Pick reference node.

(2) Label nodes: 0, 5, X, $X+3$ and $X+V_S$.

(3) Write equation for the dependent source, V_S, in terms of node voltages:



$$V_S = 10J = 10 \times \frac{X+V_S-5}{40} \Rightarrow 3V_S = X - 5$$

(4) Write KCL equations: all nodes connected by floating voltage sources and all components connecting these nodes are in the same “super-node”

$$\frac{X+V_S-5}{40} + \frac{X}{5} + \frac{X+3}{5} = 0$$

(5) Solve the two equations: $X = -1$ and $V_S = -2$

Universal Nodal Analysis Algorithm

(1) Pick any node as the voltage reference. Label its voltage as $0\,\mathrm{V}$. Label any dependent sources with V_S, I_S, $\ldots$.

(2) If any voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end. Repeat as many times as possible.

(3) Pick an unlabelled node and label it with X, Y, $\ldots$, then loop back to step (2) until all nodes are labelled.

(4) For each **dependent source**, write down an equation that expresses its value in terms of other node voltages.

(5) Write down a KCL equation for each "normal" node (i.e. one that is not connected to a floating voltage source).

(6) Write down a KCL equation for each "super-node". A super-node consists of a set of nodes that are joined by floating voltage sources and includes any other components joining these nodes.

(7) Solve the set of simultaneous equations that you have written down.

Summary

- Nodal Analysis
  - Simple Circuits (no floating or dependent voltage sources)
  - Floating Voltage Sources
    - use supernodes: all the nodes connected by floating voltage sources (independent or dependent)
  - Dependent Voltage and Current Sources
    - Label each source with a variable
    - Write extra equations expressing the source values in terms of node voltages
    - Write down the KCL equations as before
- Mesh Analysis (in most textbooks)
  - Alternative to nodal analysis but doesn't work for all circuits
  - No significant benefits $\Rightarrow$ ignore it

For further details see Hayt Ch 4 or Irwin Ch 3.

4: Linearity and Superposition

Suppose we use variables instead of fixed values for all of the *independent* voltage and current sources. We can then use nodal analysis to find all node voltages in terms of the source values.

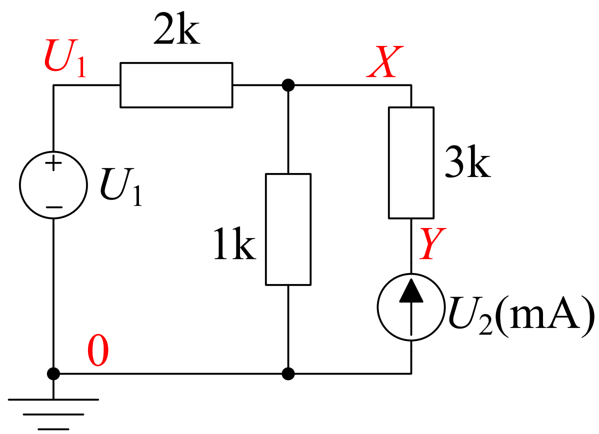
(1) Label all the nodes
(2) KCL equations

$$\frac{X-U_1}{2} + \frac{X}{1} + \frac{X-Y}{3} = 0$$

$$\frac{Y-X}{3} + (-U_2) = 0$$

(3) Solve for the node voltages

$$X = \tfrac{1}{3}U_1 + \tfrac{2}{3}U_2, \quad Y = \tfrac{1}{3}U_1 + \tfrac{11}{3}U_2$$



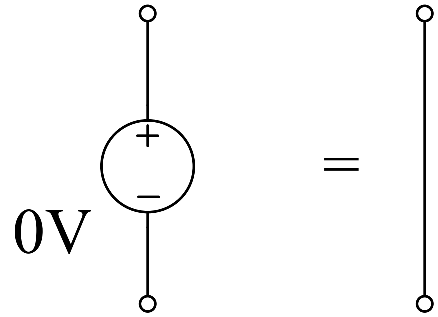
Steps (2) and (3) never involve multiplying two source values together, so:

Linearity Theorem: For any circuit containing resistors and independent voltage and current sources, every node voltage and branch current is a linear function of the source values and has the form $\sum a_i U_i$ where the U_i are the source values and the a_i are suitably dimensioned constants.

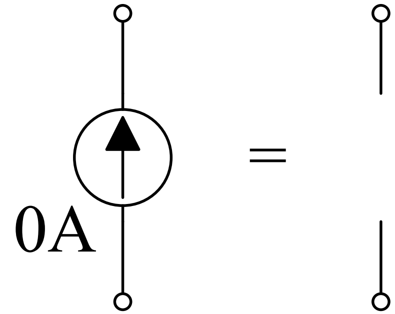
Also true for a circuit containing *dependent* sources whose values are proportional to voltages or currents elsewhere in the circuit.

Zero-value sources

A zero-valued voltage source has zero volts between its terminals for any current. It is equivalent to a *short-circuit* or piece of wire or resistor of $0\,\Omega$ (or $\infty\,\mathrm{S}$).



A zero-valued current source has no current flowing between its terminals. It is equivalent to an *open-circuit* or a broken wire or a resistor of $\infty\,\Omega$ (or $0\,\mathrm{S}$).



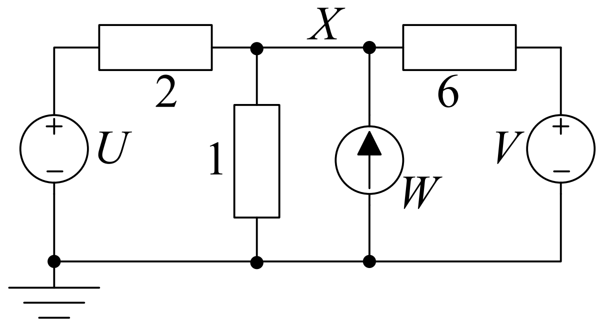
Superposition

We can use nodal analysis to find X in terms of U, V and W.

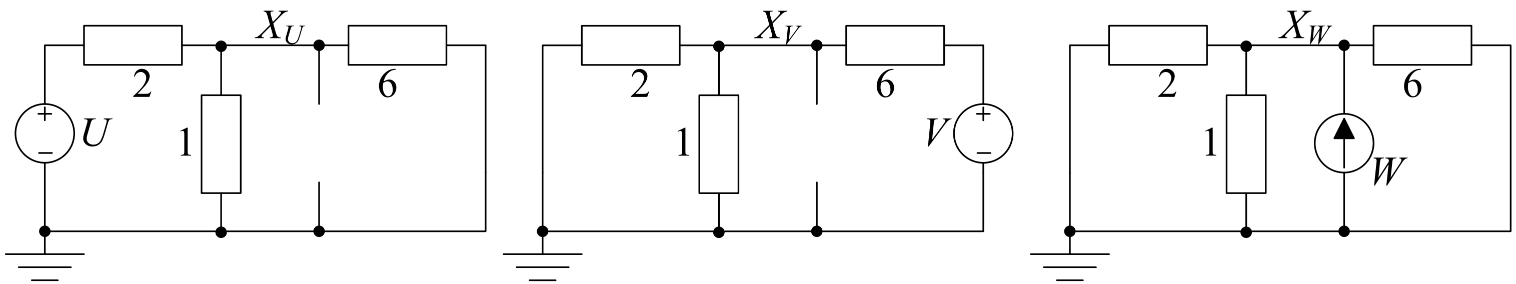
KCL: $\frac{X-U}{2}+\frac{X-V}{6}+\frac{X}{1}-W=0$

$$10X-3U-V-6W=0$$

$$X=0.3U+0.1V+0.6W$$



From the linearity theorem, we know anyway that $X=aU+bV+cW$ so all we need to do is find the values of a, b and c. We find each coefficient in turn by setting all the other sources to zero:

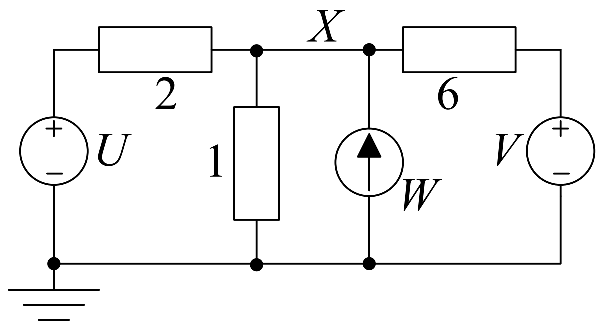


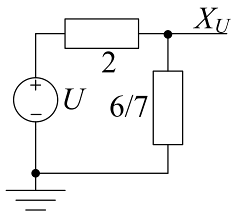
We have $X_U=aU+b\times 0+c\times 0=aU$.
Similarly, $X_V=bV$ and $X_W=cW \quad\Rightarrow\quad X=X_U+X_V+X_W$.

Superposition Calculation

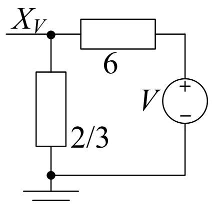
Superposition:

Find the effect of each source on its own by setting all other sources to zero. Then add up the results.

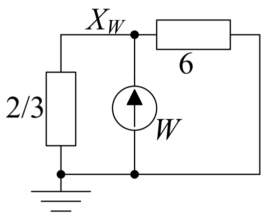




$$X_U = \frac{\frac{6}{7}}{2+\frac{6}{7}}U = \frac{6}{20}U = 0.3U$$



$$X_V = \frac{\frac{2}{3}}{6+\frac{2}{3}}V = \frac{2}{20}V = 0.1V$$

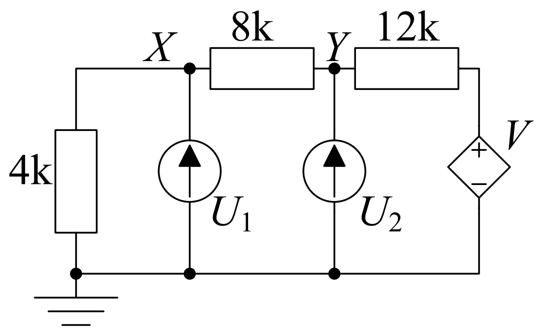


$$X_W = \frac{6}{6+\frac{2}{3}}W \times \frac{2}{3} = \frac{12}{20}W = 0.6W$$

Adding them up: $X = X_U + X_V + X_W = 0.3U + 0.1V + 0.6W$

Superposition and dependent sources

A *dependent source* is one that is determined by the voltage and/or current elsewhere in the circuit via a known equation. Here $V \triangleq Y - X$.



Step 1: Pretend all sources are independent and use superposition to find expressions for the node voltages:

$$X = \frac{10}{3}U_1 + 2U_2 + \frac{1}{6}V$$
$$Y = 2U_1 + 6U_2 + \frac{1}{2}V$$

Step 2: Express the dependent source values in terms of node voltages:

$$V = Y - X$$

Step 3: Eliminate the dependent source values from the node voltage equations:

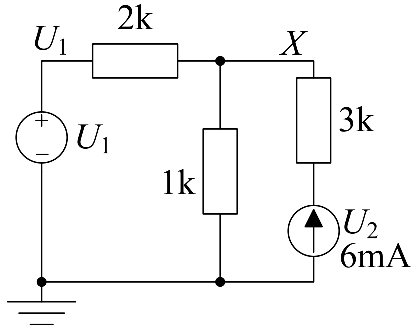
$$X = \frac{10}{3}U_1 + 2U_2 + \frac{1}{6}(Y - X) \quad \Rightarrow \frac{7}{6}X - \frac{1}{6}Y = \frac{10}{3}U_1 + 2U_2$$
$$Y = 2U_1 + 6U_2 + \frac{1}{2}(Y - X)) \quad \Rightarrow \frac{1}{2}X + \frac{1}{2}Y = 2U_1 + 6U_2$$
$$X = 3U_1 + 3U_2$$
$$Y = U_1 + 9U_2$$

Note: This is an alternative to nodal anlysis: you get the same answer.

Any current or voltage can be written $X = a_1U_1 + a_2U_2 + a_3U_3 + \ldots$.

Using nodal analysis (slide 4-2) or else superposition:

$$X = \tfrac{1}{3}U_1 + \tfrac{2}{3}U_2.$$



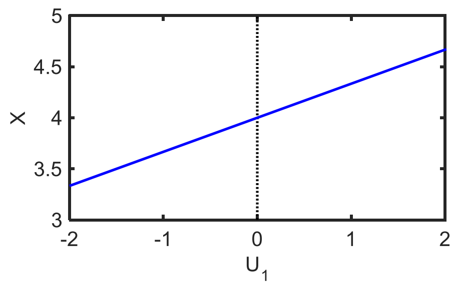
Suppose we know $U_2 = 6\,\mathrm{mA}$, then

$$X = \tfrac{1}{3}U_1 + \tfrac{2}{3}U_2 = \tfrac{1}{3}U_1 + 4.$$

If all the independent sources except for U_1 have known fixed values, then

$$X = a_1U_1 + b$$

where $b = a_2U_2 + a_3U_3 + \ldots$.



This has a straight line graph.

Superposition and Power

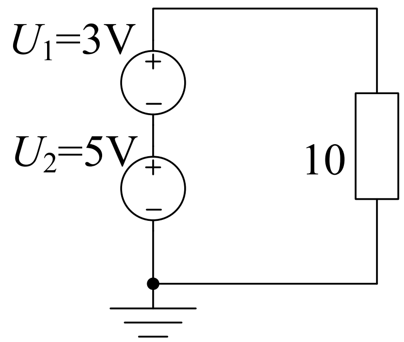
The power absorbed (or *dissipated*) by a component always equals VI where the measurement directions of V and I follow the passive sign convention.

For a resistor $VI = \frac{V^2}{R} = I^2 R$.

Power in resistor is $P = \frac{(U_1+U_2)^2}{10} = 6.4\,\text{W}$

Power due to U_1 alone is $P_1 = \frac{U_1^2}{10} = 0.9\,\text{W}$

Power due to U_2 alone is $P_2 = \frac{U_2^2}{10} = 2.5\,\text{W}$



$P \neq P_1 + P_2 \quad \Rightarrow$ Power does not obey superposition.

You must use superposition to calculate the total V and/or the total I and then calculate the power.

Proportionality

From the linearity theorem, all voltages and currents have the form $\sum a_i U_i$ where the U_i are the values of the independent sources.

If you multiply *all* the independent sources by the same factor, k, then all voltages and currents in the circuit will be multiplied by k.

The power dissipated in any component will be multiplied by k^2.

Special Case:
If there is only one independent source, U, then all voltages and currents are proportional to U and all power dissipations are proportional to U^2.

Summary

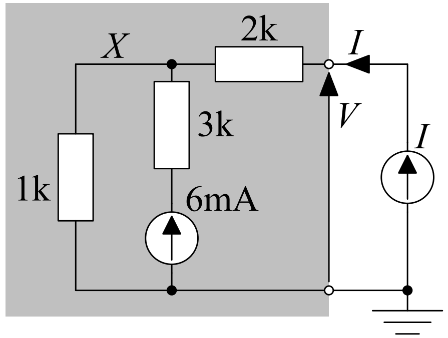
- Linearity Theorem: $X = \sum_i a_i U_i$ over all independent sources U_i
- Superposition: sometimes simpler than nodal analysis, often more insight.
  - Zero-value voltage and current sources
  - Dependent sources - treat as independent and add dependency as an extra equation
- If all sources are fixed except for U_1 then all voltages and currents in the circuit have the form $aU_1 + b$.
- Power does not obey superposition.
- Proportionality: multiplying all sources by k multiplies all voltages and currents by k and all powers by k^2.

For further details see Hayt Ch 5 or Irwin Ch 5.

5: Thévenin and Norton Equivalents

Equivalent Networks

From linearity theorem: $V = aI + b$.



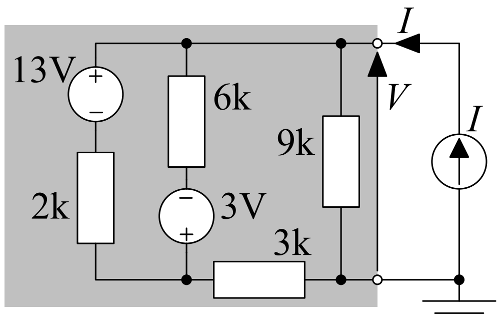
Use nodal analysis:

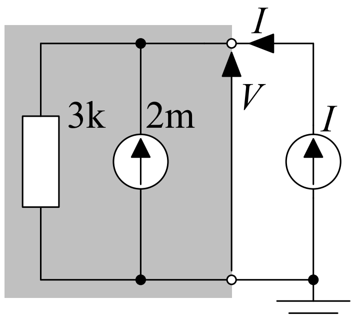
KCL@X: $\frac{X}{1} - 6 + \frac{X-V}{2} = 0$

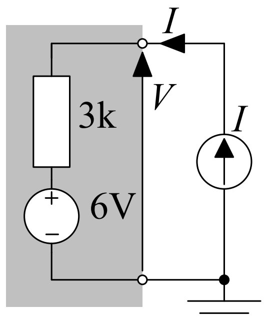
KCL@V: $\frac{V-X}{2} - I = 0$

Eliminating X gives: $V = 3I + 6$.

There are infinitely many networks with the same values of a and b:





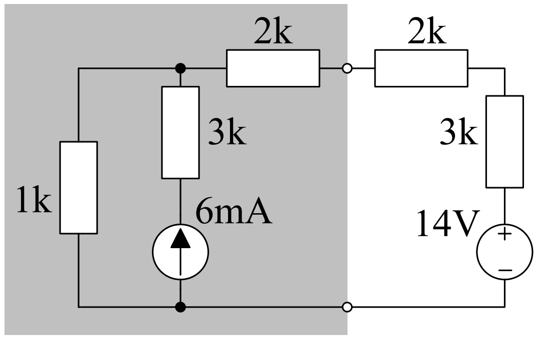


These four shaded networks are *equivalent* because the relationship between V and I is *exactly* the same in each case.

The last two are particularly simple and are respectively called the *Norton* and *Thévenin* equivalent networks.

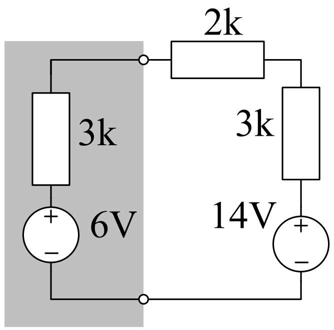
Thévenin Theorem: Any two-terminal network consisting of resistors, fixed voltage/current sources and linear dependent sources is externally equivalent to a circuit consisting of a resistor in series with a fixed voltage source.

We can replace the shaded part of the circuit with its Thévenin equivalent network.



The voltages and currents in the unshaded part of the circuit will be identical in both circuits.

The new components are called the *Thévenin equivalent resistance*, R_{Th}, and the *Thévenin equivalent voltage*, V_{Th}, of the original network.



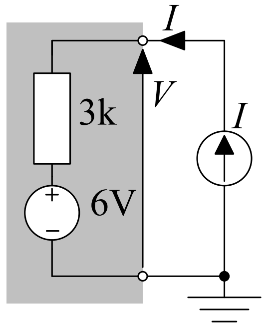
This is often a useful way to simplify a complicated circuit (provided that you do not want to know the voltages and currents in the shaded part).

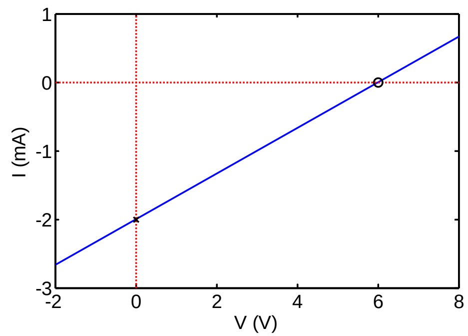
Thévenin Circuit Properties

A Thévenin equivalent circuit has a straight line characteristic with the equation:

$$V = R_{Th} I + V_{Th}$$

$$\Leftrightarrow I = \frac{1}{R_{Th}} V - \frac{V_{Th}}{R_{Th}}$$





Three important quantities are:

Open Circuit Voltage: If $I = 0$ then $V_{OC} = V_{Th}$. (X-intercept: o)

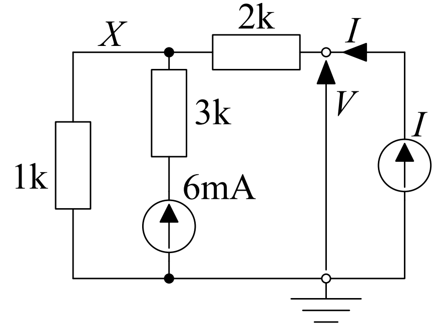
Short Circuit Current: If $V = 0$ then $I_{SC} = -\frac{V_{Th}}{R_{Th}}$ (Y-intercept: x)

Thévenin Resistance: The slope of the characteristic is $\frac{dI}{dV} = \frac{1}{R_{Th}}$.

If we know the value of any two of these three quantities, we can work out V_{Th} and R_{Th}.

In any two-terminal circuit with the same characteristic, the three quantities will have the same values. So if we can determine two of them, we can work out the Thévenin equivalent.

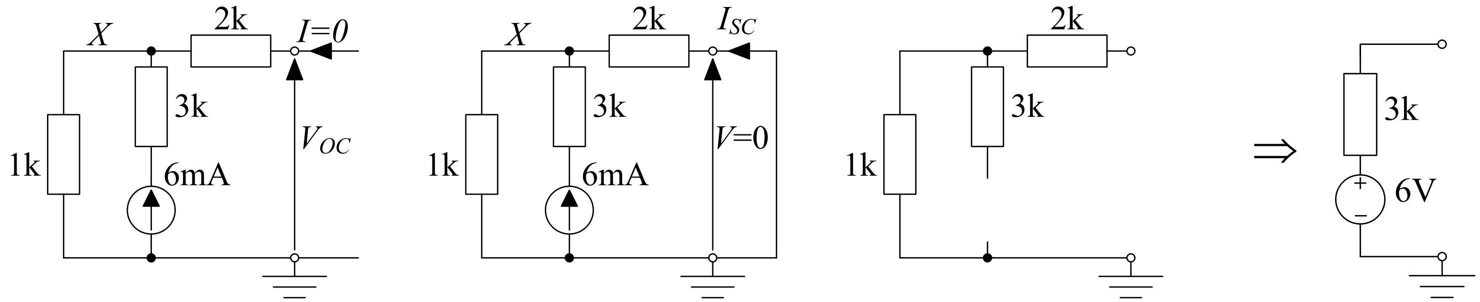
We need any two of the following:



Open Circuit Voltage: $V_{OC} = V_{Th} = 6\,\mathrm{V}$

Short Circuit Current: $I_{SC} = -\frac{V_{Th}}{R_{Th}} = -2\,\mathrm{mA}$

Thévenin Resistance: $\mathrm{R}_{Th} = 2\,\mathrm{k} + 1\,\mathrm{k} = 3\,\mathrm{k}\Omega$

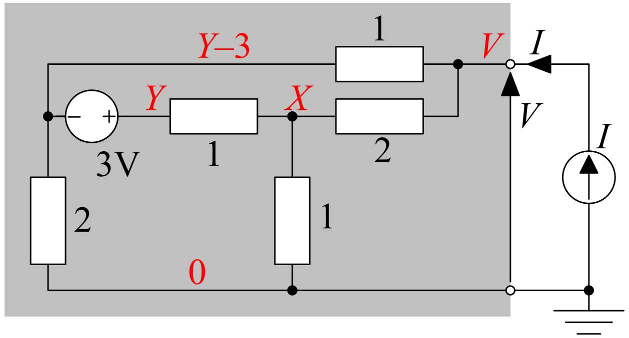


Thévenin Resistance:
We set all the independent sources to zero (voltage sources → short circuit, current sources→ open circuit). Then we find the equivalent resistance between the two terminals.
The $3\,\mathrm{k}$ resistor has no effect so $R_{Th} = 2\,\mathrm{k} + 1\,\mathrm{k} = 3\,\mathrm{k}$.
Any measurement gives the same result on an equivalent circuit.

Thévenin of Complicated Circuits

For a complicated circuit, you can use nodal analysis to find the Thévenin equivalent directly in the form:

$$V = V_{Th} + IR_{Th}.$$



Step 1: Label ground as an output terminal + label other nodes.

Step 2: Write down the equations (Y is a supernode)

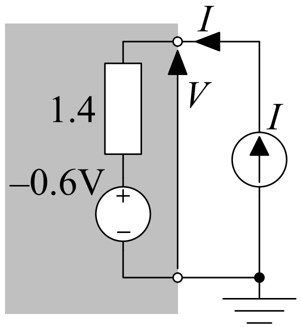
$$\frac{X-V}{2} + \frac{X}{1} + \frac{X-Y}{1} = 0$$

$$\frac{Y-3-V}{1} + \frac{Y-X}{1} + \frac{Y-3}{2} = 0$$

$$\frac{V-Y+3}{1} + \frac{V-X}{2} - I = 0$$

Step 3: Eliminate X and Y and solve for V in terms of I:

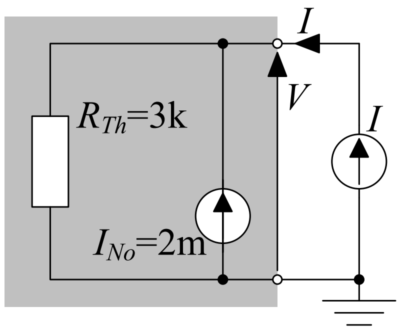
$$V = \frac{7}{5}I - \frac{3}{5} = R_{Th}I + V_{Th}$$

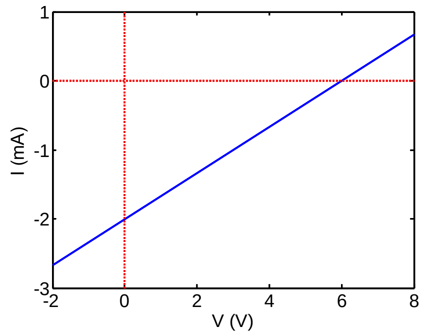


Norton Theorem: Any two-terminal network consisting of resistors, fixed voltage/current sources and linear dependent sources is externally equivalent to a circuit consisting of a resistor in parallel with a fixed current source.

KCL: $-I - I_{No} + \frac{V}{R_{Th}} = 0$
$\Leftrightarrow I = \frac{1}{R_{Th}} V - I_{No}$

c.f. Thévenin (slide 5-4):
Same R and $I_{No} = \frac{V_{Th}}{R_{Th}}$





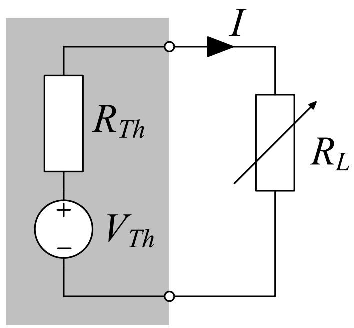
Open Circuit Voltage: If $I = 0$ then $V_{OC} = I_{No} R_{Th}$.

Short Circuit Current: If $V = 0$ then $I_{SC} = -I_{No}$

Thévenin Resistance: The slope of the characteristic is $\frac{1}{R_{Th}}$.

Easy to change between Norton and Thévenin: $V_{Th} = I_{No} R_{Th}$.
Usually best to use Thévenin for small R_{Th} and Norton for large R_{Th} compared to the other impedances in the circuit.

Suppose we connect a variable resistor, R_L, across a two-terminal network. From Thévenin's theorem, even a complicated network is equivalent to a voltage source and a resistor.



We know $I = \frac{V_{Th}}{R_{Th}+R_L}$

$\Rightarrow$ power in R_L is $P_L = I^2 R_L = \frac{V_{Th}^2 R_L}{(R_{Th}+R_L)^2}$

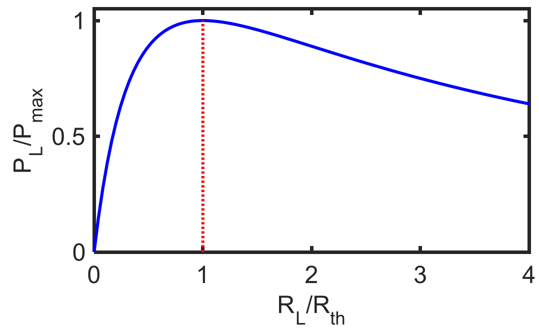
To find the R_L that maximizes P_L:

$$0 = \frac{dP_L}{dR_L} = \frac{(R_{Th}+R_L)^2 V_{Th}^2 - 2V_{Th}^2 R_L (R_{Th}+R_L)}{(R_{Th}+R_L)^4}$$

$$= \frac{V_{Th}^2 (R_{Th}+R_L) - 2V_{Th}^2 R_L}{(R_{Th}+R_L)^3}$$

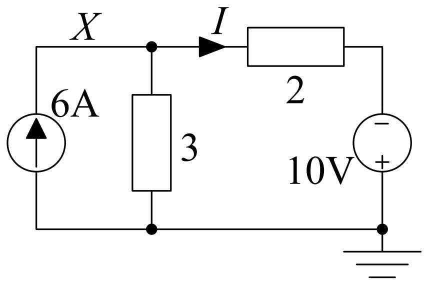
$$\Rightarrow V_{Th}^2 \left((R_{Th}+R_L) - 2R_L\right) = 0$$

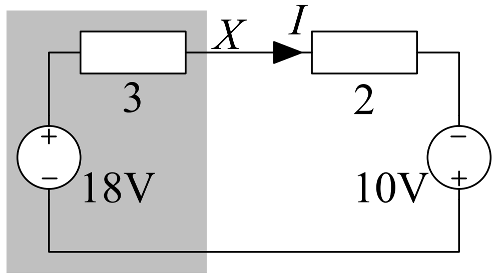
$$\Rightarrow R_L = R_{Th} \;\Rightarrow\; P_{(max)} = \frac{V_{Th}^2}{4R_{Th}}$$



For fixed R_{Th}, the maximum power transfer is when $R_L = R_{Th}$ ("*matched load*").

Sometimes changing between Thévenin and Norton can simplify a circuit.
Suppose we want to calculate I.





Norton → Thévenin on current source: $I = \frac{18-(-10)}{5} = 5.6 \text{ A}$

If you can't spot any clever tricks, you can always find out everything with nodal analysis.

$$-6 + \frac{X}{3} + \frac{X-(-10)}{2} = 0$$

$$\Rightarrow \quad 5X = 36 - 30 = 6$$

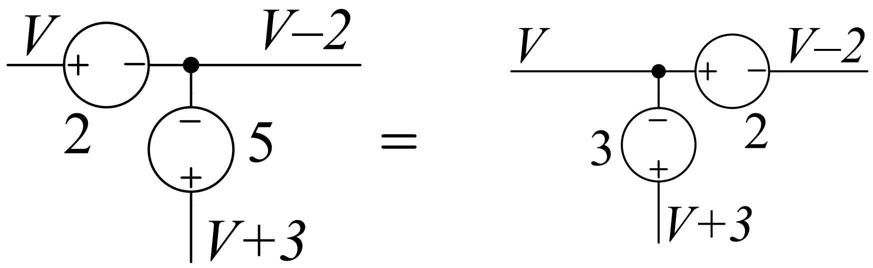
$$\Rightarrow \quad X = 1.2$$

$$\Rightarrow \quad I = \frac{X-(-10)}{2} = 5.6$$

If all but one branches connecting to a node are voltage sources or are current sources, you can choose any of the branches to be the sourceless one.

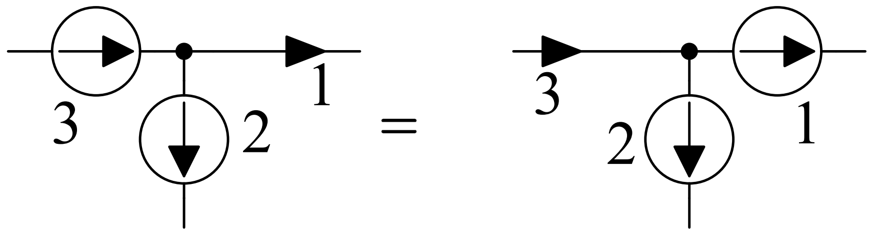
Voltage Sources:

We can use the left node as the reference



Current Sources:

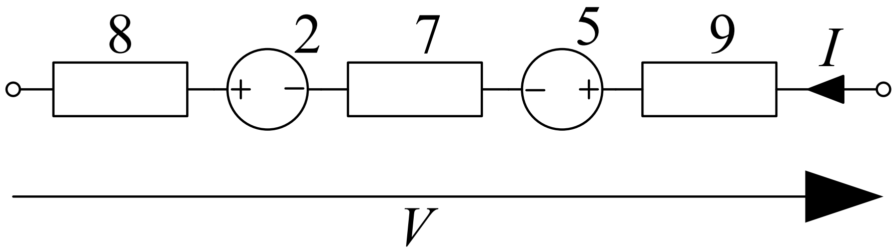
KCL gives current into rightmost node



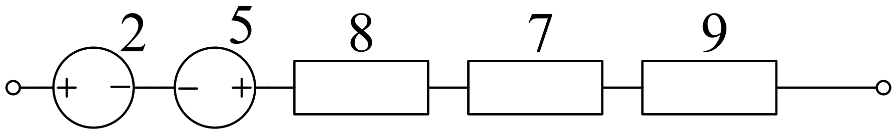
Series Rearrangement

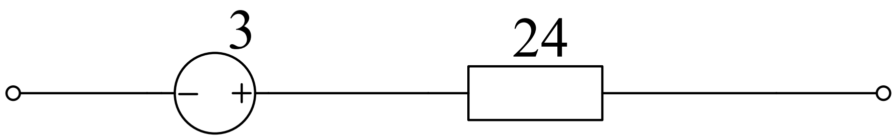
If we have any number of voltage sources and resistors in series we can calculate the total voltage across the chain as:

$$V = 8I - 2 + 7I + 5 + 9I = (-2+5) + (8+7+9)I$$
$$= 3 + 24I$$



We can arbitrarily rearrange the order of the components without affecting $V = 3 + 24I$.





If we move all the voltage sources together and all the resistors together we can merge them and then we get the Thévenin equivalent.

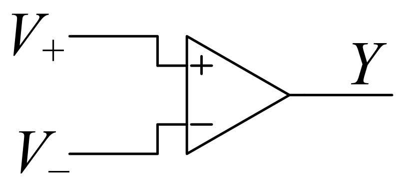
- Thévenin and Norton Equivalent Circuits
  - A network has Thévenin and Norton equivalents if:
    - only 2 terminals connect it to the outside world
    - it is made of resistors + sources + linear dependent sources
  - How to determine V_{Th}, I_{No} and R_{Th}
    - Method 1: Connect current source $\rightarrow$ Nodal analysis
    - Method 2: Find any two of:
      (a) $V_{OC} = V_{Th}$, the open-circuit voltage
      (b) $I_{SC} = -I_{No}$, the short-circuit current
      (c) R_{Th}, equivalent resistance with all sources set to zero
    - Related by Ohm's law: $V_{Th} = I_{No} R_{Th}$
- Load resistor for maximum power transfer $= R_{Th}$
- Source Transformation and Rearrangement

For further details see Hayt Ch 5 & A3 or Irwin Ch 5.

6: Operational Amplifiers

Operational Amplifier

An op amp (operational amplifier) is a circuit with two inputs and one output.

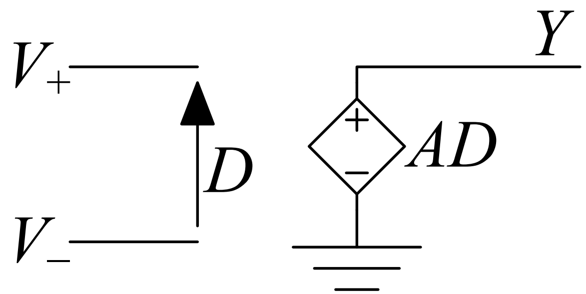


$$Y = A\,(V_+ - V_-)$$

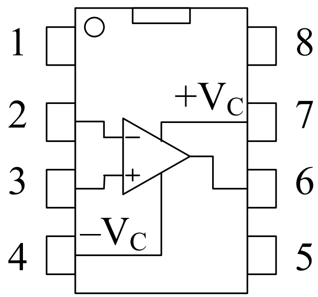
The gain, A, is usually very large: e.g. $A = 10^5$ at low frequencies.

The input currents are very small: e.g. ± 1 nA.

Internally it is a complicated circuit with about 40 components, but we can forget about that and treat it as an almost perfect dependent voltage source.



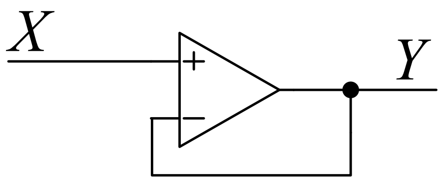
Integrated circuit pins are numbered anti-clockwise from blob or notch (when looking from above).



In a central heating system, if the temperature falls too low the thermostat turns on the heating, when it rises the thermostat turns it off again.

Negative feedback is when the occurence of an event causes something to happen that counteracts the original event.

If op-amp output Y *falls* then V_- will fall by the same amount so $(V_+ - V_-)$ will increase. This causes Y to *rise* since $Y = A(V_+ - V_-)$.



$$Y = A(X - Y)$$

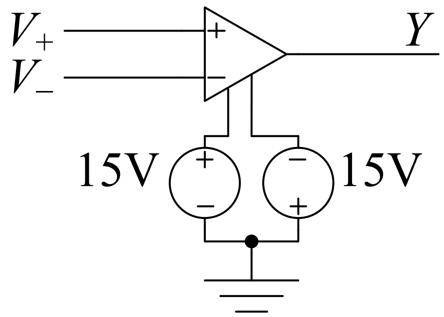
$$Y(1 + A) = AX \quad \Rightarrow \quad Y = \frac{1}{1+{}^1/_A}X \quad \to X \text{ for large } A$$

If $Y = A(V_+ - V_-)$ then $V_+ - V_- = \frac{Y}{A}$ which, since $A \simeq 10^5$, is normally *very very* small.

Golden Rule: Negative feedback adjusts the output to make $V_+ \simeq V_-$.

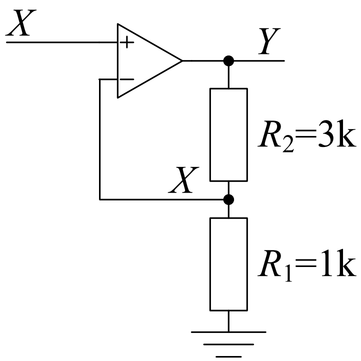
Nodal analysis is simplified by making some assumptions.

Note: The op-amp needs two power supply connections; usually $+15\,\text{V}$ and $-15\,\text{V}$. These are almost always omitted from the circuit diagram. The currents only sum to zero (KCL) if all five connections are included.



1. Check for negative feedback: to ensure that an increase in Y makes $(V_+ - V_-)$ decrease, Y must be connected (usually via other components) to V_-.
2. Assume $V_+ = V_-$: Since $(V_+ - V_-) = \frac{Y}{A}$, this is the same as assuming that $A = \infty$. Requires negative feedback.
3. Assume zero input current: in most circuits, the current at the op-amp input terminals is much smaller than the other currents in the circuit, so we assume it is zero.
4. Apply KCL at each op-amp input node separately (input currents = 0).
5. Do not apply KCL at output node (output current is unknown).

Circuit has input voltage X and output voltage Y. The circuit gain $\triangleq \frac{Y}{X}$.



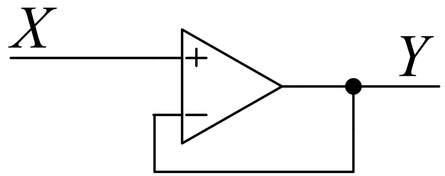
Applying steps 1 to 3:

1. Negative feedback OK.
2. $V_- = V_+ = X$
3. Zero input current at V_- means R_2 and R_1 are in series (⇒ same current) and form a voltage divider. So $X = \frac{R_1}{R_1+R_2}Y$.
   So $Y = \frac{R_1+R_2}{R_1}X = \left(1 + \frac{R_2}{R_1}\right)X = +4X$.

Non-inverting amplifier because the gain $\frac{Y}{X}$ is positive.
Consequence of X connecting to V_+ input.
Can have any gain ≥ 1 by choosing the ratio $\frac{R_2}{R_1}$.

Cause/effect reversal: Potential divider causes $V_- = \frac{1}{4}Y$.
Feedback inverts this so that $Y = 4V_+$.

A special case of the non-inverting amplifier with $R_1 = \infty$ and/or $R_2 = 0$.



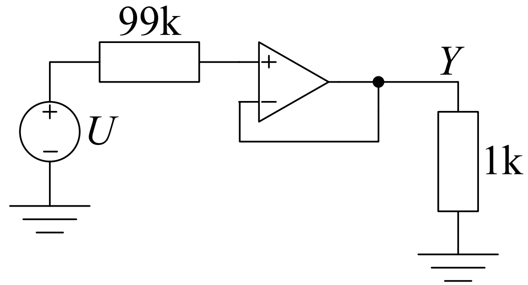
Gain is $1 + \frac{R_2}{R_1} = 1$.

Output Y "follows" input X.

Advantage: Can supply a large current at Y while drawing almost no current from X. Useful if the source supplying X has a high resistance.

Without voltage follower: $Y = 0.01U$.

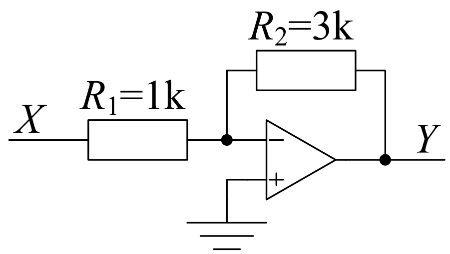
With voltage follower: $Y = U$.



Although the *voltage gain* is only 1, the power gain is much larger.

Negative feedback OK.

Since $V_+ = 0$, we must have $V_- = 0$.



KCL at V_- node: $\frac{0-X}{R_1} + \frac{0-Y}{R_2} = 0 \quad \Rightarrow \quad Y = -\frac{R_2}{R_1}X = -3X$.

Inverting Amplifier because gain $\frac{Y}{X}$ is negative. Consequence of X connecting to the V_- input (via R_1).
Can have any gain ≤ 0 by choosing the ratio $\frac{R_2}{R_1}$.

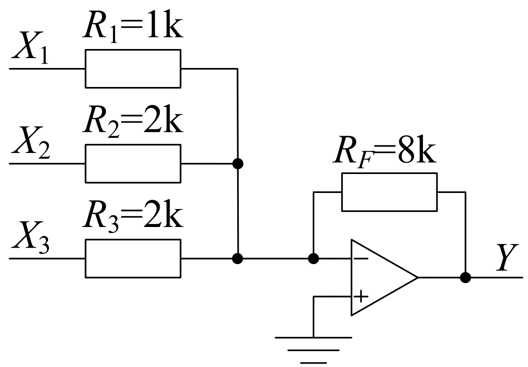
Negative feedback holds V_- very close to V_+.
If $V_+ = 0\,\mathrm{V}$, then V_- is called a *virtual earth* or *virtual ground*.

Nodal Analysis: Do KCL at V_+ and/or V_- to solve circuit. When analysing a circuit, you never do KCL at the output node of an opamp because its output current is unknown. The only exception is if you have already solved the circuit and you want to find out what the op amp output current is (e.g. to check it is not too high).

Inverting Summing Amplifier

We can connect several input signals to the inverting amplifier.

As before, $V_- = 0$ is a virtual earth due to negative feedback and $V_+ = 0$.



KCL at V_- node: $\frac{0-X_1}{R_1} + \frac{0-X_2}{R_2} + \frac{0-X_3}{R_3} + \frac{0-Y}{R_F} = 0$

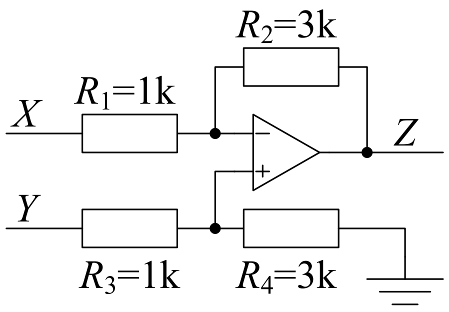
$$\Rightarrow \quad Y = -\left(\frac{R_F}{R_1}X_1 + \frac{R_F}{R_2}X_2 + \frac{R_F}{R_3}X_3\right)$$

$$\Rightarrow \quad Y = -\left(8X_1 + 4X_2 + 4X_3\right).$$

Y is a weighted sum of the input voltages with the weight of X_i equal to $-\frac{R_F}{R_i} = -G_i R_F$.

Input Isolation: The current through R_1 equals $\frac{X_1-0}{R_1}$ which is not affected by X_2 or X_3. Because V_- is held at a fixed voltage, the inputs are isolated from each other.

A 2-input circuit combining inverting and non-inverting amplifiers.



Linearity $\Rightarrow\ Z = aX + bY$.

Use superposition to find a and b.

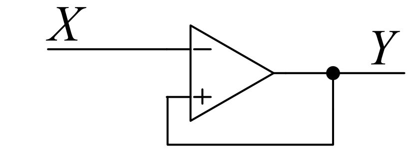
Find a: Set $Y = 0$. KCL at V_+ node $\Rightarrow\ V_+ = 0$. We now have an inverting amplifier, so $Z = -\frac{R_2}{R_1}X = -3X\ \Rightarrow\ a = -3$.

Find b: Set $X = 0$. We can redraw circuit to make it look more familiar: a potential divider followed by a non-inverting amplifier.
R_3 and R_4 are a potential divider (since current into V_+ equals zero), so $V_+ = \frac{R_4}{R_3+R_4}Y = \frac{3}{4}Y$.
The non-inverting amplifier has a gain of $\frac{R_1+R_2}{R_1} = 4$.
The combined gain is $b = \frac{R_4}{R_3+R_4} \times \frac{R_1+R_2}{R_1} = \frac{3}{4} \times 4 = +3$.

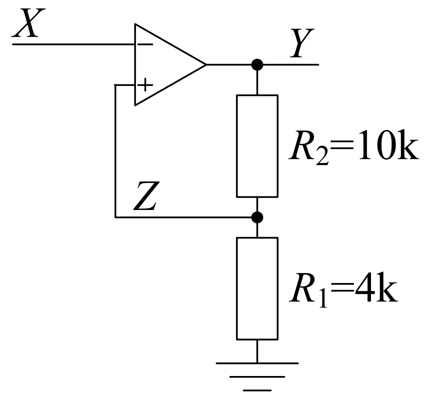
Combining the two gives $Z = 3\,(Y - X)$. The output of a *differential amplifier* is proportional to the diffference between its two inputs.

Schmitt Trigger

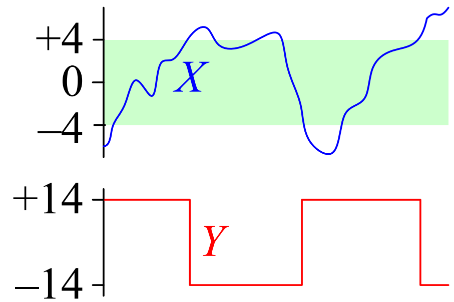
Positive feedback: If op-amp output Y *rises* then $(V_+ - V_-)$ will increase. This causes Y to *rise* even more up to its maximum value (e.g. $+14\,\mathrm{V}$).



If $Y = +14\,\mathrm{V}$, then $Z = 4$. For any $X < 4$, $(V_+ - V_-) > 0$ so the output stays at $+14\,\mathrm{V}$.
If $X > 4$, then $(V_+ - V_-) < 0$, Y will rapidly switch to its minimum value (e.g. $-14\,\mathrm{V}$).
Now $Z = -4$ and Y will only switch back to $+14$ when X falls below -4.



Negative feedback stabilizes the output to make $V_+ \simeq V_-$.
Positive feedback adjusts the output to maximize $|V_+ - V_-|$. Output will switch between its maximum and minimum values, e.g. $\pm 14\,\mathrm{V}$ (slightly less than the $\pm 15\,\mathrm{V}$ power supplies).

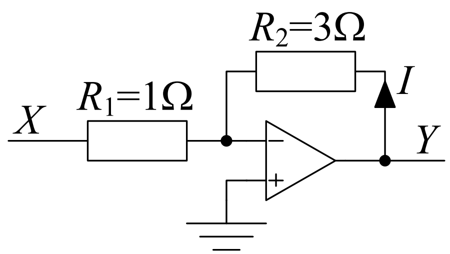


Switching will happen when $V_+ = V_-$.

The behaviour of an op-amp circuit depends on the ratio of resistor values: gain $= -R_2/R_1$. How do you choose between ${}^{3\,\Omega}/_{1\,\Omega}$, ${}^{3\,\mathrm{k}\Omega}/_{1\,\mathrm{k}\Omega}$, ${}^{3\,\mathrm{M}\Omega}/_{1\,\mathrm{M}\Omega}$ and ${}^{3\,\mathrm{G}\Omega}/_{1\,\mathrm{G}\Omega}$?

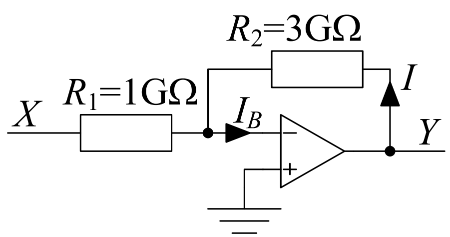
Small resistors cause large currents.

If $X = \pm 1\,\mathrm{V}$, then $Y = \mp 3\,\mathrm{V}$,
and so $I = \frac{Y-0}{R_2} = \mp 1\,\mathrm{A}$.
However typical op-amps can only supply $\pm 5\,\mathrm{mA}$, so the circuit will not work.



Large resistors increase sensitivity to interference and to op-amp input currents.

If the bias current into V_- is $I_B = 1\,\mathrm{nA}$, then KCL at V_- gives



$$\frac{0-Y}{R_2} + \frac{0-X}{R_1} + I_B = 0 \Rightarrow Y = -\frac{R_2}{R_1}X + I_B R_2 = -3X + 3$$

instead of $Y = -3X$.

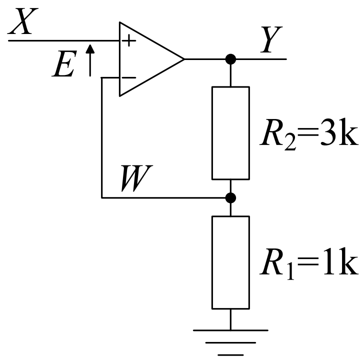
Within wide limits, the absolute resistor values have little effect. However you should avoid extremes.

- Ideal properties:
  - Zero input current
  - Infinite gain
  - Do not use KCL at output (except to determine output current).
- Negative Feedback circuits:
  - Assume $V_+ = V_-$ and zero input current
  - Standard amplifier circuits:
    - Non-inverting gain $= 1 + {}^{R_2}/_{R_1}$
    - Inverting gain $= {}^{-R_2}/_{R_1}$
    - Summing amplifier
    - Differential Amplifier
- Positive feedback circuits:
  - $V_{OUT} = \pm V_{max}$ (no good for an amplifier)
  - Schmitt Trigger: switches when $V_+ = V_-$.
- Choosing resistors: not too low or too high.

For further details see Hayt Ch 6 or Irwin Ch 4.

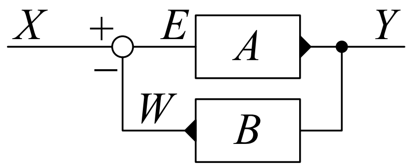
7: Negative Feedback is Wonderful

In the non-inverting op amp circuit we take a fraction of the output signal, Y, and subtract it from the input signal, X.



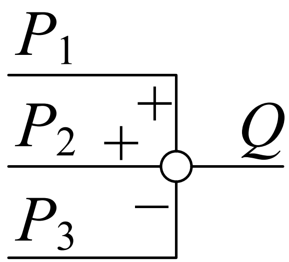
We can represent this using a block diagram:
$A = \frac{Y}{E}$: the gain of the op amp
$B = \frac{W}{Y} = \frac{1}{4}$: gain of the feedback path

The "+" and "–" signs indicate that the feedback is subtracted from X to give an "error" signal, E.



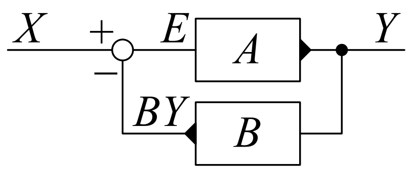
A *gain block* has one input and one output (indicated here by an arrow): $V = A \times U$

An *adder block* many inputs and one output. The signs indicate whether each input is added or subtracted: $Q = P_1 + P_2 - P_3$



Normally, inputs are on the left and outputs are on the right.

- Label inputs, output and adder outputs



- Write down equations for the output and all adder outputs

  $Y = AE$

  $E = X - BY$

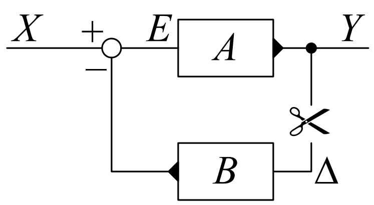
  Never use Kichoff's current law in block diagrams.

- Solve the equations by eliminating unwanted variables

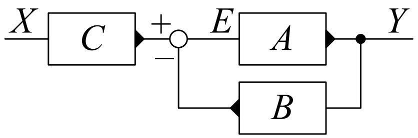
  $Y = AE = A(X - BY) = AX - ABY$

  $\Rightarrow Y(1 + AB) = AX \quad \Rightarrow \quad \frac{Y}{X} = \frac{A}{1+AB}$

AB is called the *loop gain* of the circuit. If you break the loop at any point and inject a signal Δ after the break, this will cause the other side of the break to change by $-\Delta \times AB$.



Sometimes we have an additional block at the input shown here as C.

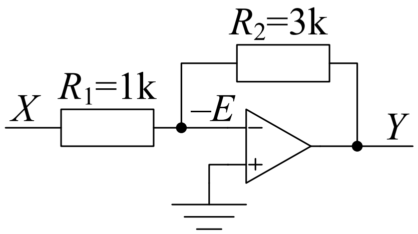


We see that $E = CX - BY$ and, as before,
$Y = AE$

Eliminating E : $\frac{Y}{X} = \frac{CA}{1+AB} = \frac{C}{A^{-1}+B} \approx \frac{C}{B}$ provided $A^{-1} \ll B$.

$\frac{Y}{X}$ equals the forward gain, CA, divided by the loop gain plus one.

Inverting Amplifier



Error signal is $E \triangleq V_+ - V_-$

Hence $V_+ = 0 \quad \Rightarrow \quad V_- = -E$

Op-amp output is $Y = AE$ where $A \approx 10^5$ is the op-amp gain.

Use superposition, nodal analysis or weighted average formula to find an expression for $-E$ in terms of X and Y:

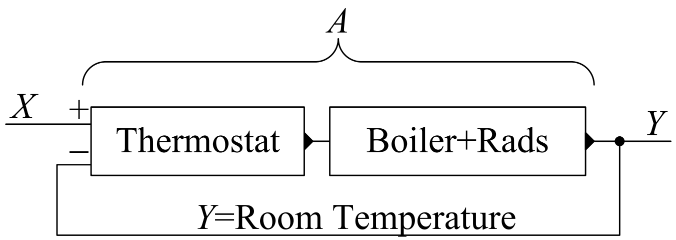
$$-E = \frac{\frac{1}{1}X + \frac{1}{3}Y}{\frac{1}{1}+\frac{1}{3}} = \frac{3}{4}X + \frac{1}{4}Y = -(CX - BY)$$

Hence $C = -\frac{3}{4}$ and $B = +\frac{1}{4}$ and $\frac{Y}{X} \approx \frac{C}{B} = -3$

Negative Feedback Examples

Central Heating:

X: Desired temperature
Y: Actual room temperature
A: Rather complicated system of boiler and radiators



Steam Engine Governor:

X: Desired Speed
Y: Actual Speed
A: Rotational speed causes weights to fly apart (centrifugal force) which adjusts the steam supply via a throttle valve.

Many Other Examples:

Economics: Demand↑ ⇒Price↑ ⇒Supply↑ ⇒Supply=Demand
Biology: More rabbits ⇒Not enough food ⇒Less rabbits ⇒Enough food

1) Gain Stabilization

The gain of a feedback system is almost entirely determined by the feedback path and not by the gain of the amplification path. This means that you can get predictable gains even when the gain of the amplification path is unknown or time-varying.

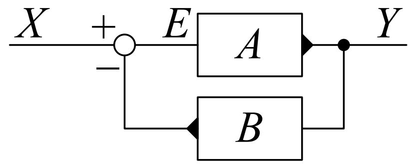
2) Distortion Reduction

High power amplifiers are often non-linear, e.g. their gain decreases at high signal amplitudes. Since the gain of a feedback system does not depend much on the gain of the amplification path, the non-linearity has little effect.

3) Interference Rejection

External disturbances have little effect on the output of a feedback system because the feedback adjusts to compensate for them.

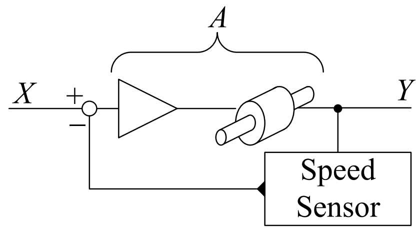
Gain is $\frac{Y}{X} = \frac{A}{1+AB} = \frac{1}{A^{-1}+B}$



If A is very large then $\frac{Y}{X} \approx \frac{1}{B}$ and the precise value of A makes no difference.

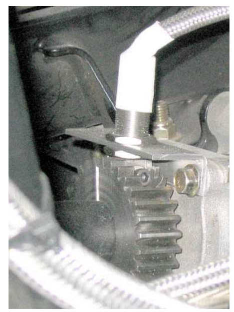
"very large" means $A^{-1} \ll B \Leftrightarrow A \gg \frac{1}{B}$. So as long as A is much larger than the desired gain, its actual value does not matter.
For an op amp $A \approx 10^5$ at low frequencies but less at high frequencies.

Motor Speed Control:



A is the "gain" of the amplifier and motor
(units = rotation speed per volt = rad.$s^{-1}V^{-1}$).
A cannot be precisely known: it depends on mechanical load and friction.

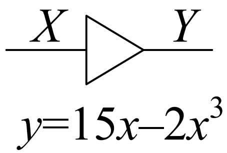
However this is OK so long as it is large enough.

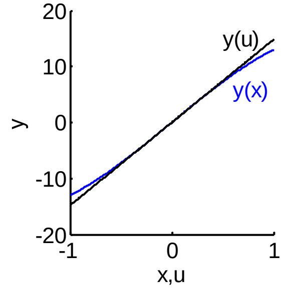


We can sense the motor speed using gear-teeth and a magnetic (Hall effect) sensor together with a circuit that converts frequency to voltage.

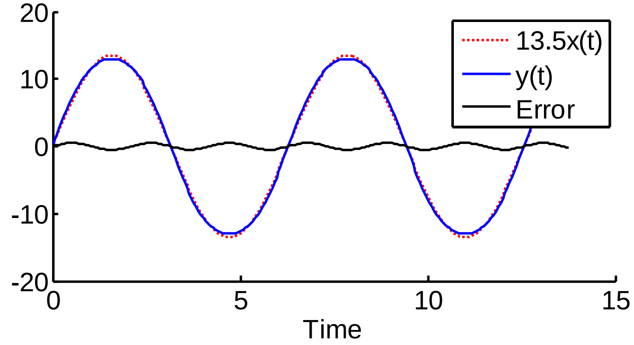
Distortion Reduction

If A includes a high-power amplifier and/or a mechanical system (e.g. a motor) it is almost always non-linear.
$y = 15x - 2x^3$: gain decreases at high $|x|$



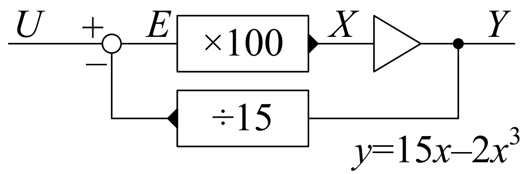


$x = \sin t \;\Rightarrow\; y = 15 \sin t - 2 \sin^3 t$
$\Rightarrow y = 13.5 \sin t + 0.5 \sin 3t$



The gain is only 13.5 instead of 15 and *harmonic distortion* is added at a multiple of the original frequency.
The total harmonic distortion (THD) is equal to $\frac{0.5^2}{13.5^2} = 0.14\%$.

Use feedback to reduce distortion



Put in feedback loop with $\times 100$ gain,
$A = \frac{Y}{E} = 100\frac{Y}{X}$ and $B = \frac{1}{15}$

Even though A depends on the signal amplitude, the gain is $\frac{Y}{U} \approx \frac{1}{B} = 15$.

[Trigonometrical Identities]

The easiest way to derive trigonometrical identities is to use De Moivre's theorem

$$\cos 3t + i\sin 3t = (\cos t + i\sin t)^3 = \cos^3 t + 3i\sin t\cos^2 t - 3\sin^2 t\cos t - i\sin^3 t.$$

Taking the imaginary part of both sides gives

$$\sin 3t = 3\sin t\cos^2 t - \sin^3 t = 3\sin t\left(1 - \sin^2 t\right) - \sin^3 t = 3\sin t - 4\sin^3 t$$

and hence

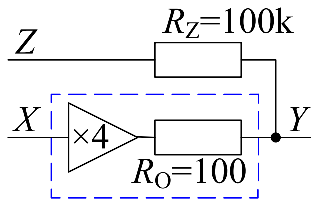
$$\sin^3 t = \tfrac{3}{4}\sin t - \tfrac{1}{4}\sin 3t.$$

Interference Rejection

The amplifier output, Y, is affected by interference, Z.
Y = average of $4X$ and Z weighted by conductances:

$$Y = \frac{\frac{1}{R_O}4X + \frac{1}{R_Z}Z}{\frac{1}{R_O}+\frac{1}{R_Z}} = 3.996X + \frac{1}{1001}Z$$

Z is often much bigger than X (e.g. mains @ 230V).
R_O is amplifier *output resistance*.



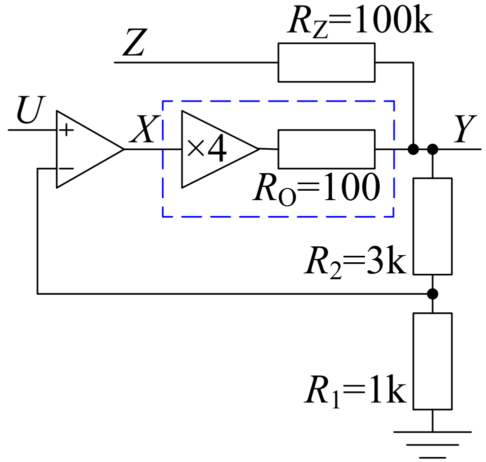
Use feedback to reject interference

Opamp gain $= A \approx 10^5 \Rightarrow X = A\left(U - \frac{Y}{4}\right)$

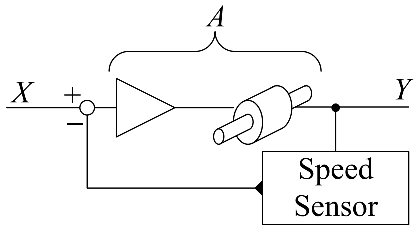
$$Y = \frac{\frac{1}{R_O}4X + \frac{1}{R_Z}Z + \frac{1}{4k}0}{\frac{1}{R_O}+\frac{1}{R_Z}+\frac{1}{4k}} = 3.899X + \frac{1}{1026}Z$$

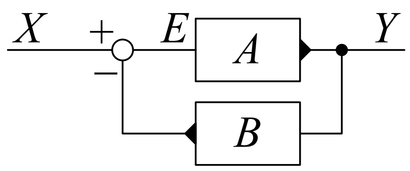
Eliminate X: $Y = 4U + \frac{1}{100001026}Z$

Interference reduced by the loop gain $\approx 10^5$.



"Interference" includes any external influence that may affect the output.
E.g. the mechanical load changing on a motor or an opened window in a heating system.

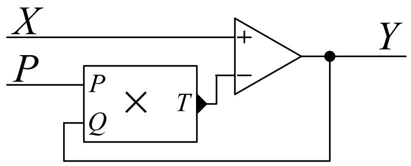




Gain is $\frac{Y}{X} = \frac{A}{1+AB} = \frac{1}{A^{-1}+B} \approx \frac{1}{B}$

If multiplying by B is easier than dividing by B, use feedback to multiply by $\frac{1}{B}$.

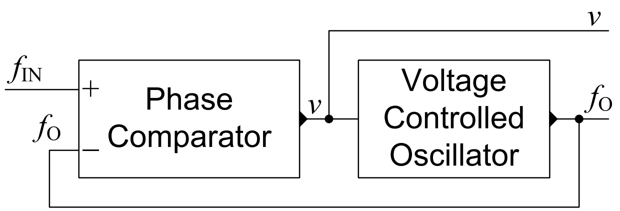
Division Circuit



Multiplier circuit is quite easy to make: $T = P \times Q$

Use in feedback loop to give $Y = \frac{X}{P}$
P must be +ve to ensure negative feedback.

Phase Lock Loop

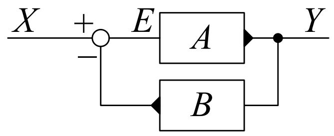


Easy to make a voltage controlled oscillator with $f_O = k \times v$

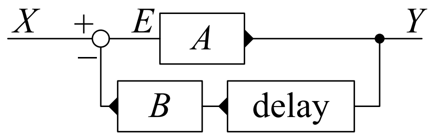
Phase comparator output is $v \propto \int (f_{IN} - f_O)\, dt$ so v increases whenever $f_O < f_{IN}$ and decreases when $f_O > f_{IN}$. When v reaches equilibrium, we must have $f_O = f_{IN}$ so $v = \frac{1}{k} \times f_{IN}$.
We have generated a voltage proportional to the input frequency.
Used in FM radios and in many other circuits.

Instability

The biggest problem of feedback systems is the possibility of instability.



Gain is $\frac{Y}{X} = \frac{A}{1+AB}$. We have four cases:

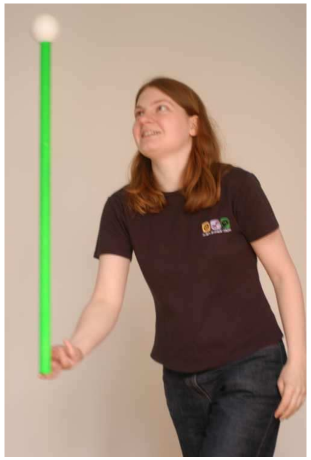


$AB > 0$	Normal: $\frac{Y}{X} \approx \frac{1}{B} < A$
$-1 < AB < 0$	Increased Gain: $\frac{Y}{X} > A$
$AB = -1$	$\frac{Y}{X} = \infty$
$AB < -1$	Usually saturates or oscillates if $AB > 0$ at DC

Delays are Death

For a sine wave, a delay anywhere within the loop of half a period (e.g. 0.5 ms for 1 kHz) is the same as multiplying by -1. At this frequency the loop gain, AB, is large and negative so the system becomes unstable and oscillates.

Quite a common problem: steering a boat, walking when drunk, balancing a stick.



© Science made simple

Why negative feedback is wonderful:

- The precise value of A does not matter as long as it is big enough because the gain is determined by the feedback, B.
- It makes no difference if A varies with time or with signal amplitude (i.e. A is non-linear).
- The effect of external interference at the output is reduced by the loop gain, AB.
- If making a gain B is easy, you can use feedback to make B^{-1}.

The one thing that can go wrong:

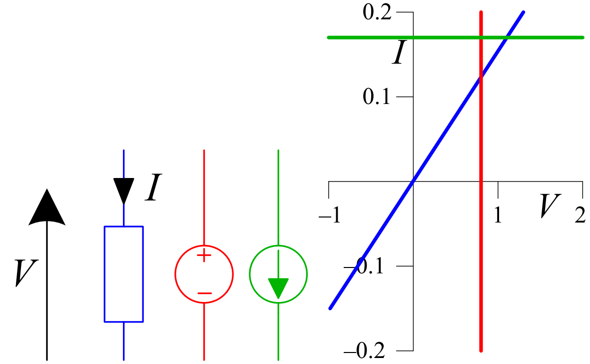
- Phase lags or delays can make a feedback system unstable (oscillate).
- Must make sure that as frequency increases, the loop gain falls below 1 before the phase shift reaches $-180°$.

8: Nonlinear Components

Ideal Diode

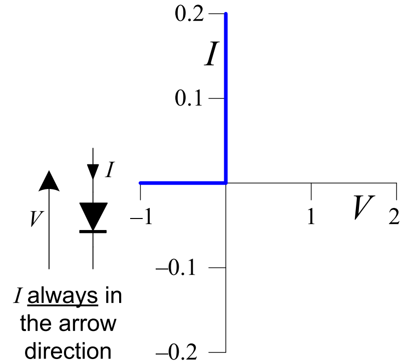
The *characteristic* of a component is a plot of I against V using the passive sign convention.

All our components have had straight-line characteristics.



An ideal *diode* allows current to flow in one direction only.

Its characteristic is not a straight line, but is made from two straight line segments: *piecewise-linear*. Each segment is a *mode of operation*.



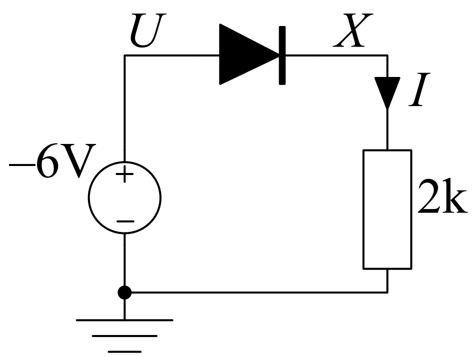
Each mode applies only when a particular condition is true:

Mode	Condition	Equation
Conducting (or "forward bias" or "on")	$I > 0$	$V = 0$
Non-conducting (or "reverse bias" or "off")	$V < 0$	$I = 0$

To analyse a circuit with a diode in it, you first guess which mode it is operating in, solve the circuit and then check the condition.
If you guessed wrongly, the condition will not be met.

Mode	Condition	Equation
Conducting	$I > 0$	$V_D = 0$
Non-conducting	$V_D < 0$	$I = 0$

Voltage across diode is $V_D = U - X$.
Current through diode is $I = \frac{X}{2}\,\mathrm{mA}$.



Assume Conducting Mode $\Rightarrow V_D = 0$

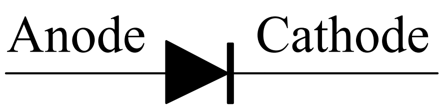
$V_D = 0 \Rightarrow X = U = -6 \Rightarrow I = -3$

but condition is $I > 0$ so bad guess

Assume Non-conducting Mode $\Rightarrow I = 0$

$I = 0 \Rightarrow X = 2I = 0 \Rightarrow V_D = U - X = -6$

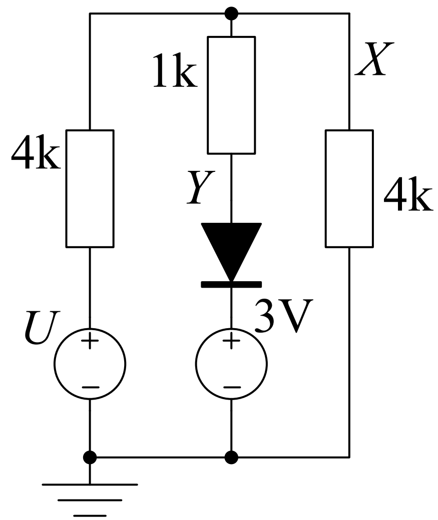
condition is $V_D < 0$ so good guess



Current flows from *anode* to *cathode*.

Switching Point

How does X change with U ?



Voltage across diode is $V_D = Y - 3$.
Current through diode is $I_D = \frac{X-Y}{1}$ mA.

Assume Conducting Mode $\Rightarrow Y = 3$

KCL: $\frac{X-U}{4} + \frac{X-3}{1} + \frac{X}{4} = 0$

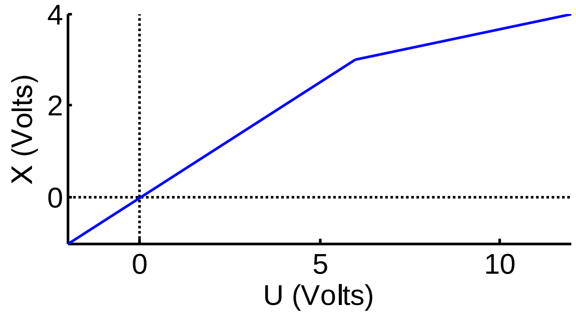
$\Rightarrow X = \frac{1}{6}U + 2$

$I_D = \frac{X-3}{1} = \frac{1}{6}U - 1$

$I_D > 0 \Leftrightarrow U > 6$

Assume Non-conducting Mode

$\Rightarrow I_D = 0$



Potential Div: $X = Y = \frac{1}{2}U$

$V_D = Y - 3 = \frac{1}{2}U - 3$

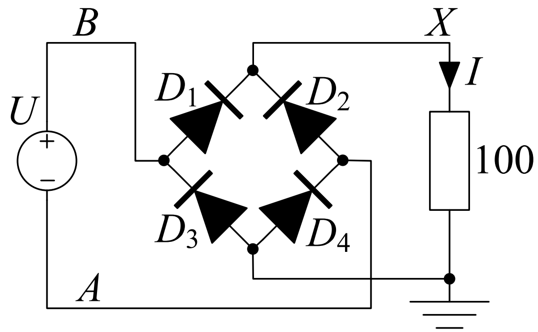
$V_D < 0 \Leftrightarrow U < 6$

Diode switches between regions where the graphs intersect $(U = 6)$.
At this point both the diode equations, $V_D = 0$ and $I_D = 0$, are true.

Bridge Rectifier: 4 diodes:

D_1 and D_2 both point towards node X.
D_3 and D_4 both point away from ground.

The input voltage is $U = B - A$.



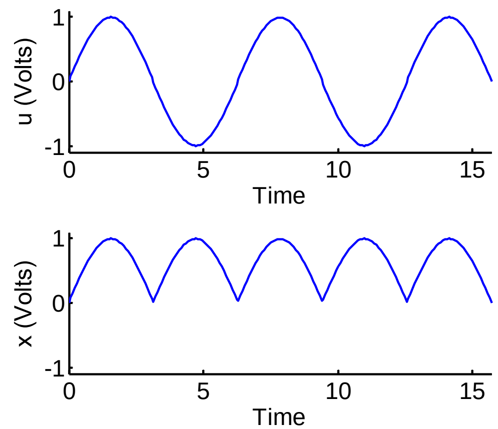
Case 1: $U > 0$. D_1,D_4 on $\Rightarrow X = U$
Check D_1, D_4: $I_1 = I_4 = I = \frac{U}{100} > 0$
Check D_2, D_3: $V_2 = V_3 = -U < 0$
All diodes OK

Case 2: $U < 0$. D_2,D_3 on $\Rightarrow X = -U$
Check D_2, D_3: $I_{2,3} = I = \frac{-U}{100} > 0$
Check D_1, D_4: $V_1 = V_4 = U < 0$
All diodes OK

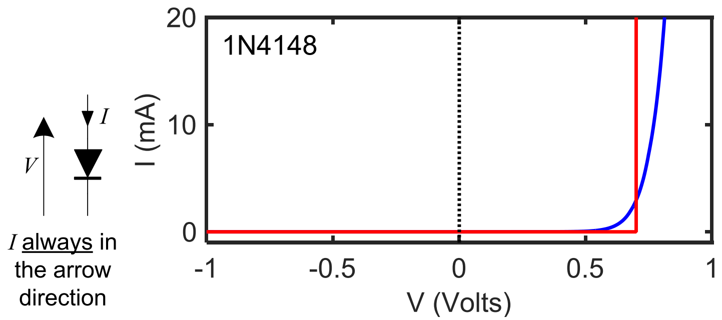
Note: I_n,V_n apply to diode n

X is always equal to $|U|$: this is an absolute value circuit.

If U is a sine wave, then X is a *full-wave rectified* sine wave with twice the frequency.



An *ideal* diode allows has $V = 0$ whenever it is "on".



A *real* diode has a voltage drop that depends approximately logarithmically on the current: it increases by about $0.1\,\text{V}$ for every 50-fold increase in current.

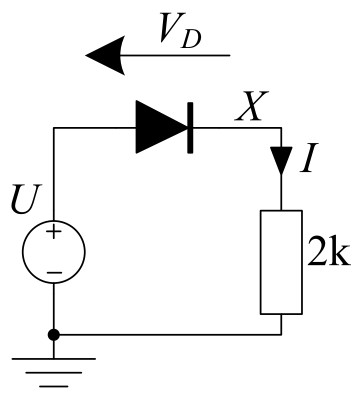
For a wide range of currents we can treat V as almost constant:

(a) For low-current circuits (e.g $I < 20\,\text{mA}$): $V \simeq 0.7\,\text{V}$.
(b) For high-current circuits: $V \simeq 1.0\,\text{V}$.

The two regions of operation are now:

Region	Condition	Equation
Conducting Mode ("on")	$I > 0$	$V = 0.7$
Non-conducting Mode ("off")	$V < 0.7$	$I = 0$

A halfwave rectifier aims for $X = \max(U, 0)$



(a) $U > 0.7$

Diode on, $X = U - 0.7$, $I = \frac{U-0.7}{2\,\text{k}} > 0$

(b) $U < 0.7$

Diode off, $I = 0$, $X = 0$, $V_D = U < 0.7$

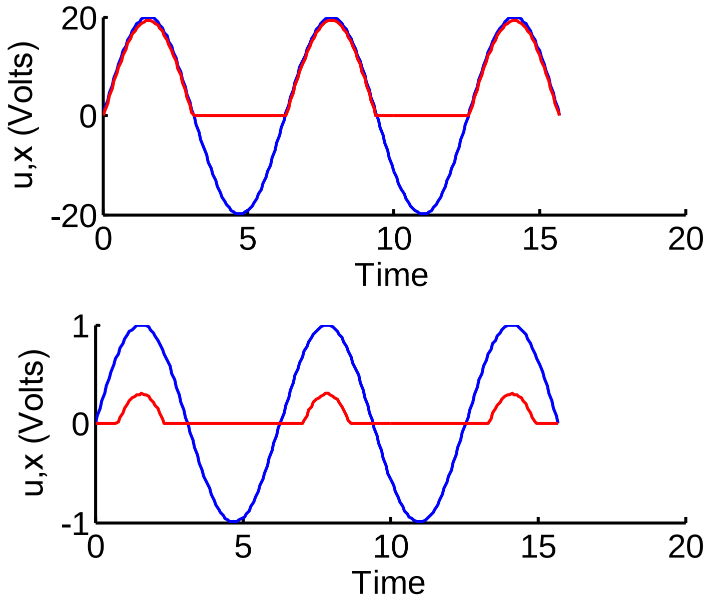
We actually have $X = \max(U - 0.7, 0)$

(1) $u(t) = 20 \sin \omega t$

The $0.7\,\text{V}$ drop makes little difference.

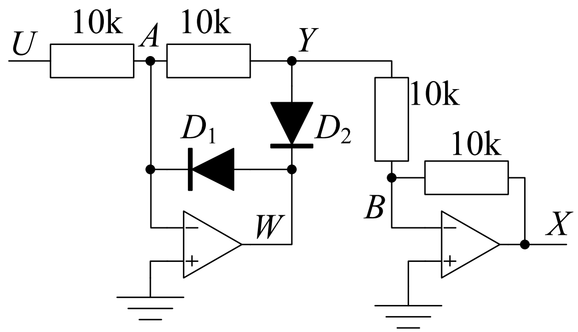
(2) $u(t) = \sin \omega t$

The $0.7\,\text{V}$ drop makes a big difference.



Precision Halfwave Rectifier

Both op-amps have negative feedback, so $A = B = 0$.
Second op-amp is an inverting amplifier so $X = -Y$.



Note: I_n,V_n apply to diode n

Case 1: $U > 0$. D_2 on $\Rightarrow W = Y - 0.7$

KCL @ A: $\frac{0-U}{10} + \frac{0-Y}{10} = 0$

$\Rightarrow Y = -U$

KCL @ Y: $\frac{Y-0}{10} + \frac{Y-0}{10} + I_2 = 0$

$\Rightarrow I_2 = \frac{U}{5} > 0$

Check D_1: $V_1 = -U - 0.7 < 0.7$

Both diodes OK

Output: $X = -Y = U$

Case 2: $U < 0$. D_1 on $\Rightarrow W = 0.7$

KCL @ Y: $\frac{Y-0}{10} + \frac{Y-0}{10} = 0 \Rightarrow Y = 0$

KCL @ A: $\frac{0-U}{10} + \frac{0-0}{10} + -I_1 = 0$

$\Rightarrow I_1 = -\frac{U}{10} > 0$

Check D_2: $V_2 = Y - W = -0.7 < 0.7$

Both diodes OK

Output: $X = -Y = 0$

So $X = \max(U, 0)$

Putting diodes in a feedback loop allows their voltage drops to be eliminated.

Summary

- Beware: a nonlinear circuit does not obey superposition
- Ideal diode:
  - Two regions of operation:
    - ▷ Conducting Mode (= "on"): $V = 0$ and $I > 0$
    - ▷ Non-conducting Mode (= "off"): $I = 0$ and $V < 0$
- Solving a diode circuit:
  - (a) Guess region
  - (b) Solve circuit: assuming $V = 0$ or $I = 0$
  - (c) Check condition: either $I > 0$ or $V < 0$
- Real diode: $V \simeq 0.7$ in Conducting Mode ($\simeq 1.0$ for high currents)
- Fullwave and halfwave rectifier circuits
- Precision Rectifier Circuit
  - Use an opamp to eliminate the $0.7\,\text{V}$ diode drop.

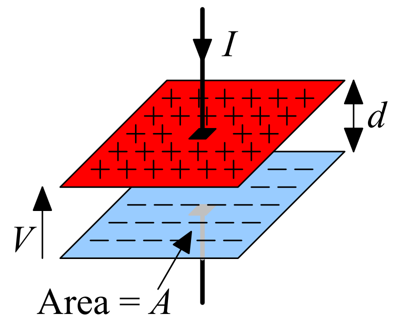
For further details see Irwin Ch 17.

9: Capacitors and Inductors

Capacitors

A capacitor is formed from two conducting plates separated by a thin insulating layer.

If a current i flows, positive change, q, will accumulate on the upper plate. To preserve charge neutrality, a balancing negative charge will be present on the lower plate.



There will be a potential energy difference (or voltage v) between the plates proportional to q.
$v = \frac{d}{A\epsilon} q$ where A is the area of the plates, d is their separation and ϵ is the permittivity of the insulating layer ($\epsilon_0 = 8.85\, {}^{pF}/_{\mathrm{m}}$ for a vacuum).

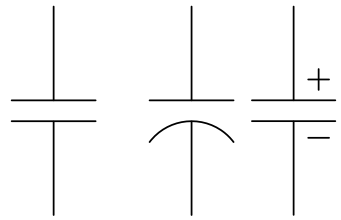
The quantity $C = \frac{A\epsilon}{d}$ is the *capacitance* and is measured in Farads (F), hence $q = Cv$.

The current, i, is the rate of charge on the plate, hence the

capacitor equation: $i = \frac{dq}{dt} = C\frac{dv}{dt}$.

Types of Capacitor

Capacitor symbol represents the two separated plates. Capacitor types are distinguished by the material used as the insulator.



Polystyrene: Two sheets of foil separated by a thin plastic film and rolled up to save space. Values: $10\,\mathrm{pF}$ to $1\,\mathrm{nF}$.

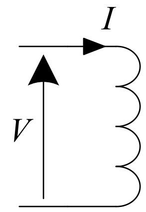
Ceramic: Alternate layers of metal and ceramic (a few $\mu\mathrm{m}$ thick). Values: $1\,\mathrm{nF}$ to $1\,\mu\mathrm{F}$.

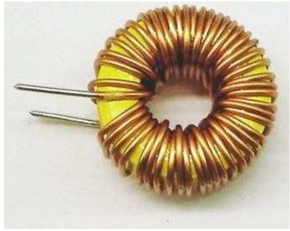
Electrolytic: Two sheets of aluminium foil separated by paper soaked in conducting electrolyte. The insulator is a thin oxide layer on one of the foils. Values: $1\,\mu\mathrm{F}$ to $10\,\mathrm{mF}$.

Electrolytic capacitors are polarised: the foil with the oxide layer must always be at a positive voltage relative to the other (else explosion). Negative terminal indicated by a curved plate in symbol or “-”.

Inductors

Inductors are formed from coils of wire, often around a steel or ferrite core.





The magnetic flux within the coil is $\Phi = \frac{\mu N A}{l} i$ where N is the number of turns, A is the cross-sectional area of the coil and l is the length of the coil (around the toroid).

μ is a property of the material that the core is made from and is called its *permeability*. For free space (or air): $\mu_0 = 4\pi \times 10^{-7} = 1.26\,{}^{\mu H}/\mathrm{m}$, for steel, $\mu \approx 4000\mu_0 = 5\,{}^{mH}/\mathrm{m}$.

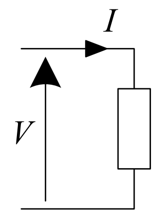
From Faraday's law: $v = N\frac{d\Phi}{dt} = \frac{\mu N^2 A}{l}\frac{di}{dt} = L\frac{di}{dt}$.

We measure the *inductance*, $L = \frac{\mu N^2 A}{l}$, in Henrys (H).

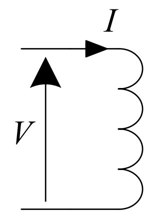
Passive Components

We can describe all three types of passive component by the relationship between V and I using, in each case, the passive sign convention.

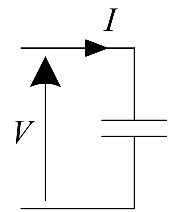
Resistor: $v = Ri$



Inductor: $v = L\frac{di}{dt}$



Capacitor: $i = C\frac{dv}{dt}$



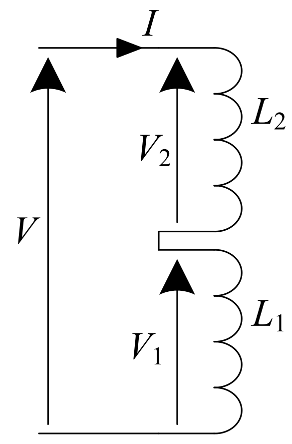
Notes: (1) There are no minus signs anywhere whatever you were taught at school.

(2) We use lower case, v, for time-varying voltages.

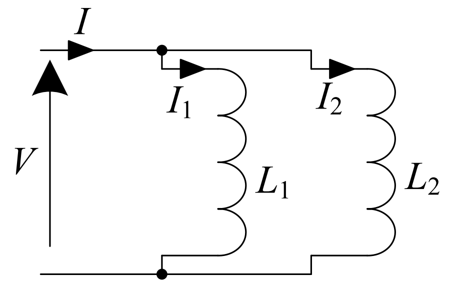
Series and Parallel Inductors

$$v = v_1 + v_2 = L_1\frac{di}{dt} + L_2\frac{di}{dt}$$
$$= (L_1 + L_2)\frac{di}{dt}$$

Same equation as a single inductor of value $L_1 + L_2$



$$\frac{di}{dt} = \frac{d(i_1+i_2)}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$
$$= \frac{v}{L_1} + \frac{v}{L_2} = v\left(\frac{1}{L_1} + \frac{1}{L_2}\right)$$
$$v = \frac{1}{\frac{1}{L_1}+\frac{1}{L_2}}\frac{di}{dt}$$

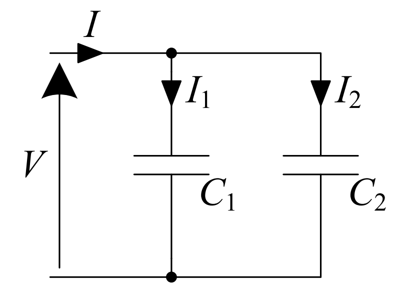


Same as a single inductor of value $\frac{1}{\frac{1}{L_1}+\frac{1}{L_2}} = \frac{L_1L_2}{L_1+L_2}$

Inductors combine just like resistors.

Series and Parallel Capacitors

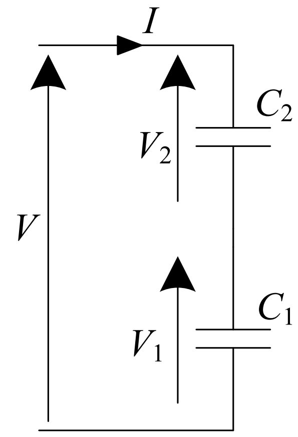
$$i = i_1 + i_2 = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt}$$
$$= (C_1 + C_2) \frac{dv}{dt}$$



Same equation as a single capacitor of value $C_1 + C_2$

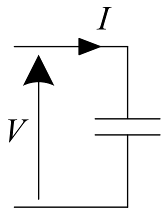
$$\frac{dv}{dt} = \frac{d(v_1+v_2)}{dt} = \frac{dv_1}{dt} + \frac{dv_2}{dt}$$
$$= \frac{i}{C_1} + \frac{i}{C_2} = i\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$
$$i = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} \frac{dv}{dt}$$

Same as a single capacitor of value $\frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} = \frac{C_1 C_2}{C_1+C_2}$



Capacitors combine just like conductances (i.e. parallel capacitors add).

Capacitor: $i = C\frac{dv}{dt}$



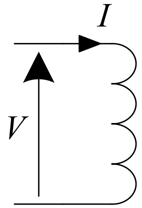
For the voltage to change abruptly

$$\frac{dv}{dt} = \infty \Rightarrow i = \infty.$$

This never happens so ...

The voltage across a capacitor never changes instantaneously.
Informal version: A capacitor "tries" to keep its voltage constant.

Inductor: $v = L\frac{di}{dt}$



For the current to change abruptly

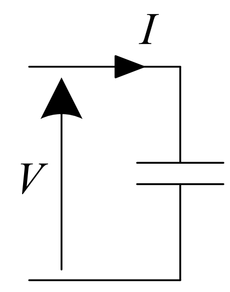
$$\frac{di}{dt} = \infty \Rightarrow v = \infty.$$

This never happens so ...

The current through an inductor never changes instantaneously.
Informal version: An inductor "tries" to keep its current constant.

Average Current/Voltage

For a capacitor $i = C\frac{dv}{dt}$. Take the average of both sides:

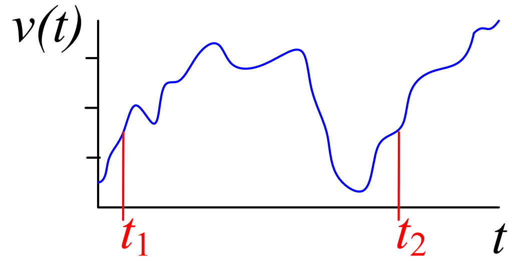


$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} i dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} C\frac{dv}{dt} dt = \frac{C}{t_2 - t_1}\int_{v(t_1)}^{v(t_2)} dv$$

$$= \frac{C}{t_2 - t_1}\left[v\right]_{v(t_1)}^{v(t_2)} = \frac{C}{t_2 - t_1}\left(v(t_2) - v(t_1)\right)$$

(1) If $v(t_1) = v(t_2)$ then the average current exactly equals zero.

(2) If v is bounded then the average current $\to 0$ as $(t_2 - t_1) \to \infty$.



The average current through a capacitor is zero and, likewise, the average voltage across an inductor is zero. The circuit symbols remind you of this.

"Average" can either be over an exact number of periods of a repetitive waveform or else the long-term average (provided v and i remain bounded).

"v is bounded" means $|v|$ always stays less than a predefined maximum value.

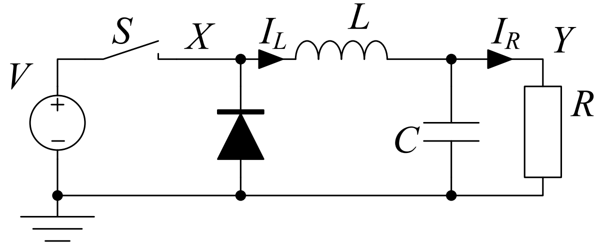
[Do not memorize this circuit]

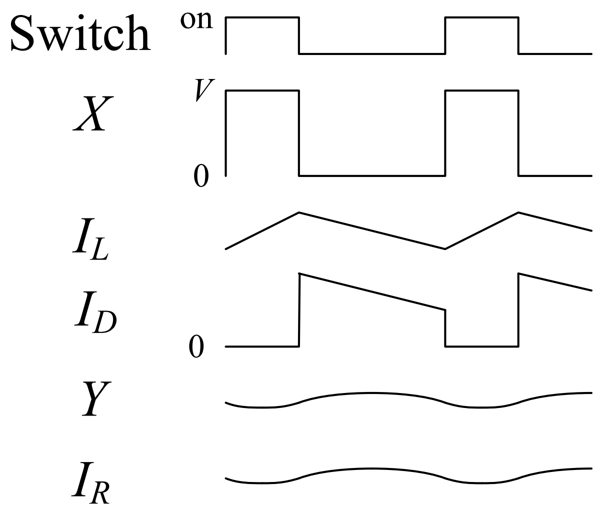
A buck converter converts a high voltage, V, into a lower one, Y.

The switch, S, closes for a fraction a of the time. a is the *duty cycle* and is $\frac{1}{3}$ in this example.

When S is closed, $x = v$, and a current i_L flows.

When S opens, the current i_L cannot change instantly and so it must flow through the diode (we assume the diode is ideal).





The average value of x is aV $\Rightarrow$ the average value of y must also be aV.

The average current through R is $\frac{aV}{R}$ so, since the average current through C must be zero, the average current i_L must also be $\frac{aV}{R}$.

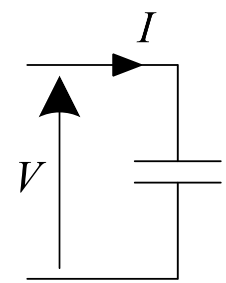
$C\frac{dy}{dt} = i_L - i_R$ $\Rightarrow$ if C is large, then the variations in y will be very small.

Power and Energy

Electrical power absorbed by any component at the instant t is $v(t) \times i(t)$.

So total energy absorbed between times t_1 and t_2 is $W = \int_{t=t_1}^{t_2} vi\, dt$.

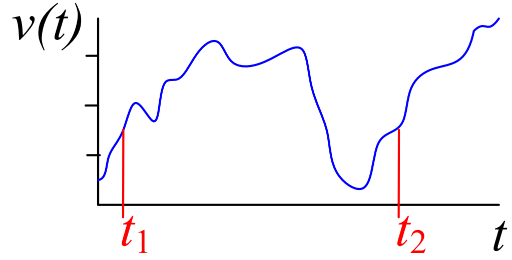
For a capacitor $i = C\frac{dv}{dt}$, so



$$W = C\int_{t=t_1}^{t_2} v\frac{dv}{dt}dt = C\int_{v=v(t_1)}^{v(t_2)} vdv$$

$$= C\left[\tfrac{1}{2}v^2\right]_{v(t_1)}^{v(t_2)} = \tfrac{1}{2}C\left(v^2(t_2) - v^2(t_1)\right)$$

If $v(t_1) = v(t_2)$ then there has been no nett energy absorbed: all the energy absorbed when the voltage rises is returned to the circuit when it falls.



The energy stored in a capacitor is $\frac{1}{2}Cv^2$ and likewise in an inductor $\frac{1}{2}Li^2$.

If v and i remain bounded, then the average power absorbed by a capacitor or inductor is always zero.

Summary

- Capacitor:
  - $i = C\frac{dv}{dt}$
  - parallel capacitors add in value
  - average i is zero, v never changes instantaneously.
  - average power absorbed is zero
- Inductor:
  - $v = L\frac{di}{dt}$
  - series inductors add in value (like resistors)
  - average v is zero, i never changes instantaneously.
  - average power absorbed is zero

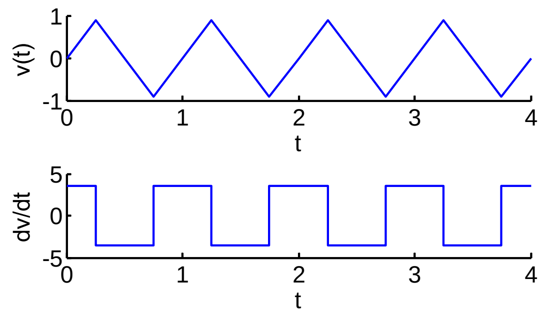
For further details see Hayt Ch 7 or Irwin Ch 6.

10: Sine waves and phasors

For inductors and capacitors $i = C\frac{dv}{dt}$ and $v = L\frac{di}{dt}$ so we need to differentiate $i(t)$ and $v(t)$ when analysing circuits containing them.

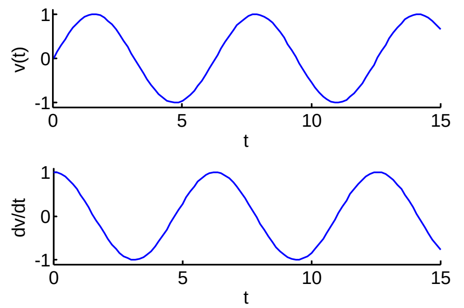
Usually differentiation changes the shape of a waveform.

For bounded waveforms there is only one exception:



$v(t) = \sin t \Rightarrow \frac{dv}{dt} = \cos t$
same shape but with a time shift.

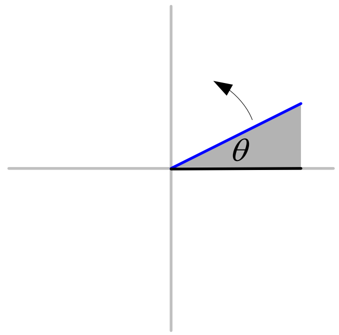
$\sin t$ completes one full period every time t increases by 2π.



$\sin 2\pi f t$ makes f complete repetitions every time t increases by 1; this gives a *frequency* of f cycles per second, or $f\,\text{Hz}$.
We often use the *angular frequency*, $\omega = 2\pi f$ instead.
ω is measured in radians per second. E.g. $50\,\text{Hz} \simeq 314\,\text{rad.s}^{-1}$.

Rotating Rod

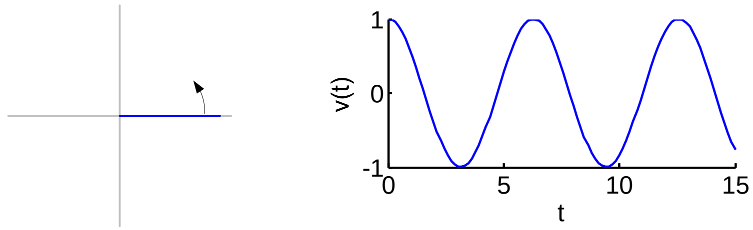
A useful way to think of a cosine wave is as the projection of a rotating rod onto the horizontal axis.

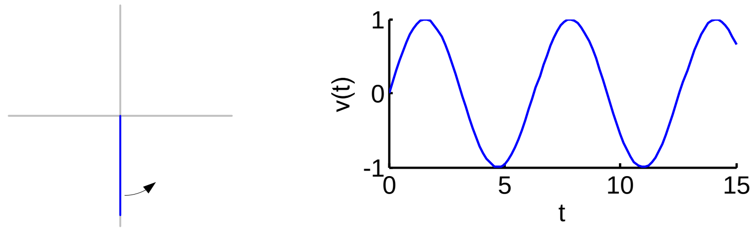


For a unit-length rod, the projection has length $\cos\theta$.

If the rod is rotating at a speed of f revolutions per second, then θ increases uniformly with time: $\theta = 2\pi ft$.

The only difference between $\cos$ and $\sin$ is the starting position of the rod:





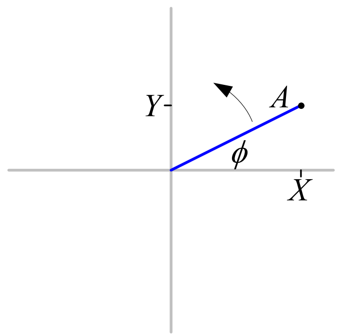
$v = \cos 2\pi ft$ $\qquad$ $v = \sin 2\pi ft = \cos\left(2\pi ft - \frac{\pi}{2}\right)$

$\sin 2\pi ft$ *lags* $\cos 2\pi ft$ by 90° (or $\frac{\pi}{2}$ radians) because its peaks occurs $\frac{1}{4}$ of a cycle later (equivalently $\cos$ *leads* $\sin$) .

Phasors

If the rod has length A and starts at an angle ϕ then the projection onto the horizontal axis is

$$\begin{aligned} &A\cos(2\pi ft+\phi)\\ &\quad = A\cos\phi\cos 2\pi ft - A\sin\phi\sin 2\pi ft\\ &\quad = X\cos 2\pi ft - Y\sin 2\pi ft \end{aligned}$$



At time $t=0$, the tip of the rod has coordinates $(X,\ Y) = (A\cos\phi,\ A\sin\phi)$.

If we think of the plane as an Argand Diagram (or complex plane), then the complex number $X+jY$ corresponding to the tip of the rod at $t=0$ is called a *phasor*.

The *magnitude* of the phasor, $A=\sqrt{X^2+Y^2}$, gives the amplitude (peak value) of the sine wave.

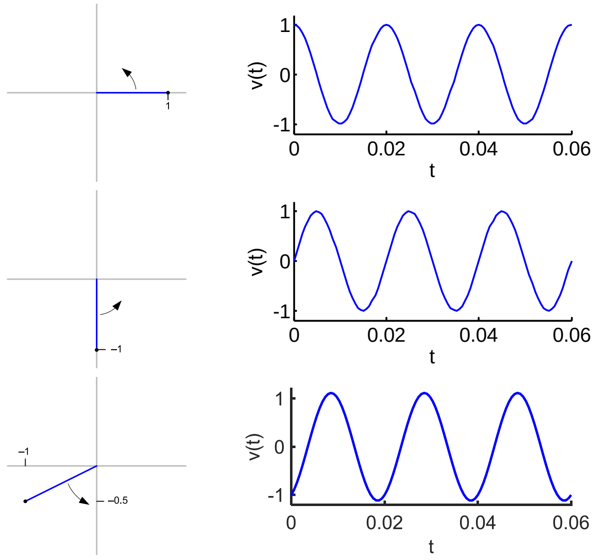
The *argument* of the phasor, $\phi = \arctan\frac{Y}{X}$, gives the phase shift relative to $\cos 2\pi ft$.
If $\phi > 0$, it is *leading* and if $\phi < 0$, it is *lagging* relative to $\cos 2\pi ft$.

Phasor Examples +

$$V = 1,\ f = 50\,\text{Hz}$$
$$v(t) = \cos 2\pi f t$$

$$V = -j$$
$$v(t) = \sin 2\pi f t$$

$$V = -1 - 0.5j = 1.12\angle -153^\circ$$
$$v(t) = -\cos 2\pi f t + 0.5 \sin 2\pi f t$$
$$= 1.12 \cos\left(2\pi f t - 2.68\right)$$



$$V = X + jY$$
$$v(t) = X \cos 2\pi f t - Y \sin 2\pi f t$$
Beware minus sign.

$$V = A\angle\phi = Ae^{j\phi}$$
$$v(t) = A \cos\left(2\pi f t + \phi\right)$$

A phasor represents an entire waveform (encompassing all time) as a single complex number. We assume the frequency, f, is known.

A phasor is not time-varying, so we use a capital letter: V.
A waveform is time-varying, so we use a small letter: $v(t)$.

Casio: $\text{Pol}(X, Y) \to A, \phi$, $\text{Rec}(A, \phi) \to X, Y$. Saved $\to X$ & Y mems.

[Algebraic Phasor↔Waveform Mapping]

A phasor is a complex number, V, that uniquely defines a waveform, $v(t)$, via the mapping $V = Ae^{j\phi} \longleftrightarrow v(t) = A\cos(2\pi ft + \phi)$. It is sometimes convenient to give an algebraic formula for this.

For the direction $V \longrightarrow v(t)$ the mapping is easy:

$$v(t) = \Re\left(Ve^{j2\pi ft}\right) = \tfrac{1}{2}\left(V + V^*\right)\cos 2\pi ft + \tfrac{1}{2}j\left(V - V^*\right)\sin 2\pi ft.$$

The reverse mapping, $V \longleftarrow v(t)$ is a bit more complicated and we use a technique that you will also use in the Maths of Fourier transforms. The mapping is given by

$$V = 2f\int_0^{\frac{1}{f}} v(t)e^{-j2\pi ft}dt.$$

To confrm that this is true, we can substitute $v(t) = A\cos(2\pi ft + \phi)$ and do the integration:

$$\begin{aligned} 2f\int_0^{\frac{1}{f}} v(t)e^{-j2\pi ft}dt &= Af\int_0^{\frac{1}{f}}\left(e^{j(2\pi ft + j\phi} + e^{-j2\pi ft - j\phi}\right)e^{-j2\pi ft}dt \\ &= Af\int_0^{\frac{1}{f}}\left(e^{j\phi} + e^{-j4\pi ft - j\phi}\right)dt = Ae^{j\phi} + Afe^{-j\phi}\int_0^{\frac{1}{f}} e^{-j4\pi ft}dt \\ &= Ae^{j\phi} + \frac{Afe^{-j\phi}}{-j4\pi f}\left[e^{-j4\pi ft}\right]_0^{\frac{1}{f}} = Ae^{j\phi} + \frac{Afe^{-j\phi}}{-j4\pi f}\left(e^{-j4\pi} - 1\right) = Ae^{j\phi} \end{aligned}$$

Phasors	Waveforms
$V = P + jQ$	$v(t) = P\cos\omega t - Q\sin\omega t$ where $\omega = 2\pi f$.
aV	$a \times v(t) \;\; = aP\cos\omega t - aQ\sin\omega t$
$V_1 + V_2$	$v_1(t) + v_2(t)$

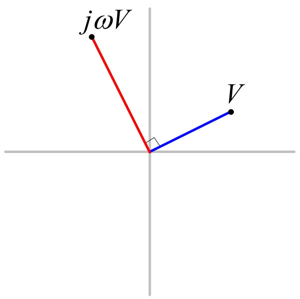
Adding or scaling is the same for waveforms and phasors.

$$\begin{aligned}\dot{V} &= (-\omega Q) + j\,(\omega P)\\ &= j\omega\,(P + jQ)\\ &= j\omega V\end{aligned}$$

$$\begin{aligned}\frac{dv}{dt} &= -\omega P\sin\omega t - \omega Q\cos\omega t\\ &= (-\omega Q)\cos\omega t - (\omega P)\sin\omega t\end{aligned}$$

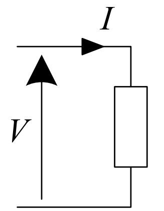
Differentiating waveforms corresponds to multiplying phasors by $j\omega$.

Rotate anti-clockwise 90° and scale by $\omega = 2\pi f$.



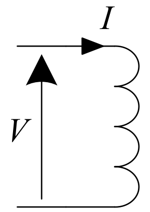
Resistor:

$v(t) = Ri(t) \Rightarrow V = RI \quad \Rightarrow \frac{V}{I} = R$



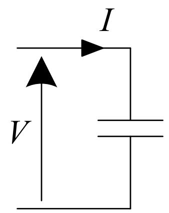
Inductor:

$v(t) = L\frac{di}{dt} \Rightarrow V = j\omega LI \quad \Rightarrow \frac{V}{I} = j\omega L$



Capacitor:

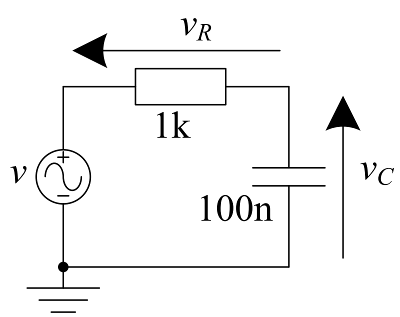
$i(t) = C\frac{dv}{dt} \Rightarrow I = j\omega CV \quad \Rightarrow \frac{V}{I} = \frac{1}{j\omega C}$



For all three components, phasors obey Ohm's law if we use the *complex impedances* $j\omega L$ and $\frac{1}{j\omega C}$ as the "resistance" of an inductor or capacitor.

If all sources in a circuit are sine waves having the same frequency, we can do circuit analysis exactly as before by using complex impedances.

Given $v = 10\sin\omega t$ where $\omega = 2\pi \times 1000$, find $v_C(t)$.

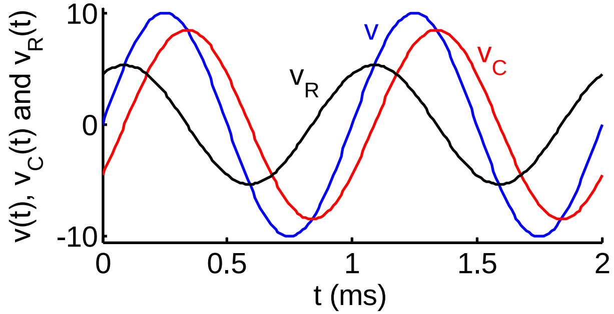


(1) Find capacitor complex impedance

$$Z = \frac{1}{j\omega C} = \frac{1}{6.28j \times 10^{-4}} = -1592j$$

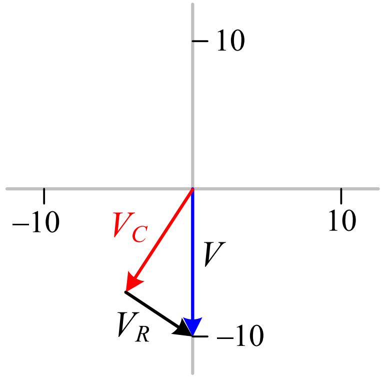
(2) Solve circuit with phasors

$$V_C = V \times \frac{Z}{R+Z}$$
$$= -10j \times \frac{-1592j}{1000-1592j}$$
$$= -4.5 - 7.2j = 8.47\angle -122°$$
$$v_C = 8.47\cos(\omega t - 122°)$$



(3) Draw a *phasor diagram* showing KVL:

$$V = -10j$$
$$V_C = -4.5 - 7.2j$$
$$V_R = V - V_C = 4.5 - 2.8j = 5.3\angle -32°$$



Phasors add like vectors

[Differential Equation Analysis]

To solve the problem form the previous slide without using phasors, we define i to be the current flowing clockwise and use the capacitor equation $i = C\frac{dv_C}{dt}$.

From KVL, we have $v = v_R + v_C = iR + v_C$.
Differentiating and applying the capacitor equation gives $\frac{dv}{dt} = 10\omega\cos\omega t = R\frac{di}{dt} + \frac{1}{C}i$.

We need to find the particular integral for the above equation. To do so, we guess that the answer will be of the form $i = A\cos\omega t + B\sin\omega t$ and substitute it into the equation (multiplied by C).

$$\begin{aligned} 10C\omega\cos\omega t &= RC\left(-A\omega\sin\omega t + B\omega\cos\omega t\right) + \left(A\cos\omega t + B\sin\omega t\right) \\ &= \left(A + RCB\omega\right)\cos\omega t + \left(B - RCA\omega\right)\sin\omega t \end{aligned}$$

which gives two siultaneous equations: $A + RC\omega B = 10C\omega$ and $-RC\omega A + B = 0$. Substituting values for R, C and ω gives $A + 0.628B = 0.00628$ and $-0.628A + B = 0$. Solving these simultaneous equations gives $A = 4.5\,\mathrm{mA}$ and $B = 2.8\,\mathrm{mA}$.

The resistor voltage is therefore $v_R = iR = 4.5\cos\omega t + 2.8\sin\omega t$ and therefore, from KVL, the capacitor votage is $v_C = v - v_R = -4.5\cos\omega t + 7.2\sin\omega t$.

Thus we get the same answer as using phasors but with more work even for a simple circuit like this. For more complicated circuits the difference is much much bigger.

Capacitors: $i = C\frac{dv}{dt} \quad \Rightarrow I$ leads V

Inductors: $v = L\frac{di}{dt} \quad \Rightarrow V$ leads I

Mnemonic: CIVIL = "In a capacitor I lead V but V leads I in an inductor".

COMPLEX ARITHMETIC TRICKS:

(1) $j \times j = -j \times -j = -1$
(2) $\frac{1}{j} = -j$
(3) $a + jb = r\angle\theta = re^{j\theta}$
where $r = \sqrt{a^2 + b^2}$ and $\theta = \arctan\frac{b}{a}$ ($\pm 180°$ if $a < 0$)
(4) $r\angle\theta = re^{j\theta} = (r\cos\theta) + j\,(r\sin\theta)$
(5) $a\angle\theta \times b\angle\phi = ab\angle(\theta + \phi)$ and $\frac{a\angle\theta}{b\angle\phi} = \frac{a}{b}\angle(\theta - \phi)$.
Multiplication and division are much easier in polar form.
(6) All scientific calculators will convert rectangular to/from polar form.

Casio fx-991 (available in all exams except Maths) will do complex arithmetic ($+, -, \times, \div, x^2, \frac{1}{x}, |x|, x^*$) in CMPLX mode.

Learn how to use this: it will save lots of time and errors.

For any network (resistors+capacitors+inductors):

(1) Impedance = Resistance + $j\times$ Reactance

$$Z = R + jX\ (\Omega)$$

$$|Z|^2 = R^2 + X^2 \qquad \angle Z = \arctan\frac{X}{R}$$

(2) Admittance = $\frac{1}{\text{Impedance}}$ = Conductance + $j\times$ Susceptance

$$Y = \frac{1}{Z} = G + jB \text{ Siemens (S)}$$

$$|Y|^2 = \frac{1}{|Z|^2} = G^2 + B^2 \qquad \angle Y = -\angle Z = \arctan\frac{B}{G}$$

Note:

$$Y = G + jB = \frac{1}{Z} = \frac{1}{R+jX} = \frac{R}{R^2+X^2} + j\frac{-X}{R^2+X^2}$$

So $G = \frac{R}{R^2+X^2} = \frac{R}{|Z|^2}$

$$B = \frac{-X}{R^2+X^2} = \frac{-X}{|Z|^2}$$

Beware: $G \neq \frac{1}{R}$ unless $X = 0$.

Summary

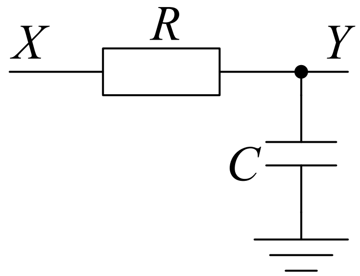
- Sine waves are the only bounded signals whose shape is unchanged by differentiation.
- Think of a sine wave as the projection of a rotating rod onto the horizontal (or real) axis.
  - A *phasor* is a complex number representing the length and position of the rod at time $t = 0$.
  - If $V = a + jb = r\angle\theta = re^{j\theta}$, then
    $$v(t) = a\cos\omega t - b\sin\omega t = r\cos(\omega t + \theta) = \Re\left(Ve^{j\omega t}\right)$$
  - The *angular frequency* $\omega = 2\pi f$ is assumed known.
- If all sources in a linear circuit are sine waves having the same frequency, we can use phasors for circuit analysis:
  - Use complex impedances: $j\omega L$ and $\frac{1}{j\omega C}$
  - Mnemonic: CIVIL tells you whether I leads V or vice versa ("leads" means "reaches its peak before").
  - Phasors eliminate time from equations ☺, converts simultaneous differential equations into simultaneous linear equations ☺☺☺.
  - Needs complex numbers ☹ but worth it.

See Hayt Ch 10 or Irwin Ch 8

11: Frequency Responses

Frequency Response

If $x(t)$ is a sine wave, then $y(t)$ will also be a sine wave but with a different amplitude and phase shift. X is an input phasor and Y is the output phasor.

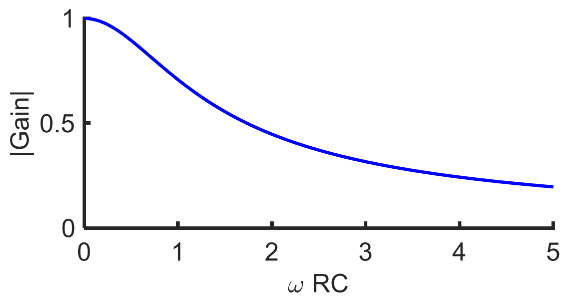


The *gain* of the circuit is $\frac{Y}{X} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{j\omega RC + 1}$

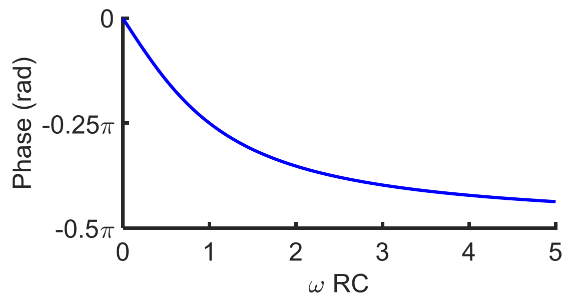
This is a complex function of ω so we plot separate graphs for:

Magnitude: $\left|\frac{Y}{X}\right| = \frac{1}{|j\omega RC+1|} = \frac{1}{\sqrt{1+(\omega RC)^2}}$

Phase Shift: $\angle\left(\frac{Y}{X}\right) = -\angle\left(j\omega RC + 1\right) = -\arctan\left(\frac{\omega RC}{1}\right)$



Magnitude Response

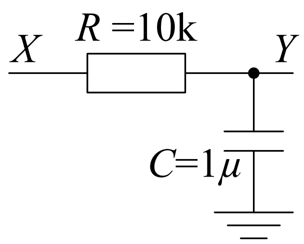


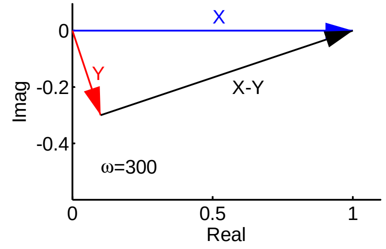
Phase Response

Sine Wave Response

$RC = 10\,\text{ms}$

$\frac{Y}{X} = \frac{1}{j\omega RC+1} = \frac{1}{0.01 j\omega+1}$

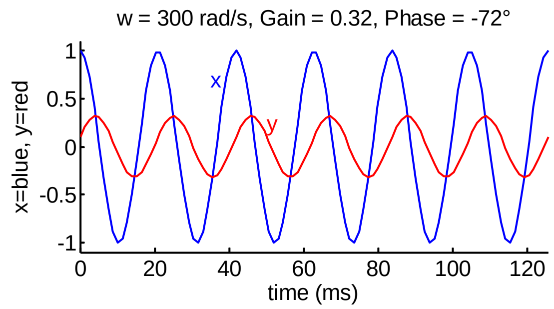


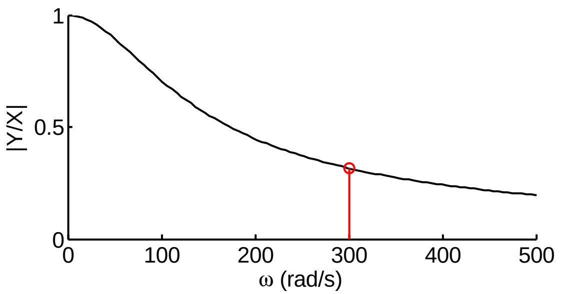


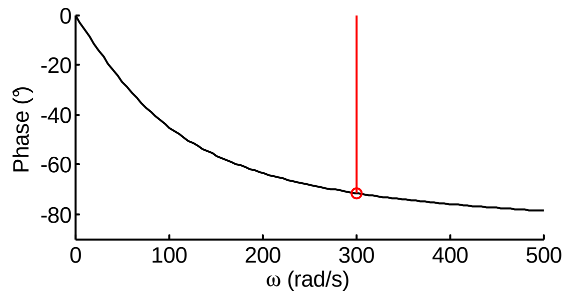
$\omega = 50 \Rightarrow \frac{Y}{X} = 0.89\angle - 27°$

$\omega = 100 \Rightarrow \frac{Y}{X} = 0.71\angle - 45°$

$\omega = 300 \Rightarrow \frac{Y}{X} = 0.32\angle - 72°$







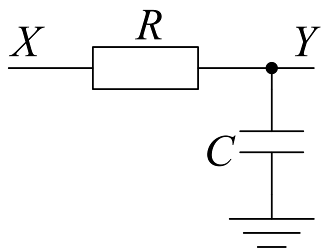
The output, $y(t)$, *lags* the input, $x(t)$, by up to $90°$.

Logarithmic axes

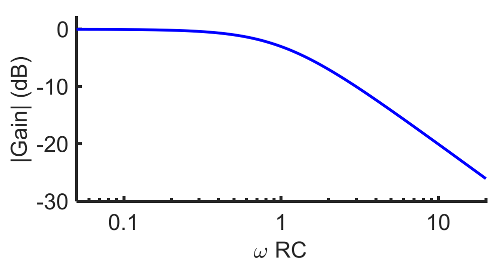
We usually use logarithmic axes for frequency and gain (but not phase) because % differences are more significant than absolute differences.
E.g. $5\,\text{kHz}$ versus $5.005\,\text{kHz}$ is less significant than $10\,\text{Hz}$ versus $15\,\text{Hz}$ even though both differences equal $5\,\text{Hz}$.

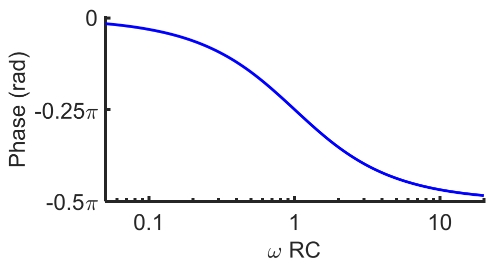
Logarithmic voltage ratios are specified in *decibels* (dB) $= 20\log_{10}\frac{|V_2|}{|V_1|}$.

Common voltage ratios:



$\frac{\lvert V_2\rvert}{\lvert V_1\rvert}$	0.1	0.5	$\sqrt{0.5}$	1	$\sqrt{2}$	2	10	100
dB	-20	-6	-3	0	3	6	20	40





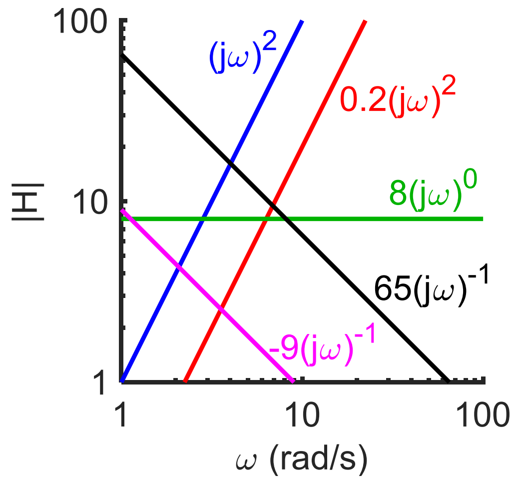
Note that 0 does not exist on a log axis and so the starting point of the axis is arbitrary.

Note: $P \propto V^2 \Rightarrow$ decibel power ratios are given by $10\log_{10}\frac{P_2}{P_1}$

$H = c\,(j\omega)^r$ has a straight-line magnitude graph and a constant phase.

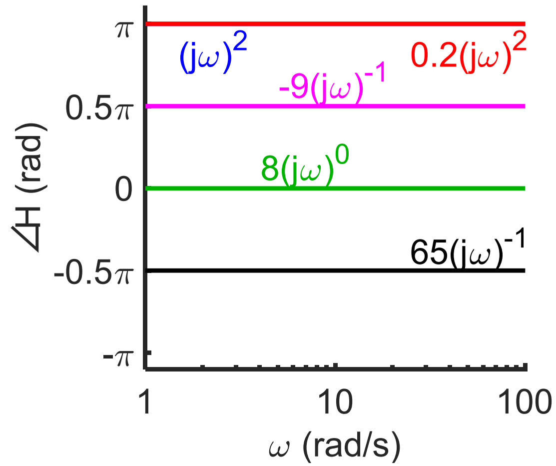
Magnitude (log-log graph):
$|H| = c\omega^r \Rightarrow \log|H| = \log|c| + r\log\omega$
This is a straight line with a slope of r.
c only affects the line's vertical position.

If $|H|$ is measured in decibels, a slope of r is called $6r$ dB/octave or $20r$ dB/decade.



Phase (log-lin graph):
$\angle H = \angle j^r + \angle c = r \times \frac{\pi}{2}$ $(+\pi$ if $c < 0)$
The phase is constant $\forall\omega$.
If $c > 0$, phase $= 90^\circ \times$ magnitude slope.
Negative c adds $\pm 180^\circ$ to the phase.

Note: Phase angles are modulo 360°, i.e. $+180^\circ \equiv -180^\circ$ and $450^\circ \equiv 90^\circ$.



[Octaves and Decades]

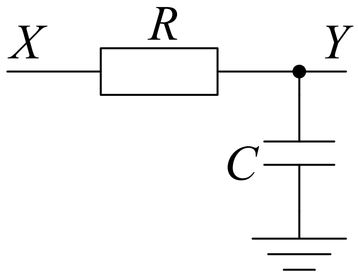
An "octave" is a factor of 2 in frequency; for example, $20\,\mathrm{Hz}$ is one octave greater than $10\,\mathrm{Hz}$. Similarly a "decade" is a factor of 10 in frequency; for example, $100\,\mathrm{Hz}$ is one decade greater than $10\,\mathrm{Hz}$.

The number of decades between any two frequencies can be calculated by taking $\log_{10}$ of the frequency ratio. Thus, for the example given above, $\log_{10}\left(\frac{100\,\mathrm{Hz}}{10\,\mathrm{Hz}}\right) = \log_{10}(10) = 1\,\mathrm{decade}$. A slightly more complicated example is $\log_{10}\left(\frac{13\,\mathrm{kHz}}{25\,\mathrm{Hz}}\right) = \log_{10}\left(\frac{13000}{25}\right) = \log_{10}(520) = 2.716\,\mathrm{decades}$ so this means that $13\,\mathrm{kHz}$ is 2.716 decades greater than $25\,\mathrm{Hz}$.

As we shall discover in this lecture, frequency response graphs can be approximated as a series of straight lines whose gradients are easy to calculate. In particular magnitude response graphs can be approximated as a series of straight lines with gradients that are integer multiples of $20\,\mathrm{dB}$ per decade and phase response graphs can be approximated as a series of straight lines with gradients that are integer multiples of 0.25π radians per decade. This means that if you know the magnitude or phase at one frequency, you can calculate how much it has changed at any other frequency by multiplying the gradient of the line by the number of decades by which the frequency has changed.

Calculating the number of *octaves* between any two frequencies is done in the same way except that you must take a base-2 log. Thus between $10\,\mathrm{Hz}$ and $100\,\mathrm{Hz}$ is $\log_2\left(\frac{100\,\mathrm{Hz}}{10\,\mathrm{Hz}}\right) = \log_{10}\left(\frac{100\,\mathrm{Hz}}{10\,\mathrm{Hz}}\right) \div \log_{10} 2 = 3.322\,\log_{10}\left(\frac{100\,\mathrm{Hz}}{10\,\mathrm{Hz}}\right) = 3.322\,\mathrm{octaves}$. Thus one decade is equal to 3.322 octaves.

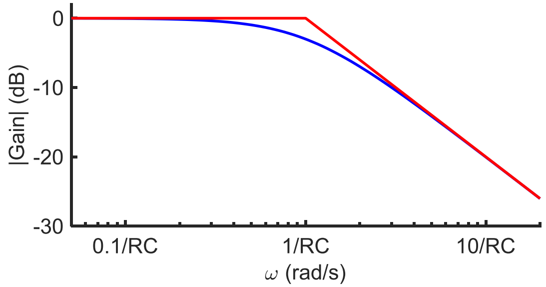
Key idea: $(aj\omega + b) \approx \begin{cases} aj\omega & \text{for } |a\omega| \gg |b| \\ b & \text{for } |a\omega| \ll |b| \end{cases}$



Gain: $H(j\omega) = \frac{1}{j\omega RC+1}$

Low frequencies ($\omega \ll \frac{1}{RC}$): $H(j\omega) \approx 1 \Rightarrow |H(j\omega)| \approx 1$
High frequencies ($\omega \gg \frac{1}{RC}$): $H(j\omega) \approx \frac{1}{j\omega RC} \Rightarrow |H(j\omega)| \approx \frac{1}{RC}\omega^{-1}$

Approximate the magnitude response as two straight lines intersecting at the *corner frequency*, $\omega_c = \frac{1}{RC}$.



At the corner frequency:

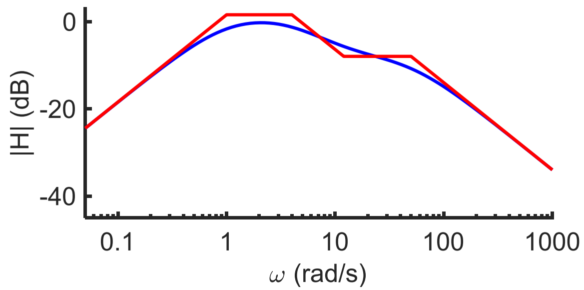
(a) the gradient changes by -1 ($= -6$ dB/octave $= -20$ dB/decade).
(b) $|H(j\omega_c)| = \left|\frac{1}{1+j}\right| = \frac{1}{\sqrt{2}} = -3$ dB (worst-case error).

A linear factor $(aj\omega + b)$ has a corner frequency of $\omega_c = \left|\frac{b}{a}\right|$.

The gain of a linear circuit is always a *rational polynomial* in $j\omega$ and is called the *transfer function* of the circuit. For example:

$$H(j\omega) = \frac{60(j\omega)^2+720(j\omega)}{3(j\omega)^3+165(j\omega)^2+762(j\omega)+600} = \frac{20j\omega(j\omega+12)}{(j\omega+1)(j\omega+4)(j\omega+50)}$$

Step 1: Factorize the polynomials
Step 2: Sort corner freqs: $1, 4, 12, 50$
Step 3: For $\omega < 1$ all linear factors equal their constant terms:
$$|H| \approx \frac{20\omega\times12}{1\times4\times50} = 1.2\omega^1.$$



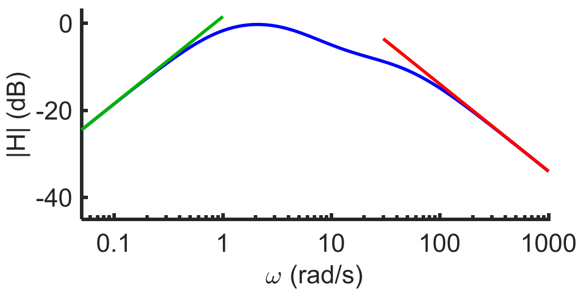
Step 4: For $1 < \omega < 4$, the factor $(j\omega+1) \approx j\omega$ so
$$|H| \approx \frac{20\omega\times12}{\omega\times4\times50} = 1.2\omega^0 = +1.58\,\text{dB}.$$
Step 5: For $4 < \omega < 12$, $|H| \approx \frac{20\omega\times12}{\omega\times\omega\times50} = 4.8\omega^{-1}$.
Step 6: For $12 < \omega < 50$, $|H| \approx \frac{20\omega\times\omega}{\omega\times\omega\times50} = 0.4\omega^0 = -7.96\,\text{dB}$.
Step 7: For $\omega > 50$, $|H| \approx \frac{20\omega\times\omega}{\omega\times\omega\times\omega} = 20\omega^{-1}$.

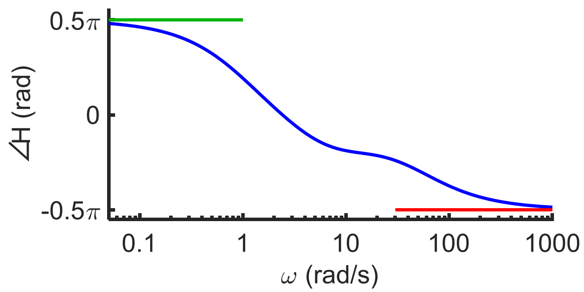
At each corner frequency, the graph is continuous but its gradient changes abruptly by $+1$ (numerator factor) or -1 (denominator factor).

Low and High Frequency Asymptotes

You can find the low and high frequency asymptotes without factorizing:

$$H(j\omega) = \frac{60(j\omega)^2+720(j\omega)}{3(j\omega)^3+165(j\omega)^2+762(j\omega)+600} = \frac{20j\omega(j\omega+12)}{(j\omega+1)(j\omega+4)(j\omega+50)}$$





Low Frequency Asymptote:

From factors: $H_{\mathrm{LF}}(j\omega) = \frac{20j\omega(12)}{(1)(4)(50)} = 1.2j\omega$

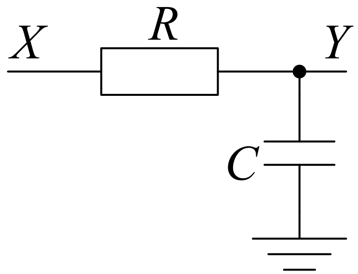
Lowest power of $j\omega$ on top and bottom: $H\left(j\omega\right) \simeq \frac{720(j\omega)}{600} = 1.2j\omega$

High Frequency Asymptote:

From factors: $H_{\mathrm{HF}}(j\omega) = \frac{20j\omega(j\omega)}{(j\omega)(j\omega)(j\omega)} = 20\left(j\omega\right)^{-1}$

Highest power of $j\omega$ on top and bottom: $H\left(j\omega\right) \simeq \frac{60(j\omega)^2}{3(j\omega)^3} = 20\left(j\omega\right)^{-1}$

Gain: $H(j\omega) = \frac{1}{j\omega RC+1}$



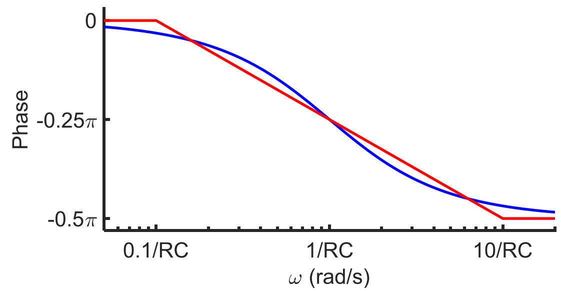
Low frequencies ($\omega \ll \frac{1}{RC}$):

$$H(j\omega) \approx 1 \Rightarrow \angle 1 = 0$$

High frequencies ($\omega \gg \frac{1}{RC}$): $H(j\omega) \approx \frac{1}{j\omega RC} \Rightarrow \angle j^{-1} = -\frac{\pi}{2}$

Approximate the phase response as three straight lines.

By chance, they intersect close to $0.1\omega_c$ and $10\omega_c$ where $\omega_c = \frac{1}{RC}$.



Between $0.1\omega_c$ and $10\omega_c$ the phase changes by $-\frac{\pi}{2}$ over two decades. This gives a gradient $= -\frac{\pi}{4}$ radians/decade.

$(aj\omega + b)$ in denominator

$$\Rightarrow \Delta\text{gradient} = \mp\frac{\pi}{4}/\text{decade at } \omega = 10^{\mp 1}\left|\frac{b}{a}\right|.$$

The sign of Δgradient is reversed for (a) numerator factors and (b) $\frac{b}{a} < 0$.

[Phase Approximation ++]

Like the magnitude response, the phase response can be approximated by a graph that consists of a sequence of straight line segments that are joined at "corners". For this to be true, we need to plot the phase response using a *linear* axis for the phase but a *logarithmic* axis for the frequency.
The previous slide showed the phase response of a filter whose frequency response, $H(z)$, has a single linear factor in the denominator. On the next slide this is extended to a more complicated frequency response.
Recall that the argument of a complex number is $\angle(a+jb) = \tan^{-1}\frac{b}{a}$ and $\angle\frac{1}{a+jb} = -\tan^{-1}\frac{b}{a}$.
Therefore if the frequency response is $H(j\omega) = \frac{1}{j\omega RC+1}$, then the phase is given by $\angle H(j\omega) = -\tan^{-1}\omega RC$ which is plotted as the blue curve. At low frequencies, this tends to zero (since $\tan^{-1} 0 = 0$) and at high frequencies it tends to $-\frac{\pi}{2}$ (since $\tan^{-1}\infty = \frac{\pi}{2}$). The magnitude response graph has a corner frequency at $\omega_c = \frac{1}{RC}$ and at this frequency, $\angle H(j\omega_c) = -\tan^{-1} 1 = -\frac{\pi}{4}$.
It turns out that we can approximate this curve with three straight lines which meet at two "phase response corner frequencies" of $0.1\omega_c$ and $10\omega_c$. Since the frequency range $0.1\omega_c$ to $10\omega_c$ is two decades (a factor of 100), the gradient of the central segment of the approximation must be $-\frac{\pi}{4}$ radians/decade. This approximation is not actually the best possible approximation using 3 straight lines but it is very close and much easier to remember that the optimum approximation.
To summarise: A linear factor of $(aj\omega + b)$ in the denominator will result in two corner frequencies in the phase response at $\omega = 10^{-1}\left|\frac{b}{a}\right|$ and $10^{+1}\left|\frac{b}{a}\right|$. At these frequencies, the gradient of the graph will change by $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$ radians/decade respectively. The signs of the gradient changes will be reversed for numerator factors and reversed again if $\frac{b}{a}$ is negative (which is rare and can only happen in the numerator).

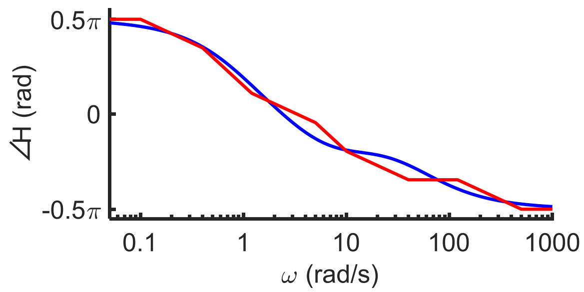
$$H(j\omega) = \frac{60(j\omega)^2+720(j\omega)}{3(j\omega)^3+165(j\omega)^2+762(j\omega)+600} = \frac{20j\omega(j\omega+12)}{(j\omega+1)(j\omega+4)(j\omega+50)}$$

Step 1: Factorize the polynomials

Step 2: List corner freqs: $\pm =$ num/den

$\omega_c = \{1^-, 4^-, 12^+, 50^-\}$

Step 3: Gradient changes at $10^{\mp 1}\omega_c$.
Sign depends on num/den and sgn $\left(\frac{b}{a}\right)$:
$.1^-, 10^+; .4^-, 40^+; 1.2^+, 120^-; 5^-, 500^+$



Step 4: Put in ascending order and calculate gaps as $\log_{10}\frac{\omega_2}{\omega_1}$ decades:
$.1^- \,(.6)\, .4^- \,(.48)\, 1.2^+ \,(.62)\, 5^- \,(.3)\, 10^+ \,(.6)\, 40^+ \,(.48)\, 120^- \,(.62)\, 500^+$.

Step 5: Find phase of LF asymptote: $\angle 1.2j\omega = +\frac{\pi}{2}$.

Step 6: At $\omega = 0.1$ gradient becomes $-\frac{\pi}{4}$ rad/decade. ϕ is still $\frac{\pi}{2}$.

Step 7: At $\omega = 0.4$, $\phi = \frac{\pi}{2} - 0.6\frac{\pi}{4} = 0.35\pi$. New gradient is $-\frac{\pi}{2}$.

Step 8: At $\omega = 1.2$, $\phi = 0.35\pi - 0.48\frac{\pi}{2} = 0.11\pi$. New gradient is $-\frac{\pi}{4}$.

Steps 9-13: Repeat for each gradient change. Final gradient is always 0.

At 0.1 and 10 times each corner frequency, the graph is continuous but its gradient changes abruptly by $\pm\frac{\pi}{4}$ rad/decade.

[Plot Phase Response ++]

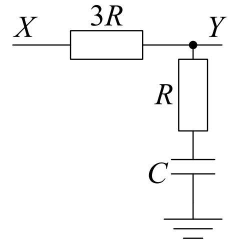
Like the magnitude response, the phase response can be approximated by a graph that consists of a sequence of straight line segments that are joined at "corners". For this to be true, we need to plot the phase response using a *linear* axis for the phase but a *logarithmic* axis for the frequency. As we saw on the previous slide, each linear factor in either the numerator or the denominator gives rise to two corners in the phase response graph. At each of these corners, the gradient of the graph changes abruptly by $\pm\frac{\pi}{4}$ radians/decade; it follows that the gradient will always be an integer multiple of $\frac{\pi}{4}$ radians/decade.
In order to plot the phase response graph, we need to determine three things: (a) the frequencies of all the corners, (b) the sign of the gradient change at each one and (c) the phase at low frequencies (i.e. frequencies less than the first corner). The example response on the slide, $H(j\omega) = \frac{20j\omega(j\omega+12)}{(j\omega+1)(j\omega+4)(j\omega+50)}$ has four linear factors: one in the numerator and three in the denominator. This means we will have a total of eight corners (two from each linear factor). Since all the factors have $\frac{b}{a} > 0$ the signs of the gradient changes will be $+$ followed by $-$for the numerator factor and $-$ followed by $+$ for the denominator factors. The two corner frequencies corresponding to a factor $(aj\omega + b)$ are at $\omega = 0.1\left|\frac{b}{a}\right|$ and $10\left|\frac{b}{a}\right|$. So, using a superscript for the sign of the gradient change, we get corners at 1.2^+ and 120^- for the numerator factor and at 0.1^-, 0.4^-, 10^+, 40^+, 5^- and 500^+ from the three denominator factors. Sorting these into ascending order of ω gives corners at 0.1^-, 0.4^-, 1.2^+, 5^-, 10^+, 40^+, 120^- and 500^+.

[Plot Phase Response ++]

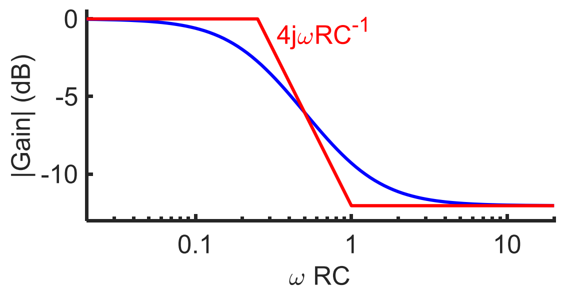
To plot the phase response, we calculate the low frequency asymptote by taking the terms with the lowest power of $j\omega$ in numerator and denominator; this gives $1.2j\omega$ which has a phase of $+\frac{\pi}{2} = 1.57$ radians. So we begin with a horizontal line at 1.57 radians until the first corner frequency at $\omega = 0.1^-$ where the gradient becomes $-\frac{\pi}{4}$. The graph will continue with this gradient until the next corner frequency which is at $\omega = 0.4^-$ where the gradient will decrease by another $\frac{\pi}{4}$ to become $-\frac{\pi}{2}$.
To work out the phase at the second corner frequency ($\omega = 0.4$) we calculate how much the phase has changed between $\omega = 0.1$ and 0.4 by multiplying the gradient of the graph ($-\frac{\pi}{4}$ radians/decade) by the separation of these two corner frequencies in decades ($\log_{10} \frac{0.4}{0.1} = 0.602$ decades). This product gives gives a phase change of -0.473 radians. So the phase is 1.571 radians at $\omega = 0.1$ and decreases by -0.473 to become 1.098 radians at $\omega = 0.4$.
The next corner is at $\omega = 1.2^+$ which is $\log_{10} \frac{1.2}{0.4} = 0.477$ decades away from $\omega = 0.4$. Since the gradient in this segment is $-\frac{\pi}{2} = -1.571$ rads/decade, the phase change between these two frequencies is $-1.571 \times 0.477 = -0.749$ radians. So the phase at $\omega = 1.2$ is $1.098 - 0.749 = 0.349$ radians.
You continue like this hopping from each corner frequency to the next. At each corner frequency, you know the new gradient (measured in radians/decade) and so you multiply this by the distance to the next corner frequency (measured in decades) to get the phase change between the two corner frequencies.
As a check, the gradient after the final corner frequency should be zero and the phase should match the phase of the high frequency asymptote. In this example, the high frequency asymptote is $20\,(j\omega)^{-1}$ which has a phase of $-\frac{\pi}{2}$. (Remember that j^r has a phase of $\left(\frac{\pi}{2}\right)^r$).

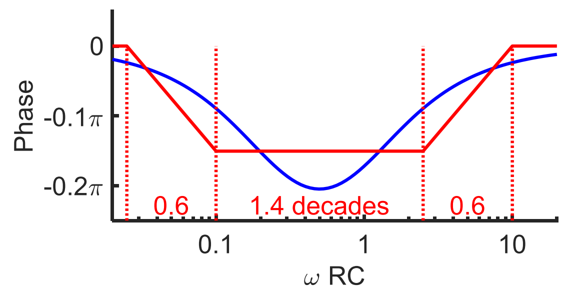
RCR Circuit

$\frac{Y}{X} = \frac{R+\frac{1}{j\omega C}}{3R+R+\frac{1}{j\omega C}} = \frac{j\omega RC+1}{4j\omega RC+1}$



Corner freqs: $\frac{0.25}{RC}^{-}, \frac{1}{RC}^{+}$ LF Asymptote: $H(j\omega) = 1$





Magnitude Response:

Gradient Changes: $-20\,\text{dB/dec}$ at $\omega = \frac{0.25}{RC}$ and $+20$ at $\omega = \frac{1}{RC}$

Line equations: $H(j\omega) =$ (a) 1, (b) $\frac{1}{4j\omega RC}$, (c) $\frac{j\omega RC}{4j\omega RC} = 0.25$

Phase Response:

LF asymptote: $\phi = \angle 1 = 0$

Gradient changes of $\pm\frac{\pi}{4}$/decade at: $\omega = \frac{0.025}{RC}^{-}, \frac{0.1}{RC}^{+}, \frac{2.5}{RC}^{+}, \frac{10}{RC}^{-}$.

At $\omega = \frac{0.1}{RC}$, $\phi = 0 - \frac{\pi}{4}\log_{10}\frac{0.1}{0.025} = -\frac{\pi}{4} \times 0.602 = -0.15\pi$

Summary

- Frequency response: magnitude and phase of $\frac{Y}{X}$ as a function of ω
  - Only applies to sine waves
  - Use log axes for frequency and gain but linear for phase
    - ▷ Decibels $= 20\log_{10}\frac{V_2}{V_1} = 10\log_{10}\frac{P_2}{P_1}$
- Linear factor $(aj\omega + b)$ gives corner frequency at $\omega = \left|\frac{b}{a}\right|$.
  - Magnitude plot gradient changes by ± 20 dB/decade $@\,\omega = \left|\frac{b}{a}\right|$.
  - Phase gradient changes in two places by:
    - ▷ $\pm\frac{\pi}{4}$ rad/decade $@\,\omega = 0.1 \times \left|\frac{b}{a}\right|$
    - ▷ $\mp\frac{\pi}{4}$ rad/decade $@\,\omega = 10 \times \left|\frac{b}{a}\right|$
- LF/HF asymptotes: keep only the terms with the lowest/highest power of $j\omega$ in numerator and denominator polynomials

For further details see Hayt Ch 16 or Irwin Ch 12.

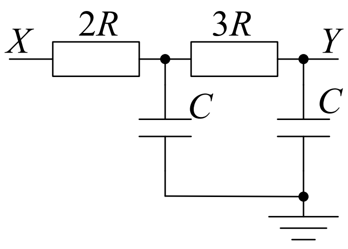
12: Resonance

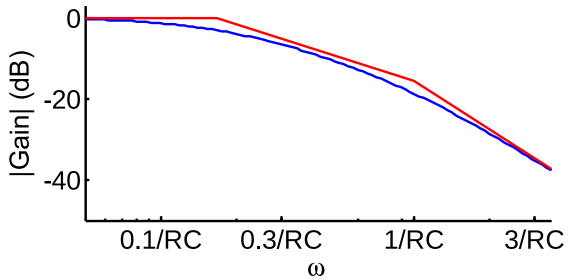
A quadratic factor in a transfer function is: $F(j\omega) = a\,(j\omega)^2 + b\,(j\omega) + c$.

Case 1: If $b^2 \geq 4ac$ then we can factorize it:

$$F(j\omega) = a(j\omega - p_1)(j\omega - p_2)$$

where $p_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.





$$\frac{Y}{X}(j\omega) = \frac{1}{6R^2C^2(j\omega)^2 + 7RCj\omega + 1}$$

$$= \frac{1}{(6j\omega RC + 1)(j\omega RC + 1)}$$

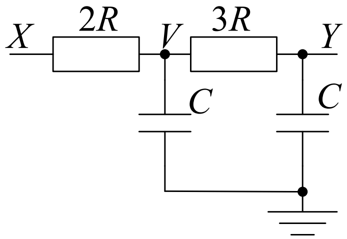
$$\omega_c = \frac{0.17}{RC},\ \frac{1}{RC} = |p_1|\,,\ |p_2|$$

Case 2: If $b^2 < 4ac$, we cannot factorize with real coefficients so we leave it as a quadratic. Sometimes called a *quadratic resonance*.

Any polynomial with real coefficients can be factored into linear and quadratic factors $\Rightarrow$ a quadratic factor is as complicated as it gets.

[Derivation of Transfer Function]

KCL at V gives



$$\frac{V-X}{2R} + j\omega CV + \frac{V-Y}{3R} = 0 \Rightarrow 3(V-X) + 6j\omega RCV + 2(V-Y) = 0$$

$$\Rightarrow (5 + 6j\omega RC)V = 3X + 2Y.$$

KCL at Y gives

$$\frac{Y-V}{3R} + j\omega CY = 0 \quad \Rightarrow \quad (1 + 3j\omega RC)Y = V.$$

Eliminating V beween these two equations gives

$$(5 + 6j\omega RC)(1 + 3j\omega RC)Y = 3X + 2Y$$

$$\Rightarrow \left(5 + 21j\omega RC + 18(j\omega RC)^2 - 2\right)Y = 3X$$

$$\Rightarrow \frac{Y}{X} = \frac{3}{3+21j\omega RC+18(j\omega RC)^2} = \frac{1}{1+7j\omega RC+6(j\omega RC)^2} = \frac{1}{(1+6j\omega RC)(1+j\omega RC)}.$$

At high frequencies, the impedance of the capacitor is much less than $3R$ so we can think of the circuit as two potential dividers one after the other (i.e. the current through the $3R$ is negligible compared to the current throught the first C). The high frequency asymptote is therefore the product of the asymptotes for the two potential dividers which gives $\frac{Y}{X} \approx \frac{1}{2j\omega RC} \times \frac{1}{3j\omega RC} = \frac{1}{6(j\omega RC)^2}$.

Suppose $b^2 < 4ac$ in $F(j\omega) = a\,(j\omega)^2 + b\,(j\omega) + c$.

Low/High freq asymptotes: $F_{\text{LF}}(j\omega) = c, \quad F_{\text{HF}}(j\omega) = a\,(j\omega)^2$

The asymptote magnitudes cross at the *corner frequency*:

$$\left|a\,(j\omega_c)^2\right| = |c| \Rightarrow \omega_c = \sqrt{\tfrac{c}{a}}.$$

We define the *damping factor*, "zeta", to be $\zeta = \frac{b}{2a\omega_c} = \frac{b\omega_c}{2c} = \frac{b\,\text{sgn}(a)}{\sqrt{4ac}}$

$$\Rightarrow F(j\omega) = c\left(\left(j\frac{\omega}{\omega_c}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_c}\right) + 1\right)$$

Properties to notice in this expression:

(a) c is just an overall scale factor.
(b) ω_c just scales the frequency axis since $F(j\omega)$ is a function of $\frac{\omega}{\omega_c}$.
(c) The shape of the $F(j\omega)$ graphs is determined entirely by ζ.
(d) The quadratic cannot be factorized $\Leftrightarrow b^2 < 4ac \Leftrightarrow |\zeta| < 1$.
(e) At $\omega = \omega_c$, asymptote gain $= c$ but $F(j\omega) = c \times 2j\zeta$.

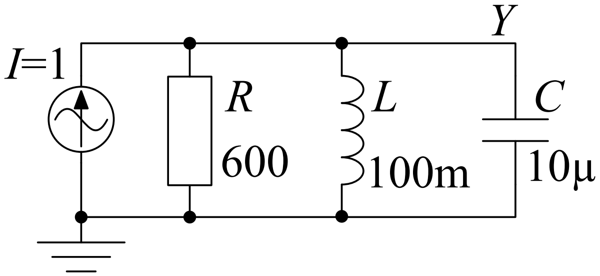
Alternatively, we sometimes use the *quality factor*, $Q \approx \frac{1}{2\zeta} = \frac{a\omega_c}{b}$.

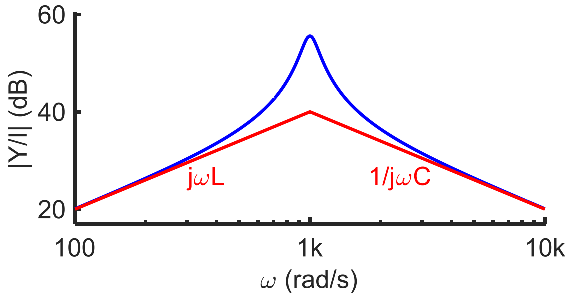
Parallel RLC

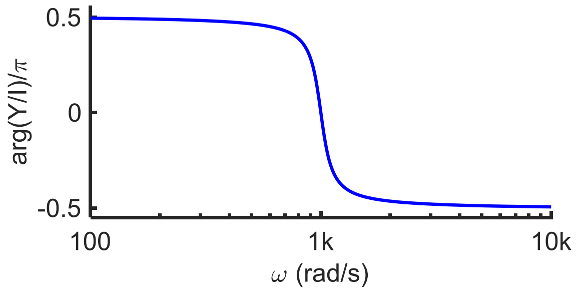
$$\frac{Y}{I} = \frac{1}{\frac{1}{R}+\frac{1}{j\omega L}+j\omega C} = \frac{j\omega L}{LC(j\omega)^2+\frac{L}{R}j\omega+1}$$

$$\omega_c = \sqrt{\frac{c}{a}} = 1000, \ \zeta = \frac{b}{2a\omega_c} = 0.083$$

Asymptotes: $j\omega L$ and $\frac{1}{j\omega C}$.



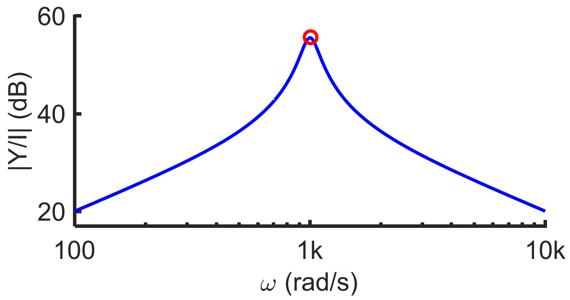


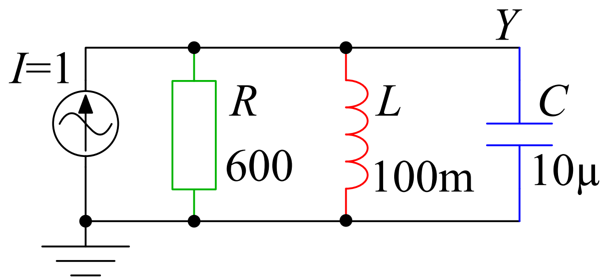


Power absorbed by resistor $\propto Y^2$. It peaks quite sharply at $\omega = 1000$. The resonant frequency, ω_r, is when the impedance is purely real:
at $\omega_r = 1000$, $Z_{RLC} = \frac{Y}{I} = R$.

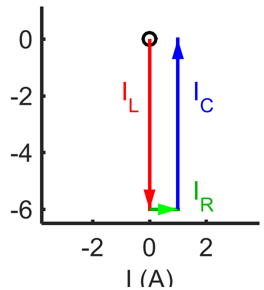
A system with a strong peak in power absorption is a *resonant* system.

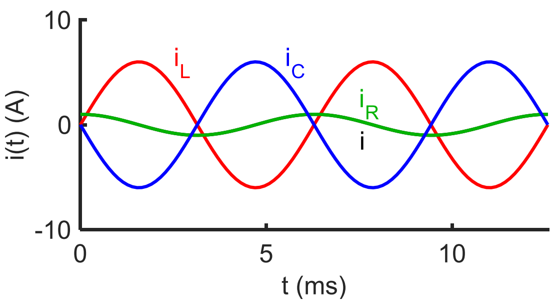
Behaviour at Resonance

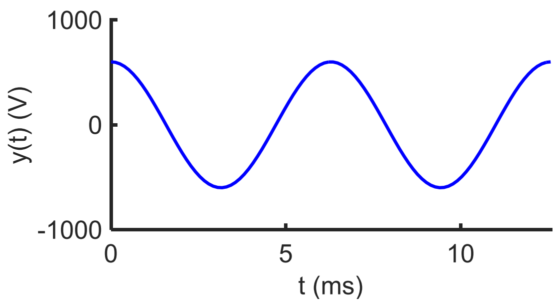




$$\omega = 1000 \Rightarrow Z_L = 100j,\ Z_C = -100j.$$
$$Z_L = -Z_C \Rightarrow I_L = -I_C$$
$$\Rightarrow I = I_R + I_L + I_C = I_R = 1$$
$$\Rightarrow Y = I_R R = 600\angle 0^\circ = 56\,\text{dBV}$$
$$\Rightarrow I_L = \frac{Y}{Z_L} = \frac{600}{100j} = -6j$$

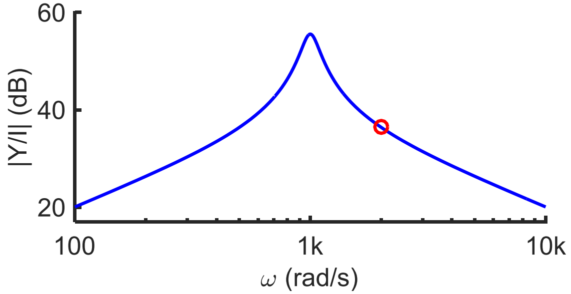


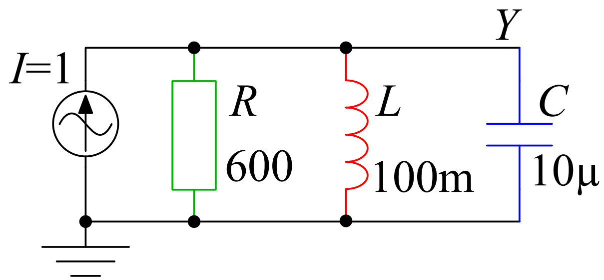




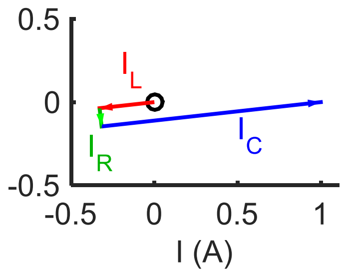
Large currents in L and C exactly cancel out $\Rightarrow I_R = I$ and $Z = R$ (real)

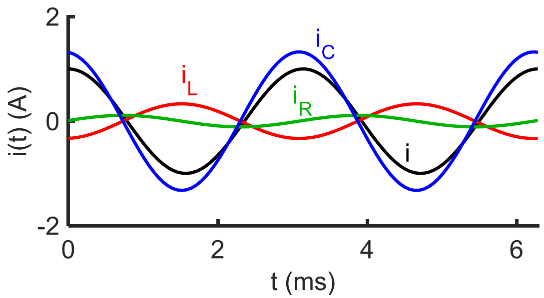
Away from resonance

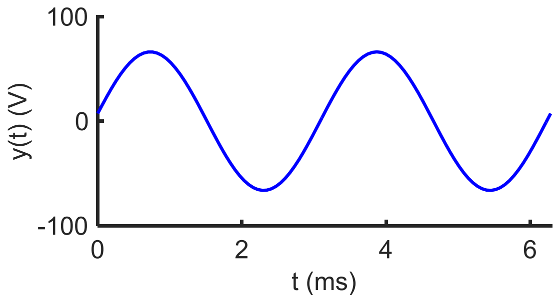




$$\omega = 2000 \Rightarrow Z_L = 200j,\ Z_C = -50j$$
$$Z = \left(\frac{1}{R} + \frac{1}{Z_L} + \frac{1}{Z_C}\right)^{-1} = 66\angle -84°$$
$$Y = I \times Z = 66\angle -84° = 36\,\mathrm{dBV}$$
$$I_R = \frac{Y}{R} = 0.11\angle -84°$$
$$I_L = \frac{Y}{Z_L} = 0.33\angle -174°,\ I_C = 1.33\angle +6°$$

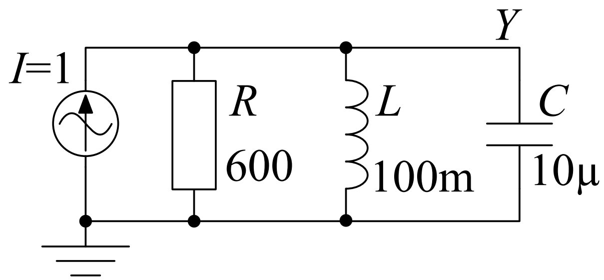




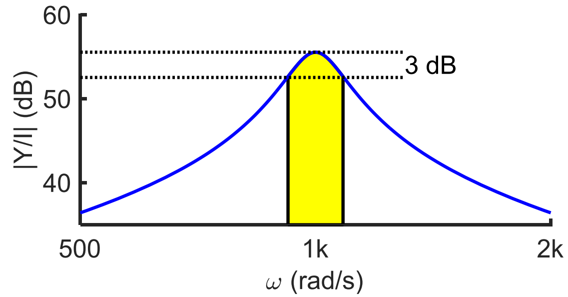


Most current now flows through C, only 0.11 through R.

$$\frac{Y}{I} = \frac{1}{1/R + j(\omega C - 1/\omega L)}$$



Bandwidth is the range of frequencies for which $\left|\frac{Y}{I}\right|^2$ is greater than half its peak.
Also called *half-power bandwidth* or *3dB bandwidth*.



$$\left|\frac{Y}{I}\right|^2 = \frac{1}{(1/R)^2 + (\omega C - 1/\omega L)^2}$$

Peak is $\left|\frac{Y}{I}(\omega_0)\right|^2 = R^2$ @ $\omega_0 = 1000$

At $\omega_{3\text{dB}}$: $\left|\frac{Y}{I}(\omega_{3\text{dB}})\right|^2 = \frac{1}{2}\left|\frac{Y}{I}(\omega_0)\right|^2$

$$\frac{1}{(1/R)^2 + (\omega_{3\text{dB}} C - 1/\omega_{3\text{dB}} L)^2} = \frac{R^2}{2} \Rightarrow 1 + \left(\omega_{3\text{dB}} RC - \frac{R}{\omega_{3\text{dB}} L}\right)^2 = 2$$

$$\omega_{3\text{dB}} RC - R/\omega_{3\text{dB}} L = \pm 1 \quad \Rightarrow \quad \omega_{3\text{dB}}^2 RLC \pm \omega_{3\text{dB}} L - R = 0$$

Positive roots: $\omega_{3\text{dB}} = \frac{\pm L + \sqrt{L^2 + 4R^2 LC}}{2RLC} = \{920,\ 1086\}$ rad/s

Bandwidth: $B = 1086 - 920 = 167\,\text{rad/s}.$

Q factor $\approx \frac{\omega_0}{B} = \frac{1}{2\zeta} = 6.$ (Q = "Quality")

Absorbed Power $=v(t)i(t)$:

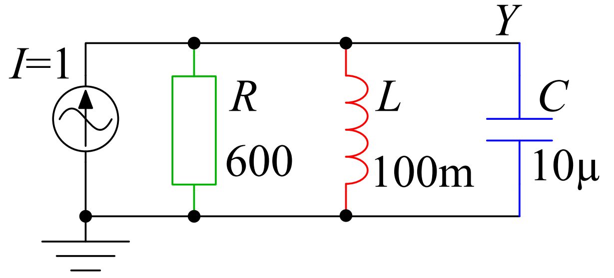
P_L and P_C opposite and $\gg P_R$.

Stored Energy $= \frac{1}{2}Li_L^2 + \frac{1}{2}Cy^2$:

sloshes between L and C.

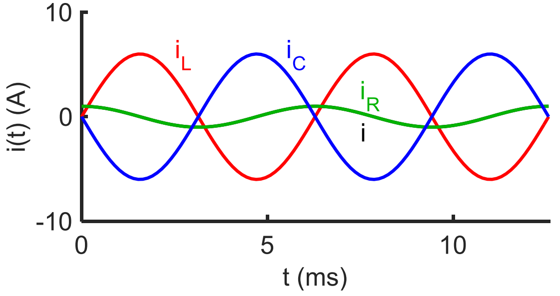
$Q \triangleq \omega \times W_{\text{stored}} \div \overline{P}_R$

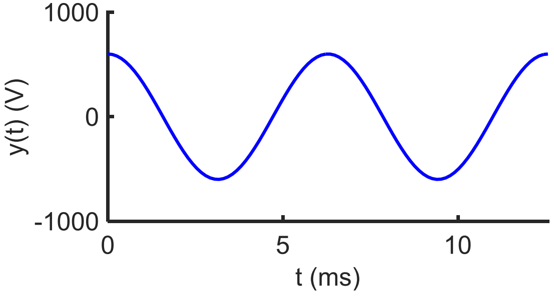
$= \omega \times \frac{1}{2}C\left|IR\right|^2 \div \frac{1}{2}\left|I\right|^2 R = \omega RC$

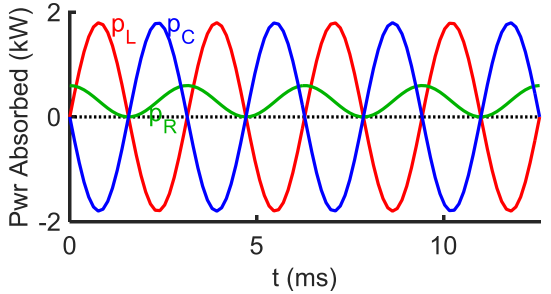


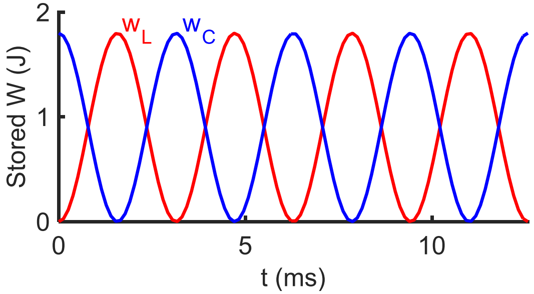
@$\omega = 1000$: $Y = 600$,

$I_R = 1$, $I_L = -6j$, $I_C = +6j$









$Q \triangleq \omega \times$ peak stored energy $\div$ average power loss.

[Derivation of Power and Energy Waveforms]

The input current is a phasor $I = 1$ (i.e. $i(t) = \cos\omega t$ where $\omega = 1000\,\mathrm{rad/s}$).

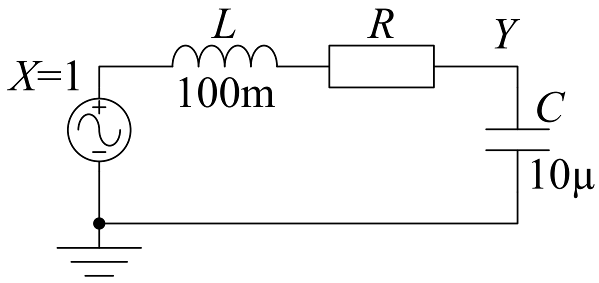
The complex impedances are $Z_L = j\omega L = 100j\,\Omega$ and $Z_C = \frac{1}{j\omega C} = -100j\,\Omega$. Using the formula for parallel impedances, the total impedance satisfies $\frac{1}{Z} = \frac{1}{600} + \frac{1}{100j} + \frac{1}{-100j} = \frac{1}{600}$. So, at the resonant frequency, the impedances of L and C cancel out and the total impedance is just $Z = 600\,\Omega$.

The voltage phasor across the three passive components is $V = IZ = 1 \times 600 = 600\,\mathrm{V}$. The waveform corresponding to this phasor is $v(t) = 600\cos\omega t$ and is plotted in the upper right graph. From knowing V, we can use Ohm's law to work out the individual current phasors in the three components as $I_R = \frac{V}{R} = \frac{600}{600} = 1$, $I_C = \frac{V}{Z_C} = \frac{600}{-100j} = 6j$ and $I_L = \frac{V}{Z_L} = \frac{600}{100j} = -6j$. The waveforms corresponding to these three phasors are plotted in the upper left graph.

Multiplying phasors together doesn't directly give the correct result and so we calculate the power waveforms directly by multiplying $v(t) \times i(t)$. For the resistor, $V = 600$ and $I_R = 1$, so $p_R(t) = 600\cos\omega t \times \cos\omega t = 600\cos^2\omega t = 300 + 300\cos 2\omega t$. For the inductor, $V = 600$ and $I_L = -6j$, so $p_R(t) = 600\cos\omega t \times 6\sin\omega t = 3600\sin\omega t\cos\omega t = 1800\sin 2\omega t$. Finally, for the capacitor, $V = 600$ and $I_L = +6j$, so $p_R(t) = 600\cos\omega t \times -6\sin\omega t = -3600\sin\omega t\cos\omega t = -1800\sin 2\omega t$. These are plotted in the lower left graph.

The energy stored in an inductor is $w_L(t) = \frac{1}{2}Li^2(t) = \frac{1}{2} \times 0.1 \times 36\sin^2\omega t = 1.8\sin^2\omega t = 0.9\,(1 - \cos 2\omega t)$. The energy stored in a capacitor is $w_C(t) = \frac{1}{2}Cv^2(t) = \frac{1}{2} \times 10^{-5} \times 600^2\cos^2\omega t = 1.8\cos^2\omega t = 0.9\,(1 + \cos 2\omega t)$. These are plotted in the lower right graph. The total stored energy in the circuit is $w_L(t) + w_C(t) = 1.8\,\mathrm{J}$ which does not vary with time.

$$\frac{Y}{X} = \frac{1/j\omega C}{R+j\omega L+\frac{1}{j\omega C}} = \frac{1}{LC(j\omega)^2+RCj\omega+1}$$



Asymptotes: 1 and $\frac{1}{LC}(j\omega)^{-2}$.

$\omega_c = \sqrt{\frac{c}{a}} = 1000$, $\zeta = \frac{b}{2a\omega_c} = \frac{R}{200}$

$@\omega_c : Z_L = -Z_C = 100j$, $I = \frac{X}{R}$, $\left|\frac{Y}{X}\right| = \frac{1}{RC\omega} = \frac{1}{2\zeta}$, $\angle\frac{Y}{X} = -\frac{\pi}{2}$

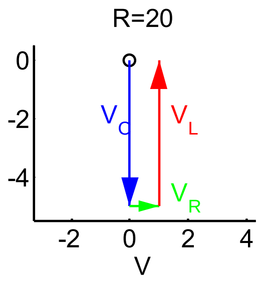
Magntitude Plot:

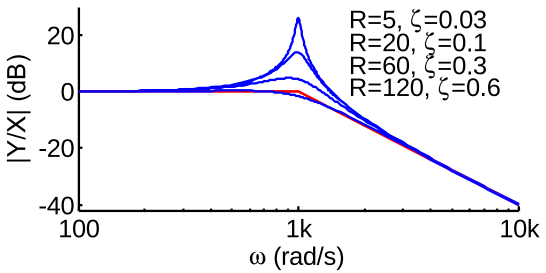
Small $\zeta \Rightarrow$ less loss, higher peak, smaller bandwidth.
Large ζ more loss, smaller peak at a lower ω, larger bandwidth.

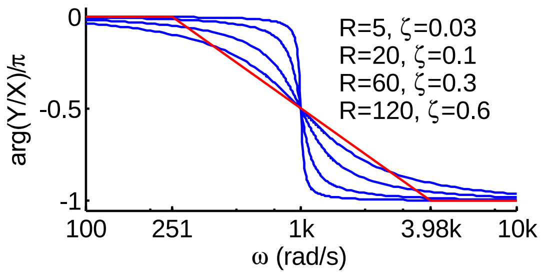
Phase Plot:

Small $\zeta \Rightarrow$ fast phase change: π over 2ζ decades.

$\angle\frac{Y}{X} \approx \frac{-\pi}{2}\left(1+\frac{1}{\zeta}\log_{10}\frac{\omega}{\omega_c}\right)$ for $10^{-\zeta} < \frac{\omega}{\omega_c} < 10^{+\zeta}$



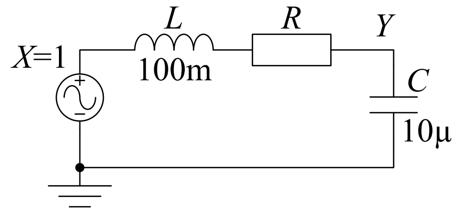




Resonance Peak for LP filter

$$\frac{Y}{X} = \frac{1}{LC(j\omega)^2+RCj\omega+1} = \frac{1}{\left(j\frac{\omega}{\omega_c}\right)^2+2\zeta j\frac{\omega}{\omega_c}+1}$$

$$\omega_c = \sqrt{\frac{c}{a}} = 1000,\ \zeta = \frac{b}{2a\omega_c} = \frac{b\omega_c}{2c} = \frac{R}{200}$$

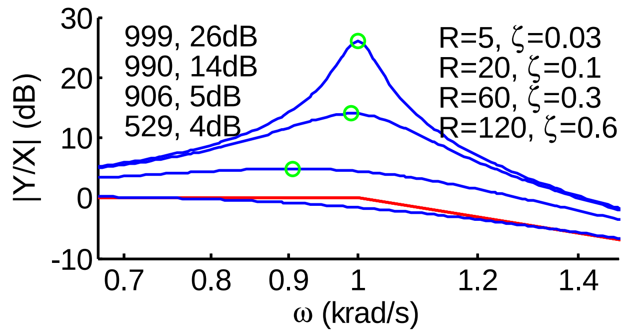


$\frac{Y}{X}$ is a function of $\frac{\omega}{\omega_c}$ so ω_c just scales frequency axis (= shift on log axis).
The *damping factor*, ζ, ("zeta") determines the shape of the peak.

Peak frequency:

$$\omega_p = \omega_c\sqrt{1-2\zeta^2}$$

$\zeta \geq 0.5 \Rightarrow$ passes under corner,
$\zeta \geq 0.71 \Rightarrow$ no peak,
$\zeta \geq 1 \Rightarrow$ can factorize



Gain relative to asymptote: @ ω_p: $\frac{1}{2\zeta\sqrt{1-\zeta^2}}$ @ ω_c: $\frac{1}{2\zeta} \approx Q$

Three frequencies: ω_p= peak, ω_c= asymptotes cross, ω_r= real impedance
For $\zeta < 0.3$, $\omega_p \approx \omega_c \approx \omega_r$. All get called the resonant frequency.

The exact relationship between ω_p, ω_c and ω_r and the gain at these frequencies is affected by any other corner frequencies in the response.

- Resonance is a peak in energy absorption
  - Parallel or series circuit has a real impedance at ω_r
    - ▷ peak response may be at a slightly different frequency
  - The quality factor, Q, of the resonance is

$$Q \triangleq \frac{\omega_0 \times \text{stored energy}}{\text{power in } R} \approx \frac{\omega_0}{3\ \text{dB bandwidth}} \approx \frac{1}{2\zeta}$$

  - 3 dB bandwidth is where power falls by $\frac{1}{2}$ or voltage by $\frac{1}{\sqrt{2}}$
  - The stored energy sloshes between L and C
- Quadratic factor: $\left(\frac{j\omega}{\omega_c}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_c}\right) + 1$
  - $a(j\omega)^2 + b(j\omega) + c \quad \Rightarrow \quad \omega_c = \sqrt{\frac{c}{a}}$ and $\zeta = \frac{b}{2a\omega_c} = \frac{b\,\operatorname{sgn}(a)}{\sqrt{4ac}}$
  - ± 40 dB/decade slope change in magnitude response
  - phase changes rapidly by 180° over $\omega = 10^{\mp\zeta}\omega_c$
  - Gain error in asymptote is $\frac{1}{2\zeta} \approx Q$ at ω_0

For further details see Hayt Ch 16 or Irwin Ch 12.

13: Filters

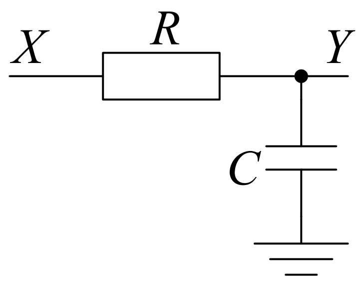
A filter is a circuit whose gain varies with frequency. Often a filter aims to allow some frequencies to pass while blocking others.

- Radio/TV: a "tuning" filter blocks all frequencies except the wanted channel
- Loudspeaker: "crossover" filters send the right frequencies to different drive units
- Sampling: an "anti-aliasing filter" eliminates all frequencies above half the sampling rate
  - Phones: Sample rate $= 8\,$kHz : filter eliminates frequencies above $3.4\,$kHz.
- Computer cables: filter eliminates interference

[Wikipedia]

1st Order Low-Pass Filter

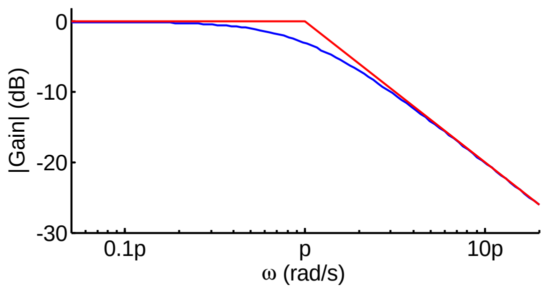
$$\frac{Y}{X} = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{j\omega RC+1} = \frac{1}{\frac{j\omega}{p}+1}$$



Corner frequency: $p = \left|\frac{b}{a}\right| = \frac{1}{RC}$

Asymptotes: 1 and $\frac{p}{j\omega}$

Very low ω: Capacitor = open circuit
Very high ω: Capacitor short circuit

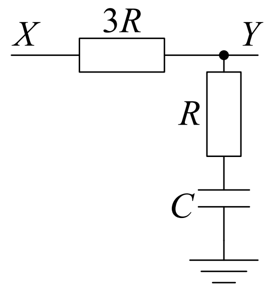


A *low-pass* filter because it allows low frequencies to pass but *attenuates* (makes smaller) high frequencies.

The *order* of a filter: highest power of $j\omega$ in the denominator.
Almost always equals the total number of L and/or C.

Low-Pass with Gain Floor

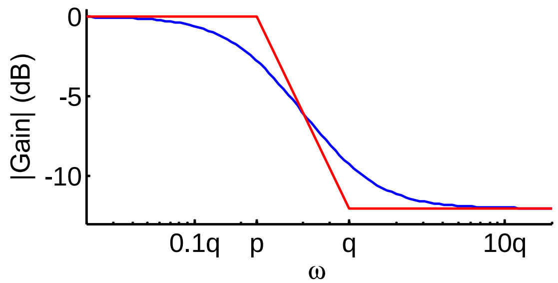
$$\frac{Y}{X} = \frac{R + 1/j\omega C}{4R + 1/j\omega C} = \frac{j\omega RC + 1}{j\omega 4RC + 1} = \frac{\frac{j\omega}{q} + 1}{\frac{j\omega}{p} + 1}$$



Corner frequencies: $p = \frac{1}{4RC}$, $q = \frac{1}{RC}$

Asymptotes: 1 and $\frac{1}{4}$

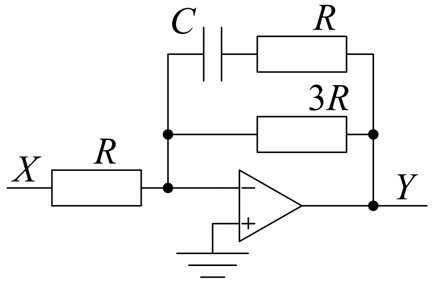
Very low ω:
  Capacitor = open circuit
  Resistor R unattached. Gain $= 1$



Very high ω:
  Capacitor short circuit
  Circuit is potential divider with gain $20\log_{10}\frac{1}{4} = -12\,\text{dB}$.

Opamp filter

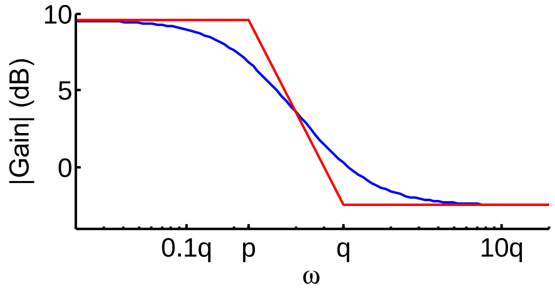
Inverting amplifier so



$$\frac{Y}{X} = -\frac{3R||(R+{}^{1}/_{j\omega C})}{R} = -\frac{3R(R+{}^{1}/_{j\omega C})}{R\times(3R+R+{}^{1}/_{j\omega C})}$$

$$= -3 \times \frac{R+{}^{1}/_{j\omega C}}{4R+{}^{1}/_{j\omega C}} = -3 \times \frac{j\omega RC+1}{j\omega 4RC+1}$$

Same transfer function as before except $\times -3 = +9.5\,\text{dB}$.



Advantages of op-amp crcuit:

1. Can have gain > 1.
2. Low output impedance - loading does not affect filter
3. Resistive input impedance - does not vary with frequency

$\frac{Y}{X} = -\frac{1/j\omega C}{R} = -\frac{1}{j\omega RC}$

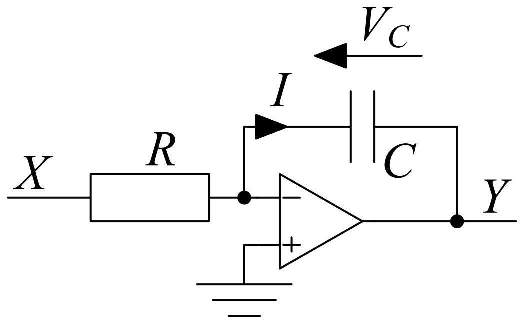
Capacitor: $i = C\frac{dv_C}{dt}$

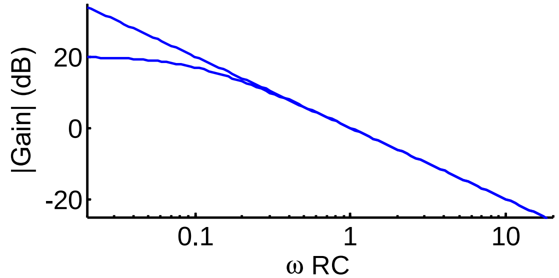
$i = \frac{x}{R} = -C\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{-1}{RC}x$

$\int_0^t \frac{dy}{dt}dt = \frac{-1}{RC}\int_0^t x dt$

$y(t) = \frac{-1}{RC}\int_0^t x dt + y(0)$



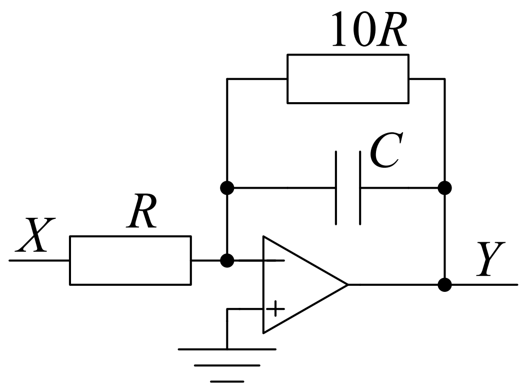


Note: if $x(t) = \cos\omega t$
$\int \cos(\omega t)dt = \frac{1}{\omega}\sin(\omega t) \Rightarrow$ gain $\propto \frac{1}{\omega}$.

We can limit the LF gain to 20 dB:

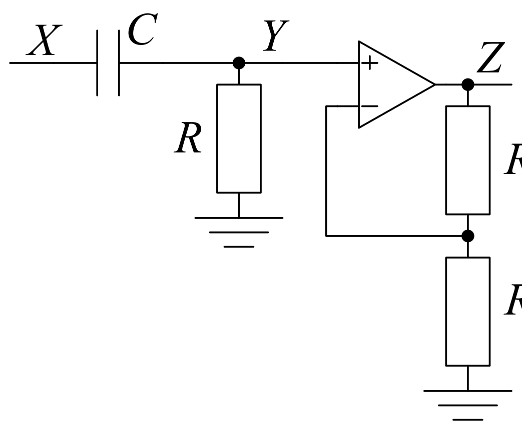
$\frac{Y}{X} = -\frac{10R||1/j\omega C}{R} = -\frac{10R\times 1/j\omega C}{R(10R+1/j\omega C)}$

$= -\frac{10}{j\omega 10RC+1} \qquad \left(\omega_c = \frac{0.1}{RC}\right)$



High Pass Filter

$\frac{Y}{X} = \frac{R}{R+{}^{1}/_{j\omega C}} = \frac{j\omega RC}{j\omega RC+1}$



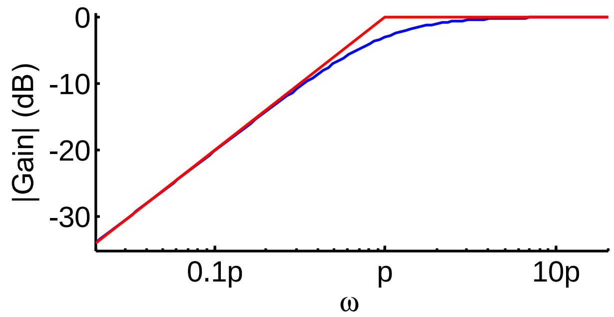
Corner Freq: $p = \frac{1}{RC}$

Asymptotes: $j\omega RC$ and 1

Very low ω: C open circuit: gain $= 0$
Very high ω: C short circuit: gain $= 1$

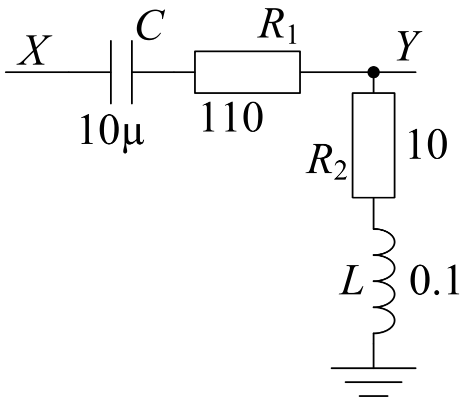
We can add an op-amp to give a low-impedance output. Or add gain:

$\frac{Z}{X} = \left(1 + \frac{R_B}{R_A}\right) \times \frac{j\omega RC}{j\omega RC+1}$



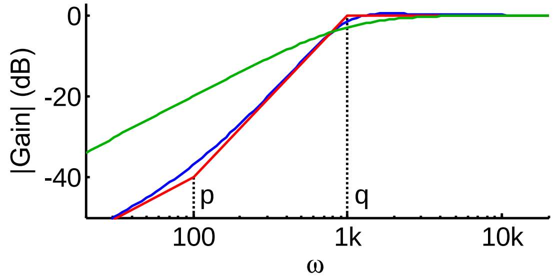
2nd order filter

$$\frac{Y}{X} = \frac{R_2 + j\omega L}{1/j\omega C + R_1 + R_2 + j\omega L}$$
$$= \frac{LC(j\omega)^2 + R_2 Cj\omega}{LC(j\omega)^2 + (R_1+R_2)Cj\omega + 1}$$
$$= \frac{j\omega C(j\omega L + R_2)}{LC(j\omega)^2 + (R_1+R_2)Cj\omega + 1}$$



Asymptotes: $j\omega R_2 C$ and 1

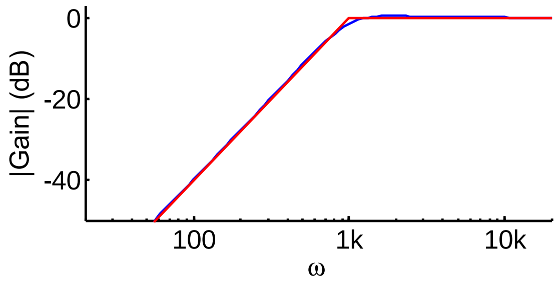
Corner frequencies:
$+20\,\text{dB/dec at } p = \frac{R_2}{L} = 100\,\text{rad/s}$
$-40\,\text{dB/dec at}$
$$q = \sqrt{\frac{c}{a}} = \frac{1}{\sqrt{LC}} = 1000\,\text{rad/s}$$

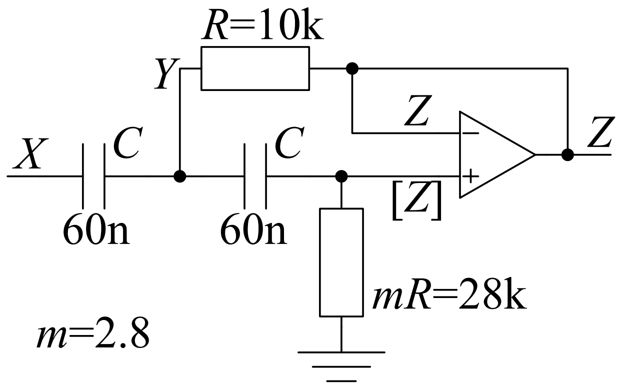


Damping factor: $\zeta = \frac{b\,\text{sgn}(a)}{\sqrt{4ac}} = \frac{qb}{2c} = \frac{q}{2}(R_1 + R_2)\,C = 0.6.$

Gain error at q is $\frac{1}{2\zeta} = Q = 0.83 = -1.6\,\text{dB}$ ($+0.04\,\text{dB}$ due to p)

Compare with 1st order:
2nd order filter attenuates more rapidly than a 1st order filter.





m=2.8

Asymptotes: $\left(\frac{j\omega}{p}\right)^2$ and 1

KCL @ Y: $\frac{Y-X}{1/j\omega C} + \frac{Y-Z}{1/j\omega C} + \frac{Y-Z}{R} = 0$ [assume $V_+ = V_- = Z$]

$\Rightarrow Y\left(1 + 2j\omega RC\right) - Z\left(1 + j\omega RC\right) = Xj\omega RC$

KCL @ V_+: $\frac{Z}{mR} + \frac{Z-Y}{1/j\omega C} = 0 \Rightarrow Z(1 + j\omega mRC) = Yj\omega mRC$

Sub Y: $Z\frac{(1+j\omega mRC)}{j\omega mRC}\left(1 + 2j\omega RC\right) - Z\left(1 + j\omega RC\right) = Xj\omega RC$

$\Rightarrow \frac{Z}{X} = \frac{m(j\omega RC)^2}{m(j\omega RC)^2 + 2j\omega RC + 1} = \frac{(j\omega/p)^2}{(j\omega/p)^2 + 2\zeta(j\omega/p) + 1}$

Corner freq: $p = \frac{1}{\sqrt{m}RC} = 996$ rad/s, $\zeta = \frac{1}{2Q} = pRC = \frac{1}{\sqrt{m}} = 0.6$

Sallen-Key: 2nd order filter without inductors. Can easily have gain >1.

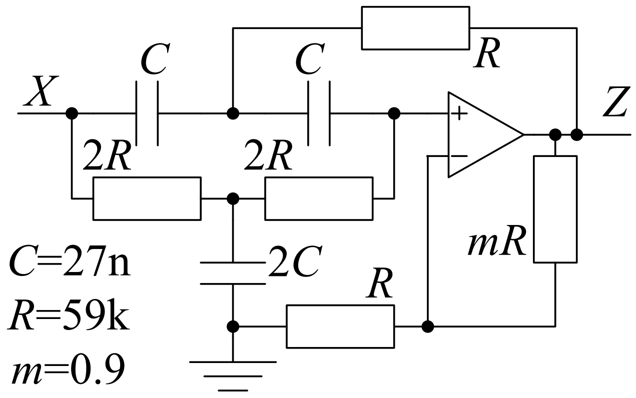
Designing: Choose $m = \zeta^{-2}$; C any convenient value; $R = \frac{\zeta}{pC}$.

After much algebra:

$$\frac{Z}{X} = \frac{(1+m)\left((2j\omega RC)^2+1\right)}{(2j\omega RC)^2+4(1-m)j\omega RC+1}$$

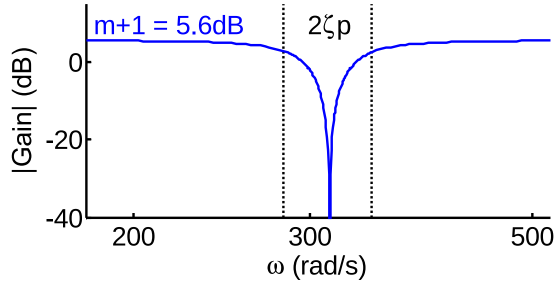
$$= \frac{(1+m)\left((^{j\omega}/_{p})^2+1\right)}{(^{j\omega}/_{p})^2+2\zeta(^{j\omega}/_{p})+1}$$

$p = \frac{1}{2RC} = 314$, $\zeta = 1 - m = 0.1$



Very low ω: C open circuit
Non-inverting amp, $\frac{Z}{X} = 1 + m$

Very high ω: C short circuit
Non-inverting amp, $\frac{Z}{X} = 1 + m$



At $\omega = p$, $\left(\frac{j\omega}{p}\right)^2 = -1$: numerator = zero resulting in infinite attenuation.
The $3\,\mathrm{dB}$ notch width is approximately $2\zeta p = 2(1-m)p$.

Used to remove one specific frequency (e.g. mains hum @ 50 Hz)

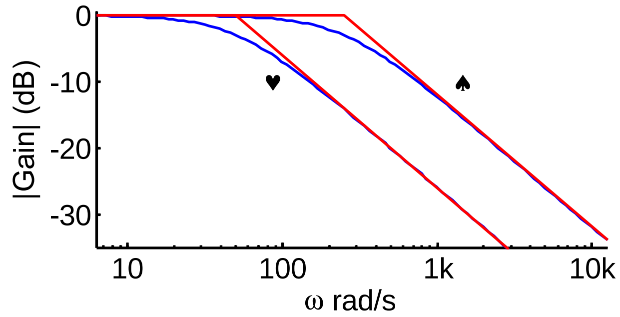
Do not try to memorize this circuit

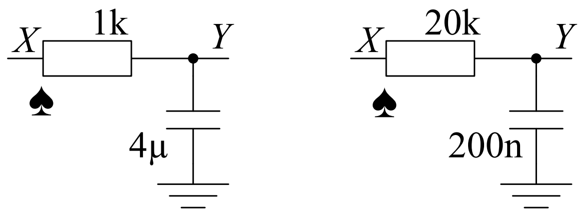
Conformal Filter Transformations (A)

A dimensionless gain, $\frac{V_Y}{V_X}$, can always be written using dimensionless impedance ratio terms: $\frac{Z_R}{Z_C} = j\omega RC$, $\frac{Z_L}{Z_R} = \frac{j\omega L}{R}$, $\frac{Z_L}{Z_C} = -\omega^2 LC$.

Impedance scaling:

Scale all impedances by k:
$R' = kR,\ C' = k^{-1}C,\ L' = kL$
Impedance ratios are unchanged so graph stays the same.
(k is arbitrary)

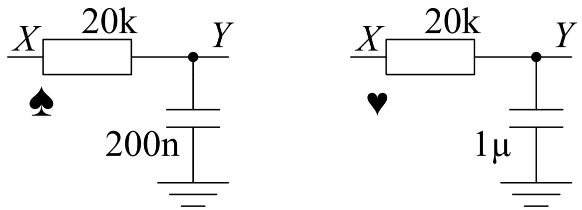




$k = 20$

Frequency Shift:

Scale reactive components by k:
$R' = R,\ C' = kC,\ L' = kL$
$\Rightarrow Z'(k^{-1}\omega) \equiv Z(\omega)$
Graph shifts left by a factor of k.



$k = 5$

Must scale all reactive components in the circuit by the same factor.

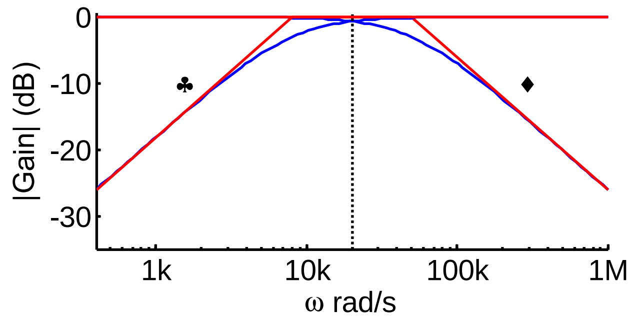
Conformal Filter Transformations (B)

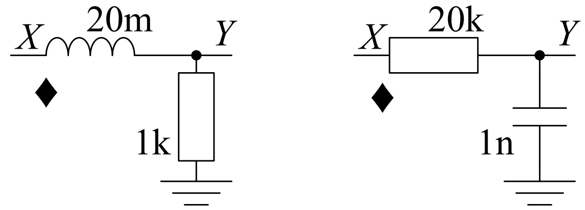
Change LR circuit to RC:

Change $R' = kL,\ C' = \frac{1}{kR}$

$\Rightarrow \frac{Z_{R'}}{Z_{C'}} = j\omega R'C' = \frac{j\omega L}{R} = \frac{Z_L}{Z_R}$

Impedance ratios are unchanged at all ω so graph stays the same. (k is arbitrary)



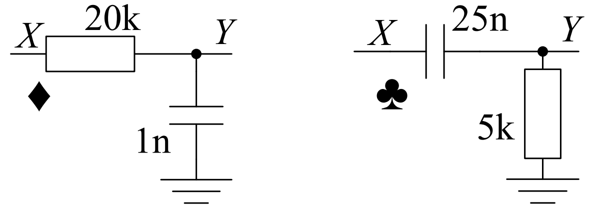


$k = 10^6$

Reflect frequency axis around ω_m:

Change $R' = \frac{k}{\omega_m C},\ C' = \frac{1}{\omega_m kR}$

$\Rightarrow \frac{Z_{R'}}{Z_{C'}}\left(\frac{\omega_m^2}{\omega}\right) = \left(\frac{Z_C}{Z_R}(\omega)\right)^*$



$k = 0.1,\ \omega_m = 20\,\text{k}$

(a) Magnitude graph flips

(b) Phase graph flips and negates since $\angle z^* = -\angle z$.
(k is arbitrary)

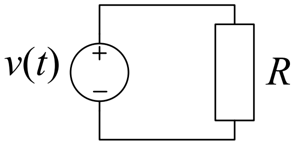
Summary

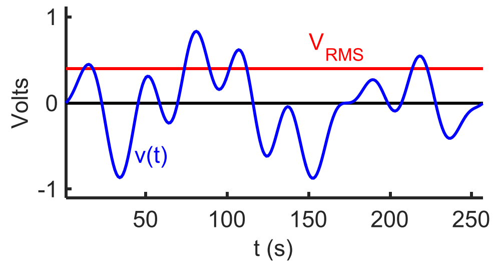
- The *order* of a filter is the highest power of $j\omega$ in the transfer function denominator.
- *Active filters* use op-amps and usually avoid the need for inductors.
  - Sallen-Key design for high-pass and low-pass.
  - Twin-T design for notch filter: gain $= 0$ at notch.
- For filters using R and C only:
  - Scale R and C: Substituting $R' = kR$ and $C' = pC$ scales frequency by $(pk)^{-1}$.
  - Interchange R and C: Substituting $R' = \frac{k}{\omega_0 C}$ and $C' = \frac{1}{k\omega_0 R}$ flips the frequency response around ω_0 $(\forall k)$.
    Changes a low-pass filter to high pass and vice-versa.

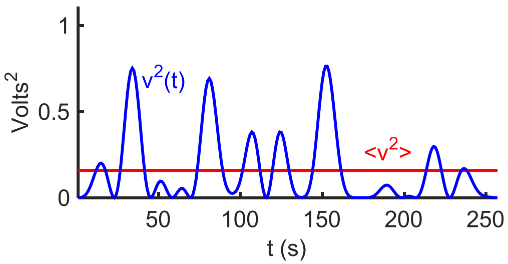
For further details see Hayt Ch 16 or Irwin Ch 12.

14: Power in AC Circuits

Average Power







Intantaneous Power dissipated in R: $p(t) = \frac{v^2(t)}{R}$

Average Power dissipated in R:

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{R} \times \frac{1}{T}\int_0^T v^2(t)dt = \frac{\langle v^2(t)\rangle}{R}$$

$\langle v^2(t)\rangle$ is the value of $v^2(t)$ averaged over time

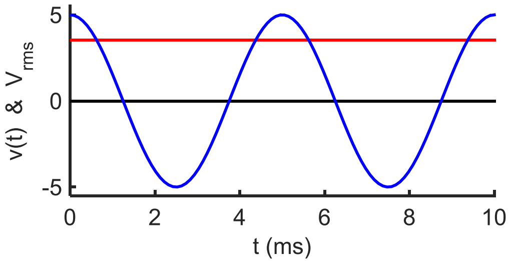
We define the *RMS Voltage* (Root Mean Square): $V_{rms} \triangleq \sqrt{\langle v^2(t)\rangle}$

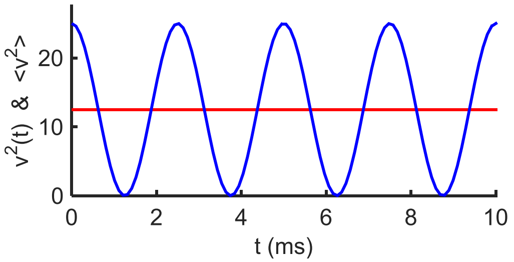
The average power dissipated in R is $P = \frac{\langle v^2(t)\rangle}{R} = \frac{(V_{rms})^2}{R}$

V_{rms} is the DC voltage that would cause R to dissipate the same power.

We use *small letters* for time-varying voltages and *capital letters* for time-invariant values.

Cosine Wave RMS





Cosine Wave: $v(t) = 5\cos\omega t$. Amplitude is $V = 5\,\text{V}$.

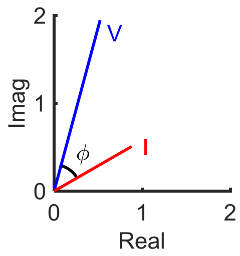
Squared Voltage: $v^2(t) = V^2\cos^2\omega t = V^2\left(\frac{1}{2} + \frac{1}{2}\cos 2\omega t\right)$

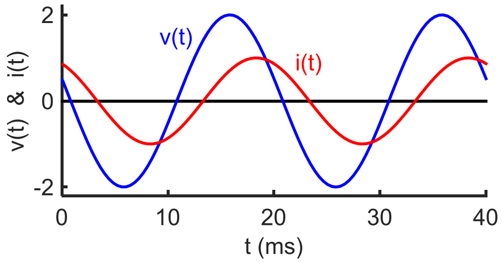
Mean Square Voltage: $\langle v^2 \rangle = \frac{V^2}{2}$ since $\cos 2\omega t$ averages to zero.

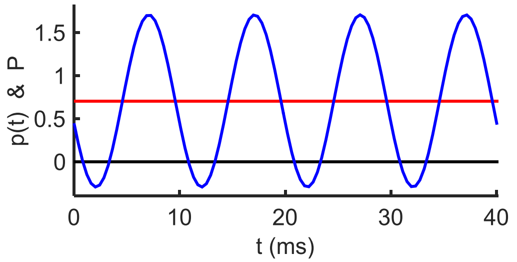
RMS Voltage: $V_{rms} = \sqrt{\langle v^2 \rangle} = \frac{1}{\sqrt{2}}V = 3.54\,\text{V} = \widetilde{V}$

Note: Power engineers *always* use RMS voltages and currents exclusively and omit the "rms" subscript.
For example UK Mains voltage = 230 V rms = 325 V peak.

In this lecture course only, a ~ overbar means $\div\sqrt{2}$: thus $\widetilde{V} = \frac{1}{\sqrt{2}}V$.



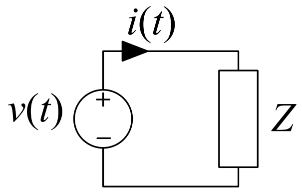




Suppose voltage and current phasors are:

$$V = |V|\, e^{j\theta_V} \quad \Leftrightarrow \quad v(t) = |V| \cos(\omega t + \theta_V)$$

$$I = |I|\, e^{j\theta_I} \quad \Leftrightarrow \quad i(t) = |I| \cos(\omega t + \theta_I)$$



Power dissipated in load Z is

$$\begin{aligned} p(t) = v(t)i(t) &= |V|\,|I| \cos(\omega t + \theta_V) \cos(\omega t + \theta_I) \\ &= |V|\,|I| \left(\tfrac{1}{2} \cos(2\omega t + \theta_V + \theta_I) + \tfrac{1}{2} \cos(\theta_V - \theta_I)\right) \\ &= \tfrac{1}{2} |V|\,|I| \cos(\theta_V - \theta_I) + \tfrac{1}{2} |V|\,|I| \cos(2\omega t + \theta_V + \theta_I) \end{aligned}$$

Average power: $P = \frac{1}{2} |V|\,|I| \cos(\phi)$ where $\phi = \theta_V - \theta_I$

$= \left|\widetilde{V}\right| \left|\widetilde{I}\right| \cos(\phi)$ $\qquad \cos\phi$ is the *power factor*

$\phi > 0 \Leftrightarrow$ a *lagging power factor* (normal case: Current lags Voltage)

$\phi < 0 \Leftrightarrow$ a *leading power factor* (rare case: Current leads Voltage)

[Multiplying Phasors]

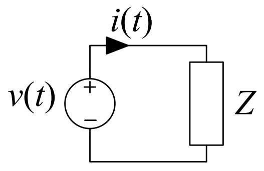
From the previous slide, if the phasor voltage and current are $V = |V|e^{j\theta_V}$ and $I = |I|e^{j\theta_I}$, then the corresponding waveforms are $v(t) = |V|\cos(\omega t + \theta_V)$ and $i(t) = |I|\cos(\omega t + \theta_I)$. When you multiply these two wavefoms together you get $p(t) = \frac{1}{2}|V|\,|I|\,\cos(\theta_V - \theta_I) + \frac{1}{2}|V|\,|I|\,\cos(2\omega t + \theta_V + \theta_I)$. This product contains two components: a constant, or DC, term that doesn't involve t and a second term that is a cosine wave of frequency 2ω.

The time-average of the second term is zero (because a cosine wave of any non-zero frequency goes symmetrically positive and negative and so averages to zero) and so the average power is just equal to the first term: $\frac{1}{2}|V|\,|I|\,\cos(\theta_V - \theta_I)$. It is easy to see that $V \times I^* = |V|e^{j\theta_V} \times |I|e^{-j\theta_I} = |V|\,|I|\,e^{j(\theta_V - \theta_I)} = |V|\,|I|\,\cos(\theta_V - \theta_I) + j|V|\,|I|\,\sin(\theta_V - \theta_I)$ and so the average power is the real part of $\frac{1}{2}V \times I^*$.

The second term is a cosine wave at a frequency of $2\omega$and so it is possible to represent this waveform, $\frac{1}{2}|V|\,|I|\,\cos(2\omega t + \theta_V + \theta_I)$, as a phasor whose value is $\frac{1}{2}V \times I = \frac{1}{2}|V|\,|I|\,e^{j(\theta_V + \theta_I)}$.

So to sum up, if you multiply together the two sinusoidal waveforms corresponding to phasors V and I, you get two components: (a) a DC component of value $\Re\left(\frac{1}{2}V \times I^*\right)$ and (b) a sinusoidal component of twice the frequency which corresponds to the phasor $\frac{1}{2}V \times I$.

If $\widetilde{V} = \frac{1}{\sqrt{2}} |V| e^{j\theta_V}$ and $\widetilde{I} = \frac{1}{\sqrt{2}} |I| e^{j\theta_I}$

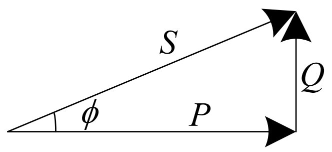


The *complex power* absorbed by Z is $S \triangleq \widetilde{V} \times \widetilde{I}^*$ where * means complex conjugate.

$$\widetilde{V} \times \widetilde{I}^* = \left|\widetilde{V}\right| e^{j\theta_V} \times \left|\widetilde{I}\right| e^{-j\theta_I} = \left|\widetilde{V}\right| \left|\widetilde{I}\right| e^{j(\theta_V - \theta_I)}$$

$$= \left|\widetilde{V}\right| \left|\widetilde{I}\right| e^{j\phi} = \left|\widetilde{V}\right| \left|\widetilde{I}\right| \cos\phi + j \left|\widetilde{V}\right| \left|\widetilde{I}\right| \sin\phi$$

$$= P + jQ$$



Complex Power: $S \triangleq \widetilde{V}\widetilde{I}^* = P + jQ$ measured in Volt-Amps (VA)

Apparent Power: $|S| \triangleq \left|\widetilde{V}\right| \left|\widetilde{I}\right|$ measured in Volt-Amps (VA)

Average Power: $P \triangleq \Re(S)$ measured in Watts (W)

Reactive Power: $Q \triangleq \Im(S)$ Measured in Volt-Amps Reactive (VAR)

Power Factor: $\cos\phi \triangleq \cos\left(\angle\widetilde{V} - \angle\widetilde{I}\right) = \frac{P}{|S|}$

Machines and transformers have capacity limits and power losses that are independent of $\cos\phi$; their ratings are always given in apparent power.

Power Company: Costs $\propto$ apparent power, Revenue $\propto$ average power.

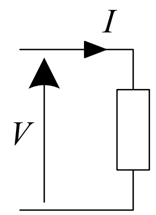
Power in R, L, C

For any impedance, Z, complex power absorbed: $S = \widetilde{V}\widetilde{I}^* = P + jQ$

Using (a) $\widetilde{V} = \widetilde{I}Z$ (b) $\widetilde{I} \times \widetilde{I}^* = \left|\widetilde{I}\right|^2$ we get $S = \left|\widetilde{I}\right|^2 Z = \frac{\left|\widetilde{V}\right|^2}{Z^*}$

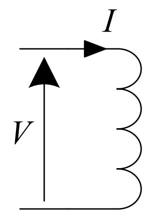
Resistor: $S = \left|\widetilde{I}\right|^2 R = \frac{\left|\widetilde{V}\right|^2}{R}$ $\qquad \phi = 0$

Absorbs average power, no VARs $(Q = 0)$



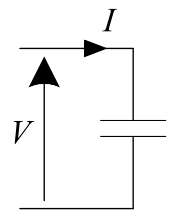
Inductor: $S = j\left|\widetilde{I}\right|^2 \omega L = j\frac{\left|\widetilde{V}\right|^2}{\omega L}$ $\qquad \phi = +90°$

No average power, Absorbs VARs $(Q > 0)$



Capacitor: $S = -j\frac{\left|\widetilde{I}\right|^2}{\omega C} = -j\left|\widetilde{V}\right|^2 \omega C$ $\qquad \phi = -90°$

No average power, Generates VARs $(Q < 0)$

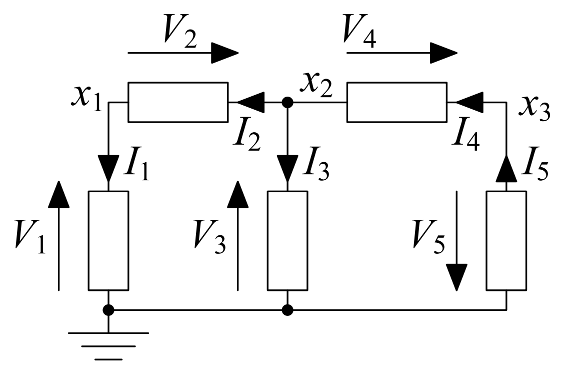


VARs are generated by capacitors and absorbed by inductors

The phase, ϕ, of the absorbed power, S, equals the phase of Z

Tellegen's Theorem: The complex power, S, dissipated in any circuit's components sums to zero.

x_n = voltage at node n
V_b, I_b = voltage/current in branch b
(obeying passive sign convention)



$$a_{bn} \triangleq \begin{cases} -1 & \text{if } V_b \text{ starts from node } n \\ +1 & \text{if } V_b \text{ ends at node } n \\ 0 & \text{else} \end{cases}$$

e.g. branch 4 goes from 2 to 3 $\Rightarrow a_{4*} = [0, -1, 1]$

branches (b) \ nodes (n): a_{bn}	1	2	3
1	1	0	0
2	–1	1	0
3	0	1	0
4	0	–1	1
5	0	0	–1

Branch voltages: $V_b = \sum_n a_{bn} x_n$ (e.g. $V_4 = x_3 - x_2$)

KCL @ node n: $\sum_b a_{bn} I_b = 0 \quad \Rightarrow \quad \sum_b a_{bn} I_b^* = 0$

Tellegen: $\sum_b V_b I_b^* = \sum_b \sum_n a_{bn} x_n I_b^*$

$$= \sum_n \sum_b a_{bn} I_b^* x_n = \sum_n x_n \sum_b a_{bn} I_b^* = \sum_n x_n \times 0$$

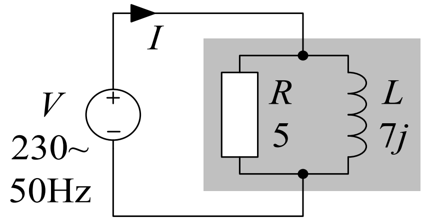
Note: $\sum_b S_b = 0 \quad \Rightarrow \quad \sum_b P_b = 0$ and also $\sum_b Q_b = 0$.

$\widetilde{V} = 230$. Motor modelled as $5||7j\,\Omega$.

$\widetilde{I} = \frac{\widetilde{V}}{R} + \frac{\widetilde{V}}{Z_L} = 46 - j32.9\,\text{A} = 56.5\angle -36°$

$S = \widetilde{V}\widetilde{I}^* = 10.6 + j7.6\,\text{kVA} = 13\angle 36°\,\text{kVA}$

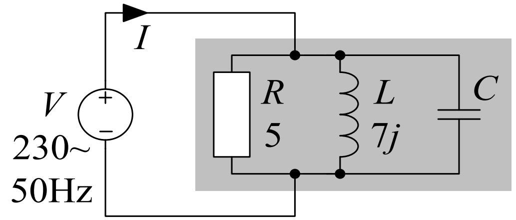
$\cos\phi = \frac{P}{|S|} = \cos 36° = 0.81$



Add parallel capacitor of $300\,\mu$F:

$Z_C = \frac{1}{j\omega C} = -10.6j\,\Omega \Rightarrow \widetilde{I}_C = 21.7j\,\text{A}$

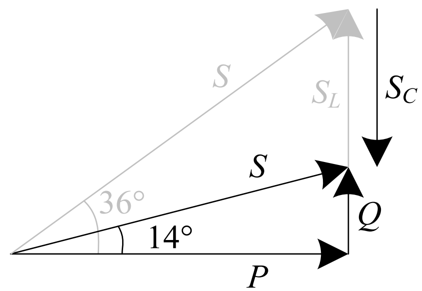
$\widetilde{I} = 46 - j11.2\,\text{A} = 47\angle -14°\,\text{A}$



$S_C = \widetilde{V}\widetilde{I}_C^* = -j5\,\text{kVA}$

$S = \widetilde{V}\widetilde{I}^* = 10.6 + j2.6\,\text{kVA} = 10.9\angle 14°\,\text{kVA}$

$\cos\phi = \frac{P}{|S|} = \cos 14° = 0.97$



Average power to motor, P, is $10.6\,\text{kW}$ in both cases.

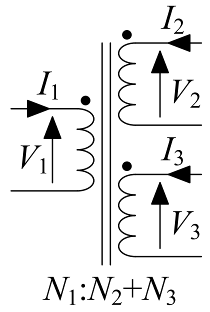
$\left|\widetilde{I}\right|$, reduced from $56.5 \searrow 47\,\text{A}$ $(-16\%) \Rightarrow$ lower losses.

Effect of C: VARs $= 7.6 \searrow 2.6\,\text{kVAR}$, power factor $= 0.81 \nearrow 0.97$.

Ideal Transformer

A transformer has ≥ 2 windings on the same magnetic core.

Ampère's law: $\sum N_r I_r = \frac{l\Phi}{\mu A}$; Faraday's law: $\frac{V_r}{N_r} = \frac{d\Phi}{dt}$.
$N_1 : N_2 + N_3$ shows the turns ratio between the windings.
The • indicates the voltage polarity of each winding.

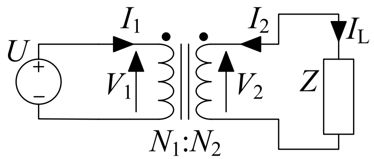


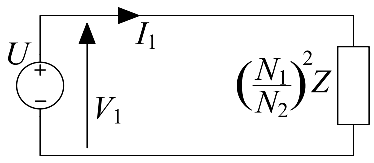
Since Φ is the same for all windings, $\frac{V_1}{N_1} = \frac{V_2}{N_2} = \frac{V_3}{N_3}$.

Assume $\mu \to \infty \Rightarrow N_1 I_1 + N_2 I_2 + N_3 I_3 = 0$

These two equations allow you to solve circuits and also imply that $\sum S_i = 0$.

Special Case:





For a 2-winding transformer this simplifies to
$V_2 = \frac{N_2}{N_1} V_1$ and $I_L = -I_2 = \frac{N_1}{N_2} I_1$

Hence $\frac{V_1}{I_1} = \left(\frac{N_1}{N_2}\right)^2 \frac{V_2}{I_L} = \left(\frac{N_1}{N_2}\right)^2 Z$

Equivalent to a *reflected impedance* of $\left(\frac{N_1}{N_2}\right)^2 Z$

Transformer Applications

Power Transmission

Suppose a power transmission cable has $1\,\Omega$ resistance.
$100\,\text{kVA}@\ 1\,\text{kV} = 100\,\text{A} \Rightarrow \widetilde{I}^2 R = 10\,\text{kW}$ losses.
$100\,\text{kVA}@\ 100\,\text{kV} = 1\,\text{A} \Rightarrow \widetilde{I}^2 R = 1\,\text{W}$ losses.

Voltage Conversion

Electronic equipment requires $\le 20\,\text{V}$ but mains voltage is $240\,\text{V} \sim$.

Interference protection

Microphone on long cable is susceptible to interference from nearby mains cables. An $N:1$ transformer reduces the microphone voltage by N but reduces interference by N^2.

Isolation

There is no electrical connection between the windings of a transformer so circuitry (or people) on one side will not be endangered by a failure that results in high voltages on the other side.

Summary

- Complex Power: $S = \widetilde{V}\widetilde{I}^* = P + jQ$ where $\widetilde{V} = V_{rms} = \frac{1}{\sqrt{2}}V$.
  - For an impedance Z: $S = \left|\widetilde{I}\right|^2 Z = \frac{\left|\widetilde{V}\right|^2}{Z^*}$
  - Apparent Power: $|S| = \left|\widetilde{V}\right| \left|\widetilde{I}\right|$ used for machine ratings.
  - Average Power: $P = \Re(S) = \left|\widetilde{V}\right| \left|\widetilde{I}\right| \cos\phi$ (in Watts)
  - Reactive Power: $Q = \Im(S) = \left|\widetilde{V}\right| \left|\widetilde{I}\right| \sin\phi$ (in VARs)
  - Power engineers *always* use $\widetilde{V}$ and $\widetilde{I}$ and omit the $\tilde{}$.
- Tellegen: In any circuit $\sum_b S_b = 0 \Rightarrow \sum_b P_b = \sum_b Q_b = 0$
- Power Factor Correction: add parallel C to generate extra VARs
- Ideal Transformer: $V_i \propto N_i$ and $\sum N_i I_i = 0$ (implies $\sum S_i = 0$)

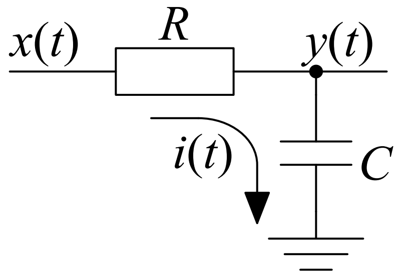
For further details see Hayt Ch 11 or Irwin Ch 9.

15: Transients (A)

Differential Equation

To find $y(t)$:

$x(t)$ constant: Nodal analysis
$x(t)$ sinusoidal: Phasors + nodal analysis
$x(t)$ anything else: Differential equation



$i(t) = C\frac{dy}{dt} = \frac{x-y}{R} \Rightarrow RC\frac{dy}{dt} + y = x$

General Solution: Particular Integral + Complementary Function

Particular Integral: Any solution to $RC\frac{dy}{dt} + y = x$

If $x(t)$ is piecewise constant or sinusoidal, we will use nodal/phasor analysis to find the *steady state solution* for $y(t)$.

Complementary Function: Solution to $RC\frac{dy}{dt} + y = 0$

Does not depend on $x(t)$, only on the circuit.

Solution is $y(t) = Ae^{-t/\tau}$

where $\tau = RC$ is the *time constant* of the circuit.

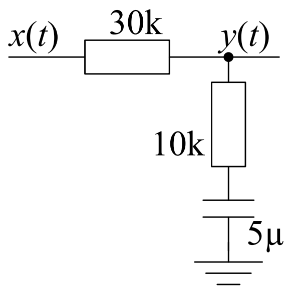
The amplitude, A, is determined by the initial conditions at $t = 0$.

Piecewise steady state inputs

We will consider input signals that are sinusoidal or constant for a particular time interval and then suddenly change in amplitude, phase or frequency.

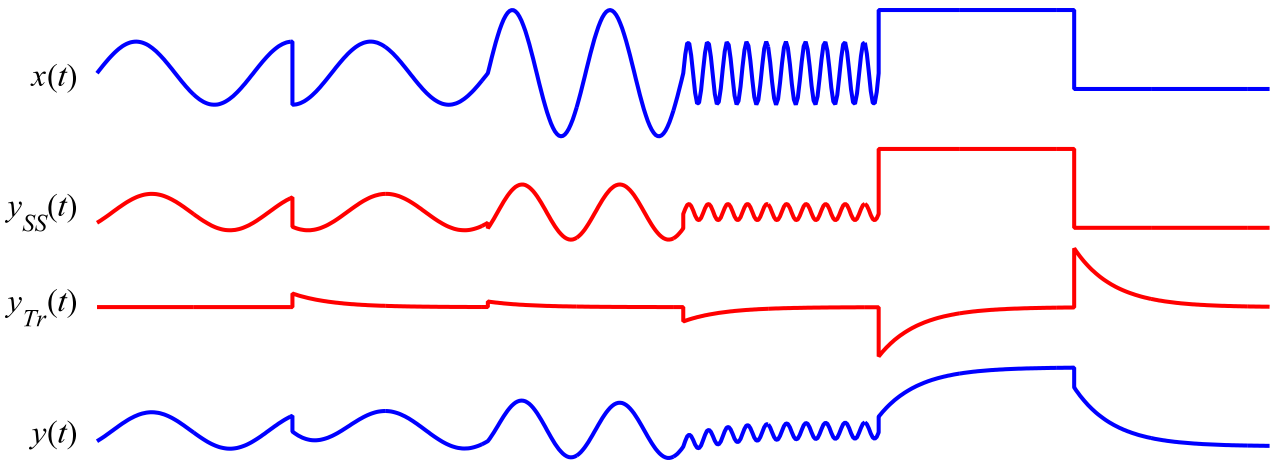
Output is the sum of the steady state and a transient:

$$y(t) = y_{SS}(t) + y_{Tr}(t)$$

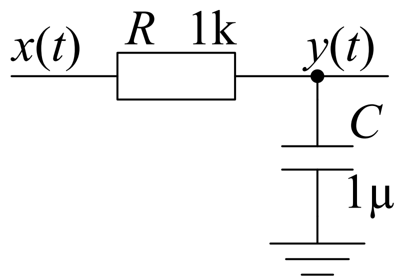


Steady state, $y_{SS}(t)$, is the same frequency as the input; use phasors + nodal analysis.

Transient is always $y_{Tr}(t) = Ae^{-\frac{t}{\tau}}$ at each change.

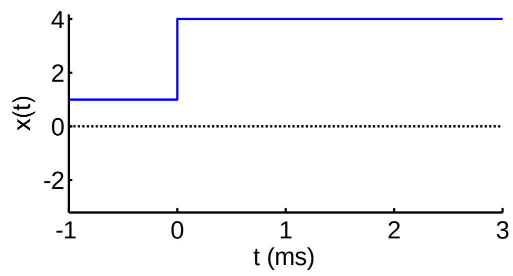


For $t < 0$, $y(t) = x(t) = 1$
For $t \geq 0$, $RC\frac{dy}{dt} + y = x = 4$
Time Const: $\tau = RC = 1\,\text{ms}$



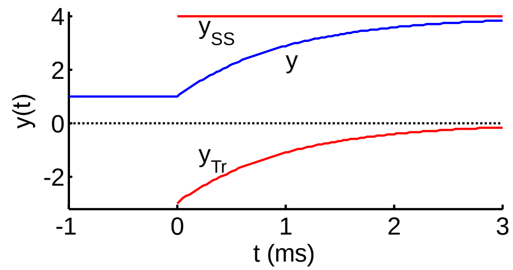
Steady State (Particular Integral)
$y_{SS}(t) = x(t) = 4$ for $t \geq 0$

Transient (Complementary Function)
$y_{Tr}(t) = Ae^{-t/\tau}$



Steady State + Transient
$y(t) = y_{SS} + y_{Tr} = 4 + Ae^{-t/\tau}$

To find A, use capacitor property:
Capacitor voltage never changes abruptly



$y(0+) = 4 + A$ and $y(0-) = 1 \Rightarrow 4 + A = 1 \Rightarrow A = -3$

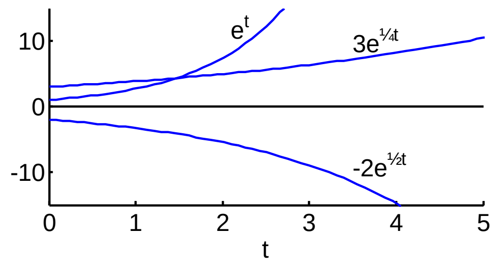
So transient: $y_{Tr}(t) = -3e^{-t/\tau}$ and total $y(t) = 4 - 3e^{-t/\tau}$

Transient amplitude $\Leftarrow$ capacitor voltage continuity: $v_C(0+) = v_C(0-)$

Negative exponentials

Positive exponentials grow to $\pm\infty$:

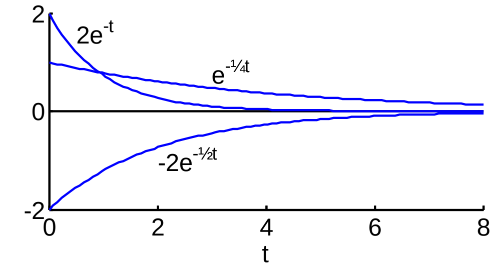
e^{t}, $3e^{t/4}$, $-2e^{t/2}$



Negative exponentials decay to 0:

$2e^{-t}$, $e^{-t/4}$, $-2e^{-t/2}$

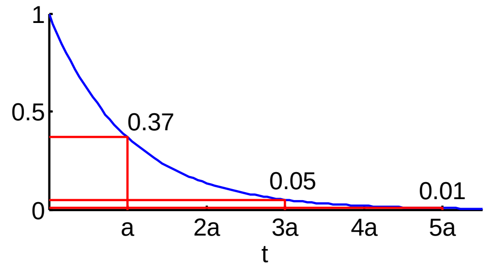
Transients are negative exponentials.



Decay rate of $e^{-t/a}$

37% after 1 time constant

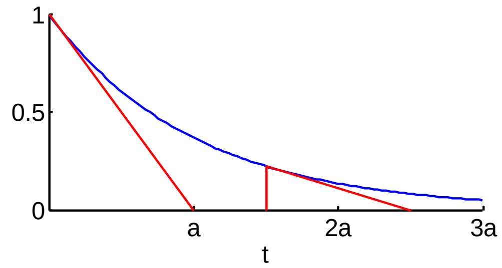
5% after 3, <1% after 5



Gradient of $e^{-t/a}$

Gradient at t hits zero at $t + a$.

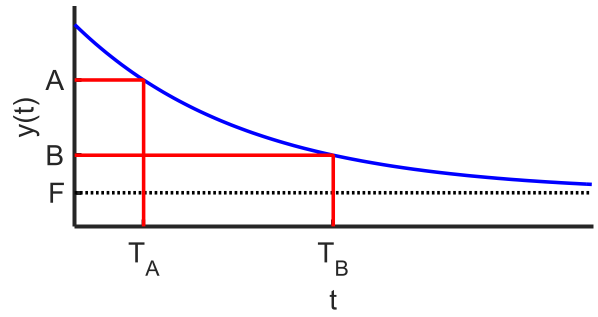
True for any t.



Exponential Time Delays

Negative exponential with a final value of F.

$$y(t) = F + (A - F)\, e^{-(t - T_A)/\tau}$$



How long does it take to go from A to B ?

At $t = T_B$:

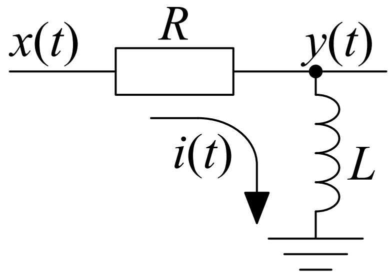
$$y(T_B) = B = F + (A - F)\, e^{-(T_B - T_A)/\tau}$$

$$\frac{B-F}{A-F} = e^{-(T_B - T_A)/\tau}$$

Hence $T_B - T_A = \tau \ln\left(\frac{A-F}{B-F}\right) = \tau \ln\left(\frac{\text{initial distance to } F}{\text{final distance to } F}\right)$

Useful formula - worth remembering.

We know $i = \frac{x-y}{R}$
$y(t) = L\frac{di}{dt} = \frac{L}{R} \times \frac{d(x-y)}{dt} = \frac{L}{R}\frac{dx}{dt} - \frac{L}{R}\frac{dy}{dt}$
$\Rightarrow \frac{L}{R}\frac{dy}{dt} + y = \frac{L}{R}\frac{dx}{dt}$



Solution: Particular Integral + Complementary Function

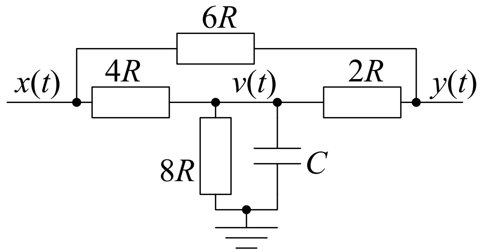
Particular Integral: Any solution to $\frac{L}{R}\frac{dy}{dt} + y = \frac{L}{R}\frac{dx}{dt}$
If $x(t)$ is piecewise constant or sinusoidal, we will use nodal/phasor analysis to find the *steady state solution*, $y_{SS}(t)$.

Complementary Function: Solution to $\frac{L}{R}\frac{dy}{dt} + y = 0$
Does not depend on $x(t)$, only on the circuit.
Solution is $y_{Tr}(t) = Ae^{-t/\tau}$
where $\tau = \frac{L}{R}$ is the *time constant* of the circuit.

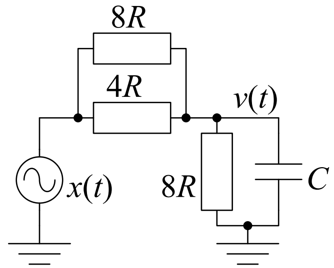
1st order transient is *always* $y_{Tr}(t) = Ae^{-t/\tau}$ where $\tau = RC$ or $\frac{L}{R}$
Amplitude $A \Leftarrow$ no abrupt change in capacitor voltage or inductor current.

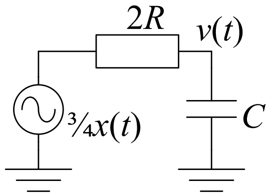
Linearity

1st order circuit has only one C or L.
Make a Thévenin equivalent of the network connected to the terminals of C. Remember x is a voltage source but y is not.

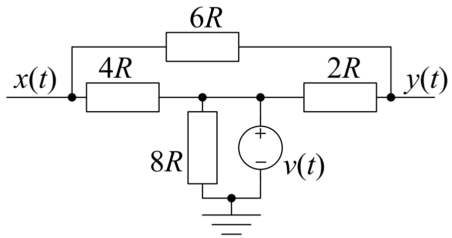


Now $v(t) = v_{SS}(t) + v_{Tr}(t)$
$= v_{SS}(t) + Ae^{-t/\tau}$
Time constant is $\tau = R_{Th}C$
where R_{Th} is the Thévenin resistance.





Replace the capacitor with a voltage source $v(t)$; all voltages and currents in the circuit will remain unchanged.



Linearity: $y = ax + bv = ax + bv_{SS} + bv_{Tr} = y_{SS} + bv_{Tr}$

All voltages and currents in a circuit have the same transient (but scaled).

The *circuit's time constant* is $\tau = R_{Th}C$ or $\frac{L}{R_{Th}}$ where R_{Th} is the Thévenin resistance of the network connected to C or L.

Transient Amplitude

Find Steady State (DC $\Rightarrow Z_L = 0$)

Potential divider: $y_{SS} = \frac{1}{2}x$

$y_{SS}(0-) = 1,\ y_{SS}(0+) = 3$

Inductor Current Continuity

$i_{SS}(0-) = 1\,\text{mA} \Rightarrow i_L(0+) = 1\,\text{mA}$

At $t = 0+$

$x - y = 1\,\text{mA} \times 1\,\text{k} = 1$

$y(0+) = x(0+) - 1 = 5$

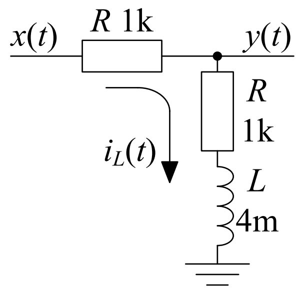
Time Constant

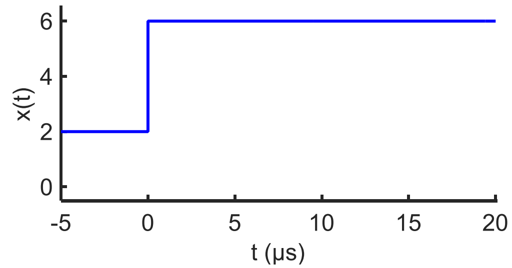
Set $x \equiv 0 \to R_{Th} = 2\,\text{k}$

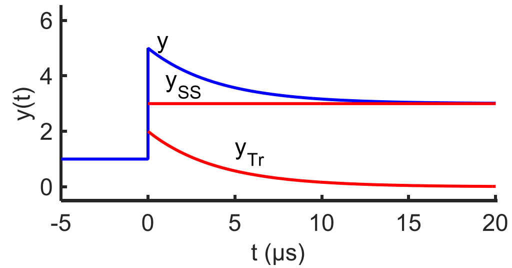
$\tau = \frac{L}{R_{Th}} = 2\,\mu\text{s}$

Result

$$
\begin{aligned}
y &= y_{SS} + \left(y\left(0+\right) - y_{SS}\left(0+\right)\right) e^{-t/\tau} \\
&= 3 + \left(5 - 3\right) e^{-t/\tau} \\
&= 3 + 2e^{-t/\tau}
\end{aligned}
$$







Capacitor Voltage Continuity

Find Steady State (DC $\Rightarrow Z_C = \infty$)

KCL @ V: $\frac{v-x}{4R} + \frac{v}{8R} + \frac{v-y}{2R} = 0$

KCL @ Y: $\frac{y-v}{2R} + \frac{y-x}{6R} = 0$

$v_{SS} = \frac{3}{4}x,\ y_{SS} = \frac{13}{16}x$

Capacitor Voltage Continuity

$v_{SS}(0-) = -3 \Rightarrow v(0+) = -3$

At $t = 0+$: $x = 4$ and $v = -3$

KCL @ Y: $\frac{y-(-3)}{2R} + \frac{y-4}{6R} = 0$

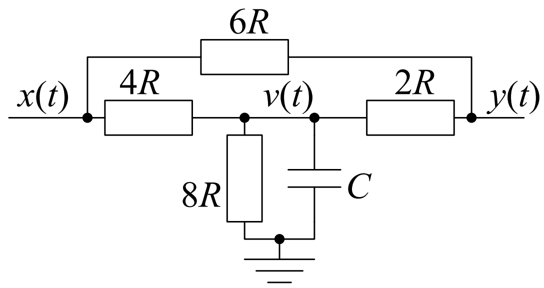
$y(0+) = \frac{-9+4}{4} = -\frac{5}{4}$

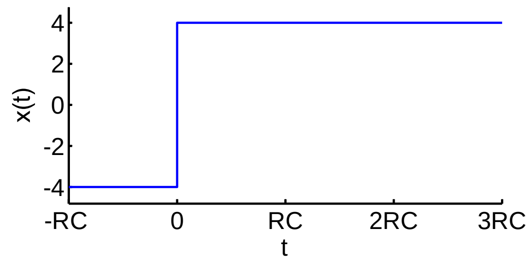
Time Constant

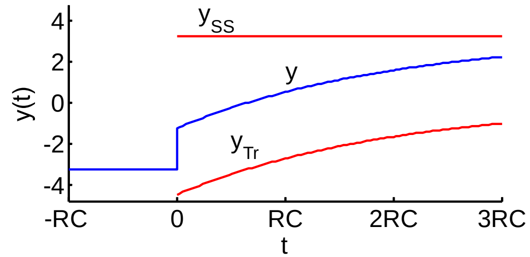
$\tau = R_{Th}C = 2RC$ (from earlier slide)

Result

$$y = y_{SS} + \left(y\left(0+\right) - y_{SS}\left(0+\right)\right)e^{-t/\tau}$$
$$= \frac{13}{4} + \left(-\frac{5}{4} - \frac{13}{4}\right)e^{-t/\tau}$$
$$= \frac{13}{4} - \frac{18}{4}e^{-t/\tau} = 3\tfrac{1}{4} - 4\tfrac{1}{2}e^{-t/2RC}$$







Summary

- 1st order circuits: include one C or one L.
  - v_C or i_L never change abruptly. The output, y, is not necessarily continuous unless it equals v_C.
- Circuit time constant: $\tau = R_{Th}C$ or $\frac{L}{R_{Th}}$
  - R_{Th} is the Thévenin resistance seen by C or L.
  - Same τ for all voltages and currents.
- Output = Steady State + Transient
  - Steady State: use nodal/Phasor analysis when input is piecewise constant or piecewise sinusoidal. The steady state has the same frequency as the input signal.
  - Transient: Find $v_C(0-)$ or $i_L(0-)$: unchanged at $t = 0+$
    Find $y(0+)$ assuming source of $v_C(0+)$ or $i_L(0+)$
    Amplitude never complex, never depends on t.
  - $y(t) = y_{SS}(t) + (y(0+) - y_{SS}(0+))\, e^{-t/\tau}$

See Hayt Ch 8 or Irwin Ch 7.

16: Transients (B)

Piecewise steady state inputs

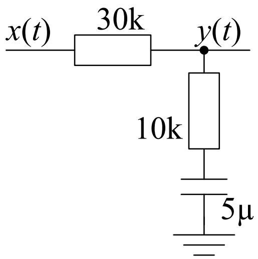
We will consider input signals that are sinusoidal or constant for a particular time interval and then suddenly change in amplitude, phase or frequency.

Output is the sum of the steady state and a transient:

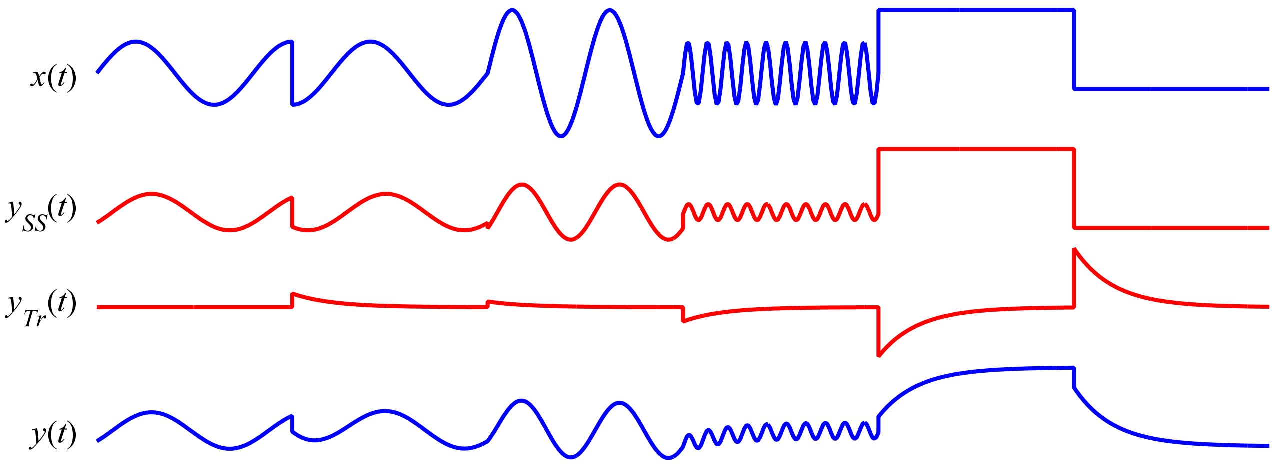
$$y(t) = y_{SS}(t) + y_{Tr}(t)$$

Steady state, $y_{SS}(t)$, is the same frequency as the input; use phasors + nodal analysis.

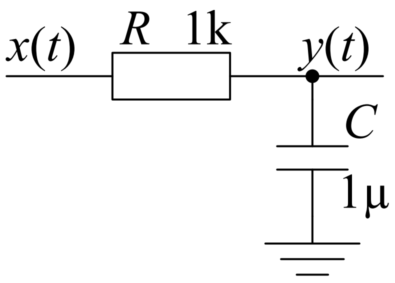
Transient is always $y_{Tr}(t) = Ae^{-\frac{t}{\tau}}$ at each change.



[only one C or L]

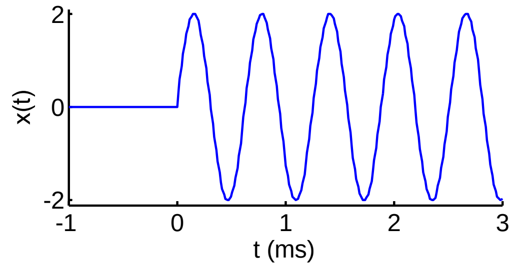


For $t < 0$: $y(t) = x(t) = 0$
For $t \geq 0$: $x = 2\sin\omega t \Rightarrow X = -2j$
$\tau = RC = 1\,\text{ms},\ \omega = 10\,\text{krad/s}$



Steady State (for $t \geq 0$)

$$\frac{Y}{X} = \frac{1}{j\omega RC+1} = 0.1\angle-84^\circ$$
$$Y = X \times \frac{Y}{X} = -2j \times 0.1\angle-84^\circ$$
$$y_{SS}(t) = 0.2\cos(\omega t - 174^\circ)$$

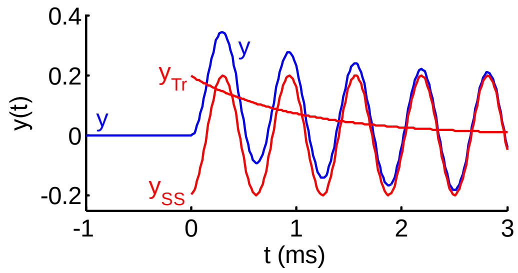


Steady State + Transient

$$y(t) = 0.2\cos(\omega t - 174^\circ) + Ae^{-t/\tau}$$

Transient Amplitude

$$y(0+) = 0.2\cos(-174^\circ) + A$$
$$= -0.198 + A$$



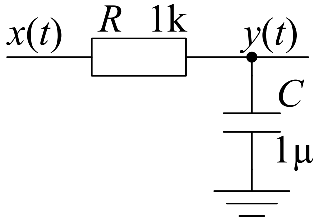
$$y(0+) = y(0-) = 0 \Rightarrow A = 0.198 \Rightarrow y_{Tr}(t) = 0.198e^{-t/\tau}$$

Complete Expression for $y(t)$

$$y(t) = 0.2\cos(\omega t - 174^\circ) + 0.198e^{-t/\tau}$$

Multiple Discontinuities

For $0 \leq t < 0.2\pi$ ms: $X = -2j$, $\omega_1 = 10\,\text{k}$, $\tau = 1\,\text{ms}$
prev page $\Rightarrow y(t) = 0.2\cos(\omega t - 174^\circ) + 0.198e^{-t/\tau}$



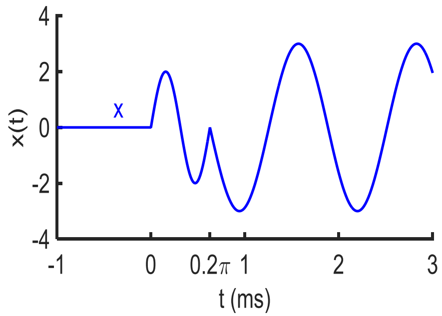
Steady State (for $t \geq 0.0002\pi = 0.63\,\text{ms}$)

$X = -3j$, $\omega_2 = 5\,\text{k}$

$\frac{Y}{X} = \frac{1}{j\omega_2 RC+1} = 0.2\angle - 79^\circ$

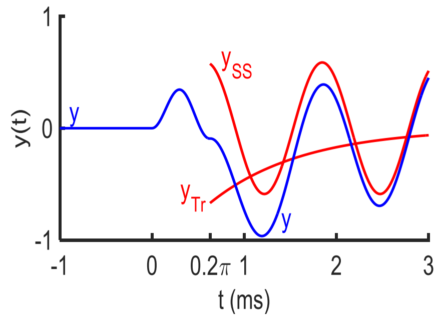
$Y = X \times \frac{Y}{X} = -3j \times 0.2\angle - 79^\circ$

$y_{SS}(t) = 0.59\cos(\omega_2 t - 169^\circ)$



Steady State + Transient (for $t \geq 0.63\,\text{ms}$)

$y = 0.59\cos(\omega_2 t - 169^\circ) + Be^{-(t-0.00063)/\tau}$



Transient Amplitude (at $t = 0.63\,\text{ms}$)

$y(0.00063+) = 0.59\cos(0.00063\omega_2 - 169^\circ) + B$
$= 0.577 + B$

$y(0.00063-) = 0.2\cos(0.00063\omega_1 - 174^\circ) + 0.198e^{-0.00063/\tau} = -0.092$

$\Rightarrow 0.577 + B = -0.092 \Rightarrow B = -0.67 \Rightarrow y_{Tr} = -0.67e^{-(t-0.00063)/\tau}$

Complete Expression for $y(t)$ (for $t \geq 0.63\,\text{ms}$)

$y(t) = 0.59\cos(\omega_2 t - 169^\circ) - 0.67e^{-(t-0.00063)/\tau}$

Operating the switch changes τ:

Closed: $\tau_C = (1\,\mathrm{k}\,||\,9\,\mathrm{k}) \times C = 0.9\,\mathrm{ms}$

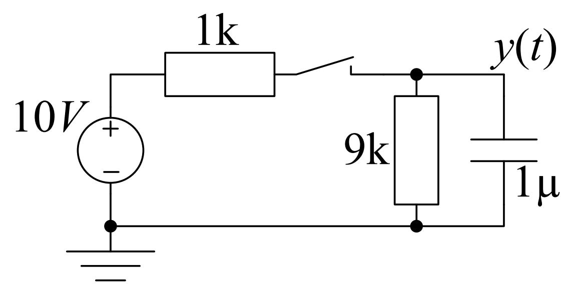
Open: $\tau_O = 9\,\mathrm{k} \times C = 9\,\mathrm{ms}$

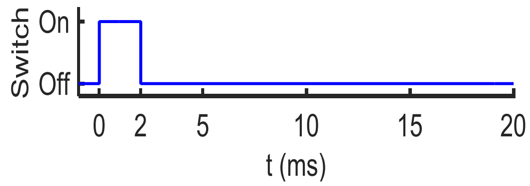
Switch closed at $t = 0$.

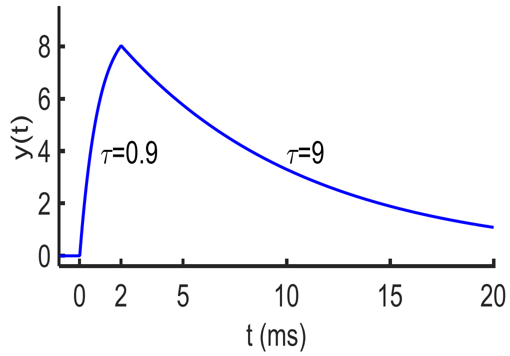
$$y_{SS} = 10 \times \frac{9}{10} = 9\,\mathrm{V}$$
$$y(t) = 9 - 9e^{-t/\tau_C}$$
$$y(2-) = 9 - 9e^{-2/0.9} = 8.02$$

Switch opened at $t = 2$.

$$y_{SS} = 0\,\mathrm{V}$$
$$y(t) = 0 + Ae^{-(t-2)/\tau_O}$$
$$y(2+) = A = y(2-) = 8.02$$
$$y(20) = 8.02e^{-(20-2)/9} = 1.09$$

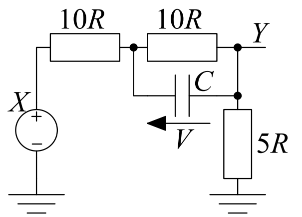






Transfer Function

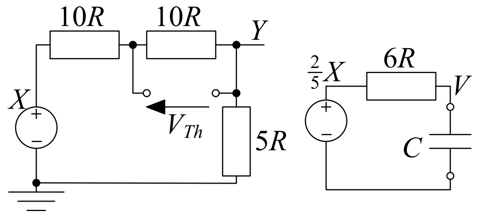
Phasor nodal analysis:



$$\frac{Y}{X} = \frac{5R}{15R+\frac{10R}{1+10j\omega RC}} = \frac{10j\omega RC+1}{30j\omega RC+5} = 0.2\frac{\frac{j\omega}{p}+1}{\frac{j\omega}{q}+1}$$

Corner freqencies: $p = \frac{1}{10RC}$, $q = \frac{1}{6RC}$, HF gain $= \frac{1}{3}$

Thévenin Equivalent driving C:

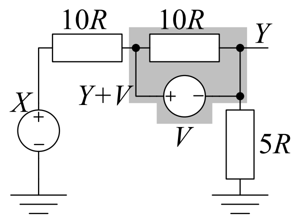


$V_{Th} = \frac{2}{5}X$, $R_{Th} = 10R||15R = 6R$, $\tau = 6RC$

$V = \frac{2}{5}X \times \frac{1}{6j\omega RC+1} = \frac{2}{5}X \times \frac{1}{j\omega\tau+1}$

Denominator is always $(j\omega\tau + 1)$

Linearity: $Y = aX + bV$



KCL @ supernode: $\frac{(Y+V)-X}{10R} + \frac{Y}{5R} = 0 \Rightarrow 3Y + V - X = 0$

$Y = \frac{1}{3}X - \frac{1}{3}V = \frac{1}{3}X - \frac{2}{15}X\left(\frac{1}{j\omega\tau+1}\right) = \frac{X}{15}\left(\frac{5j\omega\tau+3}{j\omega\tau+1}\right)$

Denominator of bV is unchanged by adding aX

(1) Denominator corner frequency is always $\frac{1}{\tau}$ for any transfer function in the circuit.

(2) $V = 0$ at $\omega = \infty$, so since $Y = aX + bV$, $a = \frac{Y}{X}\Big|_{\omega=\infty}$(= HF-gain)

V is never discontinuous so ΔY discontinuity = HF-gain$\times\Delta X$ discontinuity

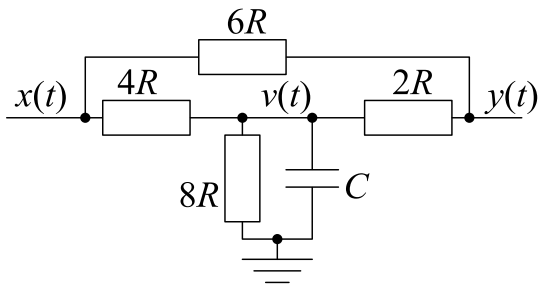
Calculate Transfer Function

KCL @ V: $\frac{V-X}{4R} + \frac{V}{8R} + j\omega CV + \frac{V-Y}{2R} = 0$

KCL @ Y: $\frac{Y-V}{2R} + \frac{Y-X}{6R} = 0$

$\rightarrow$ Transfer Function: $\frac{Y}{X} = \frac{8j\omega RC+13}{32j\omega RC+16}$

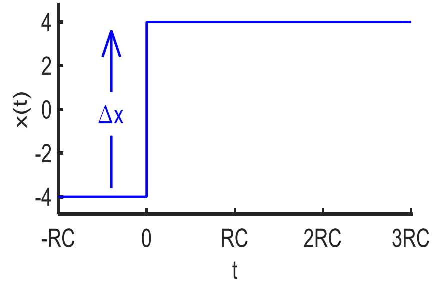
DC gain: $\frac{13}{16}$, HF gain: $\frac{8}{32} = \frac{1}{4}$, $\tau = \frac{32RC}{16} = 2RC$



Steady State

$t < 0$: $y_{SS}(t) = \frac{13}{16}x(t) = \frac{13}{16} \times -4 = -3\frac{1}{4}$

$t \geq 0$: $y_{SS}(t) = \frac{13}{16}x(t) = \frac{13}{16} \times +4 = +3\frac{1}{4}$



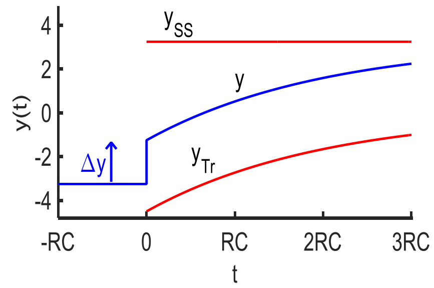
Steady State + Transient (for $t > 0$)

$t \geq 0$: $y = 3\frac{1}{4} + Ae^{-t/\tau}$

Discontinuity Gain (= HF Gain @ $\omega = \infty$)

$\Delta y = y(0+) - y(0-) = \frac{1}{4}\Delta x = \frac{1}{4} \times 8 = 2$

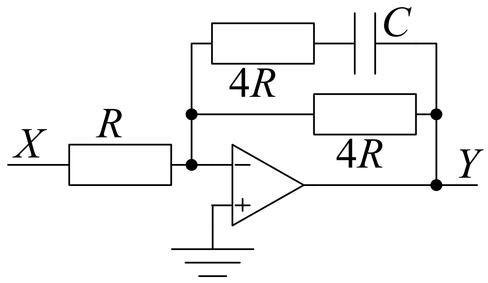
$\left(3\frac{1}{4} + A\right) - \left(-3\frac{1}{4}\right) = 2 \Rightarrow A = -4\frac{1}{2}$



Complete Expression

$t \geq 0$: $y(t) = 3\frac{1}{4} - 4\frac{1}{2}e^{-t/2RC}$

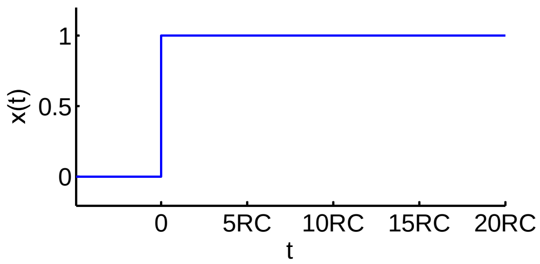
Opamp Circuit Transient



Calculate Transfer Function (Inverting Amplifier)

$$\frac{Y}{X} = -\frac{Z_F}{R} = -\frac{1}{R} \times \frac{4R\left(4R+\frac{1}{j\omega C}\right)}{4R+\left(4R+\frac{1}{j\omega C}\right)} = -4\frac{4j\omega RC+1}{8j\omega RC+1}$$

DC gain: -4, HF gain: -2, $\tau = 8RC$



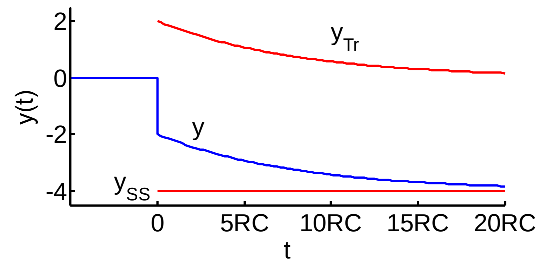
Steady State

$t < 0$: $y_{SS}(t) = -4v(t) = 0$

$t \geq 0$: $y_{SS}(t) = -4v(t) = -4 \times 1 = -4$

Steady State + Transient

$t \geq 0$: $y = -4 + Ae^{-t/\tau}$



Discontinuity Gain (= HF Gain)

$y(0+) - y(0-) = -2\left(x(0+) - x(0-)\right) = -2$

$(-4 + A) - (0) = -2 \Rightarrow A = 2$

Complete Expression

$t \geq 0$: $y(t) = -4 + 2e^{-t/8RC}$

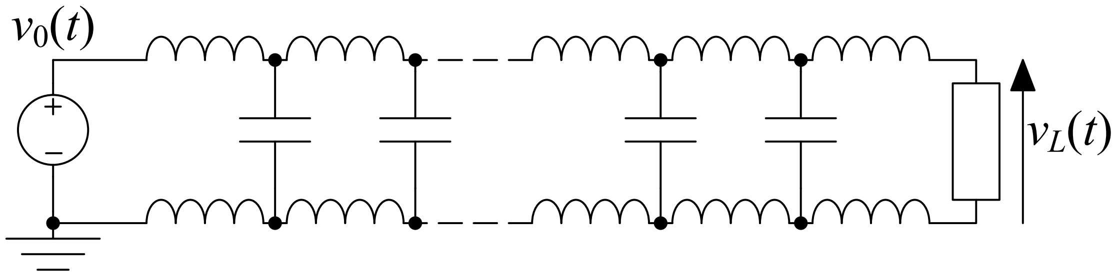
For opamp circuits get τ from the transfer function because R_{Th} is difficult to work out.

Summary

- 1st order transients: circuits with only one C or L
- Transients arise from abrupt changes in the frequency, phase or amplitude of the input signal or else a switch changing
- Output is steady state + transient
- Steady state: nodal analysis $\rightarrow$ transfer function
- Transient: $Ae^{-t/\tau}$ where:
  - Two methods to find τ:
    - ▷ Thévenin seen by L or C: $\tau = R_{Th}C$ or $\frac{L}{R_{Th}}$
    - ▷ Transfer function denominator: $(aj\omega + b) \Rightarrow \tau = \frac{1}{\omega_c} = \frac{a}{b}$
  - Two methods to find A:
    - ▷ Continuity: $\Delta V_C = 0$ or $\Delta I_L = 0$
    - ▷ Discontinuity gain: Δoutput $=$ HF gain $\times\Delta$input

For further details see Hayt Ch 8 or Irwin Ch 7.

17: Transmission Lines



Previously assume that any change in $v_0(t)$ appears instantly at $v_L(t)$.

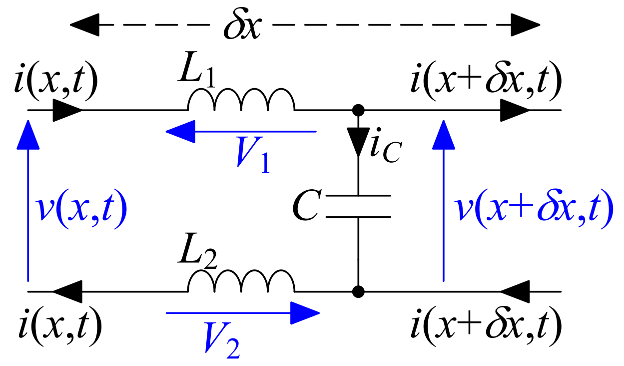
<u>This is not true.</u>

If fact signals travel at around half the speed of light ($c = 30$ cm/ns).

Reason: all wires have capacitance to ground and to neighbouring conductors and also self-inductance. It takes time to change the current through an inductor or voltage across a capacitor.

A *transmission line* is a wire with a uniform goemetry along its length: the capacitance and inductance of any segment is proportional to its length. We represent as a large number of small inductors and capacitors spaced along the line.
The signal speed along a transmisison line is predictable.

A short section of line δx long:



$v(x,t)$ and $i(x,t)$ depend on both position and time.

Small $\delta x \Rightarrow$ ignore 2nd order derivatives:

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial v(x+\delta x,t)}{\partial t} \triangleq \frac{\partial v}{\partial t}.$$

Basic Equations

KVL: $\quad v(x,t) = V_2 + v(x+\delta x,t) + V_1$

KCL: $\quad i(x,t) = i_C + i(x+\delta x,t)$

Capacitor equation: $\quad C\frac{\partial v}{\partial t} = i_C = i(x,t) - i(x+\delta x,t) = -\frac{\partial i}{\partial x}\delta x$

Inductor equation (L_1 and L_2 have the same current):

$$(L_1 + L_2)\frac{\partial i}{\partial t} = V_1 + V_2 = v(x,t) - v(x+\delta x,t) = -\frac{\partial v}{\partial x}\delta x$$

Transmission Line Equations

$$C_0\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}$$
$$L_0\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}$$

where $C_0 = \frac{C}{\delta x}$ is the capacitance per unit length (Farads/m) and $L_0 = \frac{L_1+L_2}{\delta x}$ is the total inductance per unit length (Henries/m).

[Partial Derivatives]

When we differentiate a function of two variables, we keep one of the variables fixed while differentiating with respect to the other; this is called a partial derivative and is written with a curly version of the letter "d". Thus

$$\frac{\partial v}{\partial x} \triangleq \lim_{\delta x \to 0} \frac{v(x+\delta x, t) - v(x,t)}{\delta x} \quad \text{and} \quad \frac{\partial v}{\partial t} \triangleq \lim_{\delta t \to 0} \frac{v(x, t+\delta t) - v(x,t)}{\delta t}.$$

Higher order derivatives may be obtained by differentiating the partial derivatives again to give

$$\frac{\partial^2 v}{\partial x^2} \triangleq \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right), \quad \frac{\partial^2 v}{\partial t^2} \triangleq \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}\right) \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial t} \triangleq \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right).$$

Provided the second order partial derivatives are continuous, the order of differentiation doesn't matter so that $\frac{\partial^2 v}{\partial x \partial t} = \frac{\partial^2 v}{\partial t \partial x}$.

If we take the normal Taylor series with respect to x, $v(x+\delta x,\, t) = v(x,t) + \frac{\partial v(x,t)}{\partial x}\delta x + O\left(\delta x^2\right)$, and differentiate each term with respect to t, we get

$$\frac{\partial v(x+\delta x,\, t)}{\partial t} = \frac{\partial v(x,t)}{\partial t} + \frac{\partial^2 v(x,t)}{\partial t \partial x}\delta x + O\left(\delta x^2\right).$$

If $\delta x \to 0$, then we get $\frac{\partial v(x+\delta x,\, t)}{\partial t} \to \frac{\partial v(x,t)}{\partial t}$ as assumed on the previous slide.

[Deriving the Transmission Line Equations]

This note provides slightly more detail about how we derive the transmission line equations. By expanding $v(x+\delta x, t)$ and $i(x+\delta x, t)$ as Taylor Series in x, we can write

$$v(x+\delta x,t)=v(x,t)+\delta x\frac{\partial v}{\partial x}(x,t)+O(\delta x^2) \quad \text{and} \quad i(x+\delta x,t)=i(x,t)+\delta x\frac{\partial i}{\partial x}(x,t)+O(\delta x^2).$$

From the diagram on the previous page, the voltage across the capacitor is $v(x+\delta x, t)$ and so the capacitor equation is

$$C\frac{\partial v}{\partial t}(x+\delta x,t)=i(x,t)-i(x+\delta x,t).$$

Substituting in the Taylor series expansions for $v(x+\delta x, t)$ and $i(x+\delta x, t)$ and also substituting $C = C_0\delta x$ results in

$$\begin{aligned} C_0\delta x\left(\frac{\partial v}{\partial t}(x,t)+\delta x\frac{\partial^2 v}{\partial x\partial t}(x,t)+O(\delta x^2)\right) &= -\delta x\frac{\partial i}{\partial x}(x,t)-O(\delta x^2) \\ \Rightarrow \quad C_0\left(\frac{\partial v}{\partial t}(x,t)+\delta x\frac{\partial^2 v}{\partial x\partial t}(x,t)+O(\delta x^2)\right) &= -\frac{\partial i}{\partial x}(x,t)-O(\delta x). \end{aligned}$$

Finally, we let $\delta x \to 0$ and so all the terms that are $O(\delta x)$ or smaller disappear which leaves

$$C_0\frac{\partial v}{\partial t}(x,t)=-\frac{\partial i}{\partial x}(x,t).$$

The inductor equation, $L_0\frac{\partial i}{\partial t}=-\frac{\partial v}{\partial x}$, can be derived in a similar way.

Transmission Line Equations: $C_0 \frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}$ $\quad L_0 \frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}$

General solution:
$$v(t,x) = f(t - \tfrac{x}{u}) + g(t + \tfrac{x}{u})$$
$$i(t,x) = \frac{f(t-\frac{x}{u}) - g(t+\frac{x}{u})}{Z_0}$$

where $u = \sqrt{\frac{1}{L_0 C_0}}$ and $Z_0 = \sqrt{\frac{L_0}{C_0}}$.

u is the *propagation velocity* and Z_0 is the *characteristic impedance*.

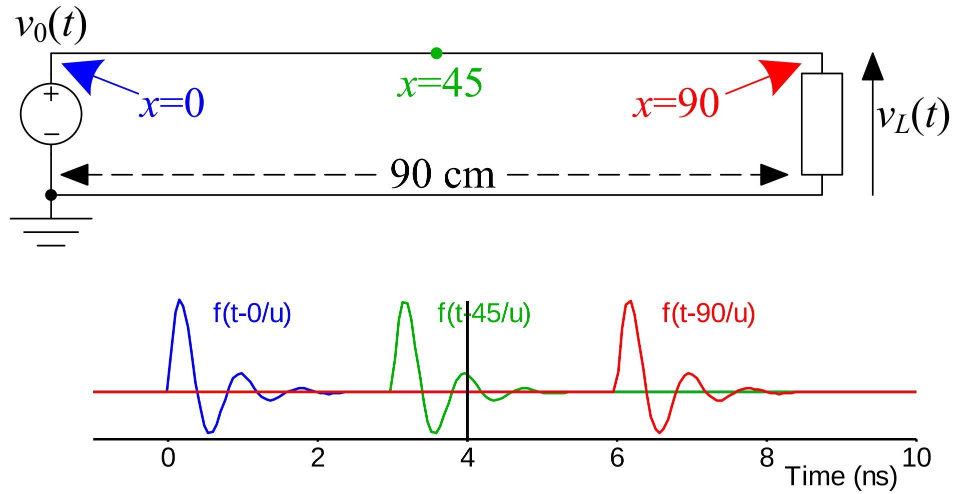
$f()$ and $g()$ can be *any* differentiable functions.

Verify by substitution:

$$-\frac{\partial i}{\partial x} = -\left(\frac{-f'(t-\frac{x}{u}) - g'(t+\frac{x}{u})}{Z_0} \times \frac{1}{u}\right)$$
$$= C_0 \left(f'(t - \tfrac{x}{u}) + g'(t + \tfrac{x}{u})\right) = C_0 \frac{\partial v}{\partial t}$$

Forward Wave

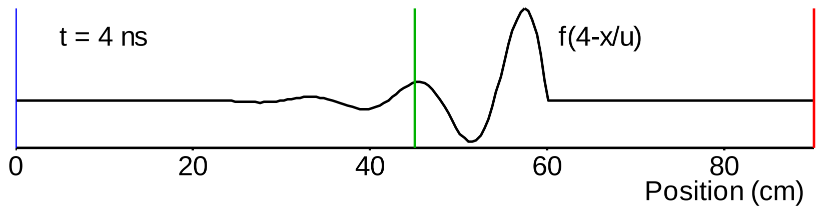
Suppose:
$u = 15\,\mathrm{cm/ns}$
and $g(t) \equiv 0$
$\Rightarrow v(x,t) = f\left(t - \frac{x}{u}\right)$



- At $x = 0$ cm [▲], $v_S(t) = f(t - \frac{0}{u})$
- At $x = 45$ cm [▲], $v(45,t) = f(t - \frac{45}{u})$
  $f(t - \frac{45}{u})$ is exactly the same as $f(t)$ but delayed by $\frac{45}{u} = 3$ ns.
- At $x = 90$ cm [▲], $v_R(t) = f(t - \frac{90}{u})$; now delayed by $6\,\mathrm{ns}$.

Waveform at $x = 0$ completely determines the waveform everywhere else.

Snapshot at $t_0 = 4\,\mathrm{ns}$:
the waveform has just arrived at the point $x = ut_0 = 60$ cm.

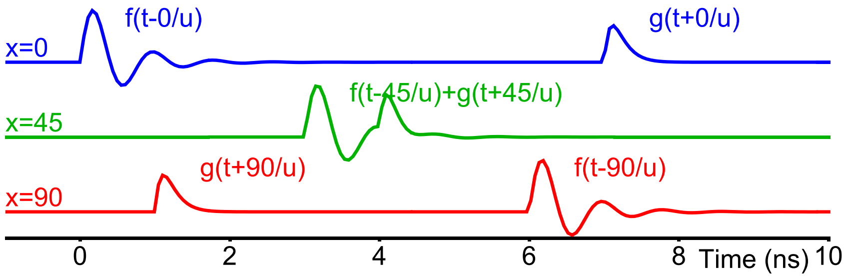


$f(t - \frac{x}{u})$ is a wave travelling forward (i.e. towards +x) along the line.

Similarly $g(t + \frac{x}{u})$ is a wave travelling backwards, i.e. in the $-x$ direction.

$v(x,t) =$
$\quad f(t - \frac{x}{u}) + g(t + \frac{x}{u})$

At $x = 0$ cm [▲],
$\quad v_S(t) = f(t) + g(t)$

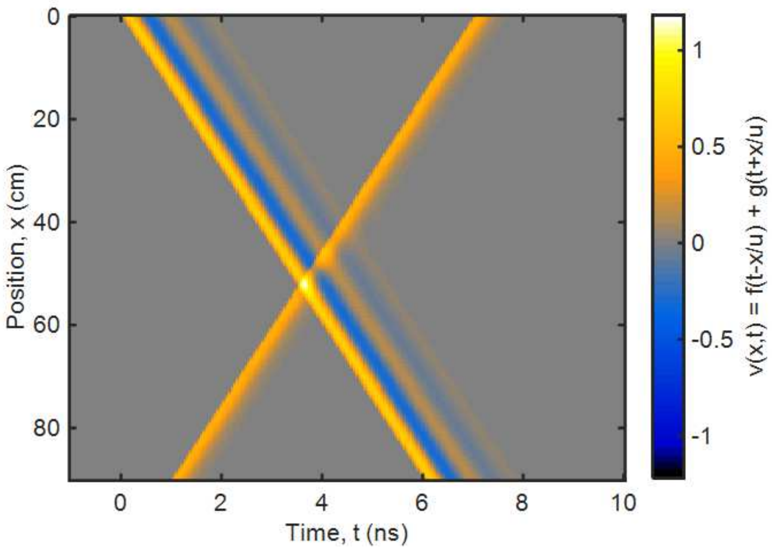


At $x = 45$ cm [▲], g is only 1 ns behind f and they add together.
At $x = 90$ cm [▲], g starts at $t = 1$ and f starts at $t = 6$.

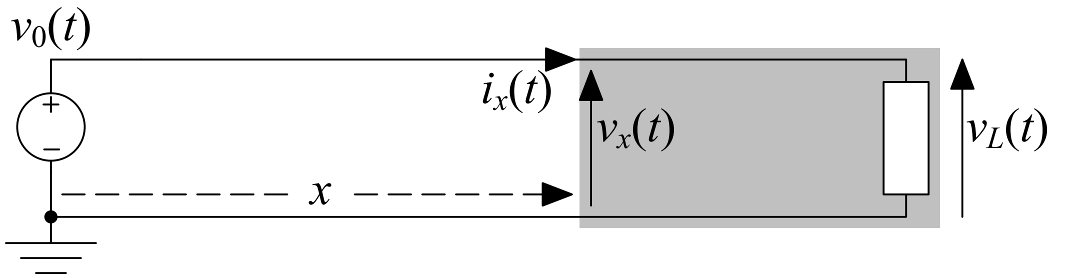
A vertical line on the diagram gives a snapshot of the entire line at a time instant t.

f and g first meet at $t = 3.5$ and $x = 52.5$.

Magically, f and g pass through each other entirely unaltered.



Define $f_x(t) = f\left(t - \frac{x}{u}\right)$ and $g_x(t) = g\left(t + \frac{x}{u}\right)$ to be the forward and backward waveforms at any point, x.



i is always measured in the +ve x direction.

Then $\quad v_x(t) = f_x(t) + g_x(t) \quad$ and $\quad i_x(t) = Z_0^{-1}\left(f_x(t) - g_x(t)\right)$.

Note: Knowing the waveform $f_x(t)$ or $g_x(t)$ at any position x, tells you it at all other positions: $f_y(t) = f_x\left(t - \frac{y-x}{u}\right)$ and $g_y(t) = g_x\left(t + \frac{y-x}{u}\right)$.

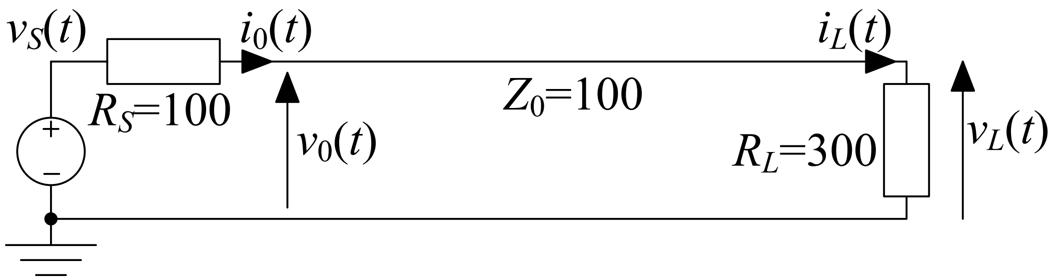
Power Flow

The power transferred into the shaded region across the boundary at x is

$$P_x(t) = v_x(t)i_x(t) = Z_0^{-1}\left(f_x(t) + g_x(t)\right)\left(f_x(t) - g_x(t)\right)$$
$$= \frac{f_x^2(t)}{Z_0} - \frac{g_x^2(t)}{Z_0}$$

f_x carries power into shaded area and g_x carries power out independently.
Power travels in the same direction as the wave.
The same power as would be absorbed by a [ficticious] resistor of value Z_0.

Reflections



$$v_x = f_x + g_x$$
$$i_x = Z_0^{-1}\left(f_x - g_x\right)$$

From Ohm's law at $x = L$, we have $v_L(t) = i_L(t)R_L$

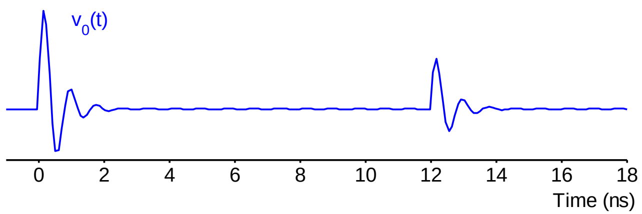
Hence $\left(f_L(t) + g_L(t)\right) = Z_0^{-1}\left(f_L(t) - g_L(t)\right)R_L$

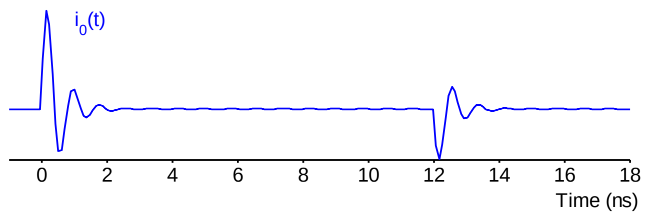
From this: $g_L\left(t\right) = \frac{R_L - Z_0}{R_L + Z_0} \times f_L\left(t\right)$

We define the *reflection coefficient*: $\rho_L = \frac{g_L(t)}{f_L(t)} = \frac{R_L - Z_0}{R_L + Z_0} = +0.5$

Substituting $g_L\left(t\right) = \rho_L f_L\left(t\right)$ gives

$v_L(t) = \left(1 + \rho_L\right) f_L(t)$ and $i_L(t) = \left(1 - \rho_L\right) Z_0^{-1} f_L(t)$





At source end: $g_0(t) = \rho_L f_0\left(t - \frac{2L}{u}\right)$ i.e. delayed by $\frac{2L}{u} = 12$ ns.

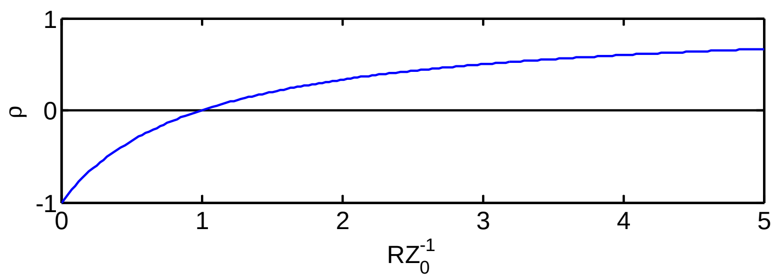
Note that the reflected current has been multiplied by $-\rho$.

Reflection Coefficients

$$\rho = \frac{R-Z_0}{R+Z_0} = \frac{\frac{R}{Z_0}-1}{\frac{R}{Z_0}+1}$$

$$\frac{v_L(t)}{f(t)} = 1+\rho$$

$$\frac{i_L(t)Z_0}{f(t)} = 1-\rho$$



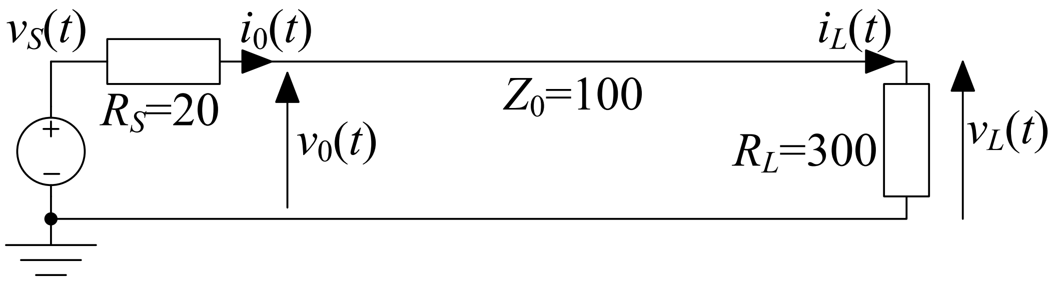
ρ depends on the ratio $\frac{R}{Z_0}$.

$\frac{R}{Z_0}$	ρ	$\frac{v_L(t)}{f(t)}$	$\frac{i_L(t)Z_0}{f(t)}$	Comment
∞	$+1$	2	0	Open circuit: $v_L = 2f$, $i_L \equiv 0$
3	$+0.5$	1.5	0.5	$R > Z_0 \Rightarrow \rho > 0$
1	0	1	1	Matched: No reflection at all
$\frac{1}{3}$	-0.5	0.5	1.5	$R < Z_0 \Rightarrow \rho < 0$
0	-1	0	2	Short circuit: $v_L \equiv 0$, $i_L = \frac{2f}{Z_0}$

Note: Reverse mapping is $R = \frac{v_L}{i_L} = \frac{1+\rho}{1-\rho} \times Z_0$

Remember: $\rho \in \{-1, +1\}$ and increases with R.

Driving a line



$$v_x = f_x + g_x$$
$$i_x = \frac{f_x - g_x}{Z_0}$$

From Ohm's law at $x = 0$, we have $v_0(t) = v_S(t) - i_0(t)R_S$ where R_S is the Thévenin resistance of the voltage source.

Substituting $v_0(t) = f_0 + g_0$ and $i_0(t) = \frac{f_0 - g_0}{Z_0}$ leads to:

$$f_0(t) = \frac{Z_0}{R_S + Z_0} v_S(t) + \frac{R_S - Z_0}{R_S + Z_0} g_0(t) \triangleq \tau_0 v_S(t) + \rho_0 g_0(t)$$

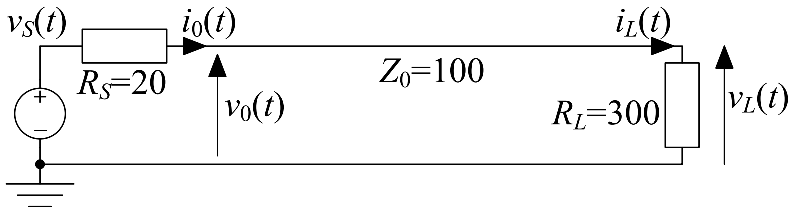
So $f_0(t)$ is the superposition of two terms:

(1) Input $v_S(t)$ multiplied by $\tau_0 = \frac{Z_0}{R_S + Z_0}$ which is the same as a potential divider if you replace the line with a [ficticious] resistor Z_0.

(2) The incoming backward wave, $g_0(t)$, multiplied by a reflection coefficient: $\rho_0 = \frac{R_S - Z_0}{R_S + Z_0}$.

For $R_S = 20$: $\tau_0 = \frac{100}{20+100} = 0.83$ and $\rho_0 = \frac{20-100}{20+100} = -0.67$.

Multiple Reflections



$$\rho_0 = -\frac{2}{3}$$
$$\rho_L = \frac{1}{2}$$
$$v_x = f_x + g_x$$

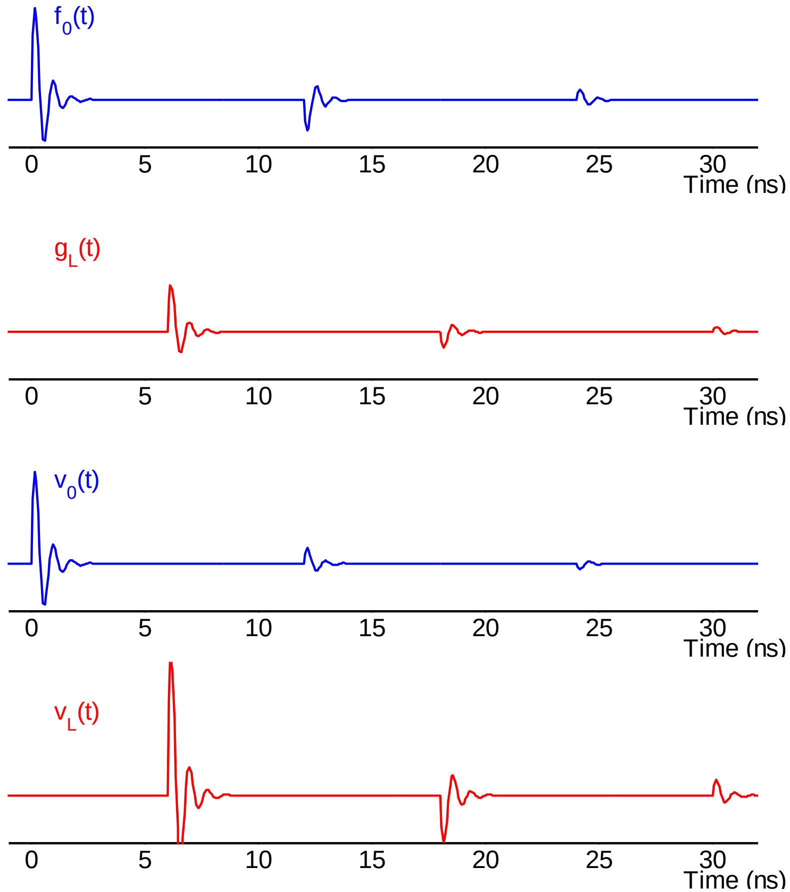
Each extra bit of f_0 is delayed by $\frac{2L}{u}$ ($=12\,\text{ns}$) and multiplied by $\rho_L \rho_0$:

$$f_0(t) = \sum_{i=0}^{\infty} \tau_0 \rho_L^i \rho_0^i v_S\left(t - \frac{2Li}{u}\right)$$

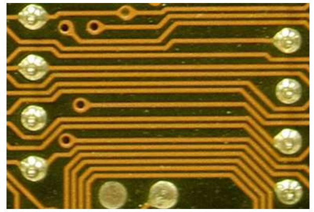
$$g_L(t) = \rho_L f_0\left(t - \frac{L}{u}\right)$$

$$v_0(t) = f_0(t) + g_L\left(t - \frac{L}{u}\right)$$

$$v_L(t) = f_0\left(t - \frac{L}{u}\right) + g_L(t)$$

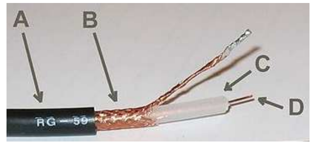


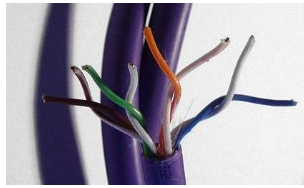
Integrated circuits & Printed circuit boards



High speed digital or high frequency analog interconnections
$Z_0 \approx 100\,\Omega$, $u \approx 15\,\text{cm/ns}$.

Long Cables





Coaxial cable ("coax"): unaffacted by external fields; use for antennae and instrumentation.
$Z_0 = 50$ or $75\,\Omega$, $u \approx 25\,\text{cm/ns}$.
Twisted Pairs: cheaper and thinner than coax and resistant to magnetic fields; use for computer network and telephone cabling. $Z_0 \approx 100\,\Omega$, $u \approx 19\,\text{cm/ns}$.

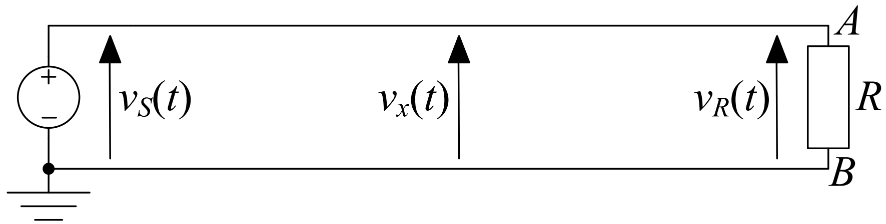
When do you have to bother?

Answer: long cables or high frequencies. You can completely ignore transmission line effects if $\text{length} \ll \frac{u}{\text{frequency}} = \text{wavelength}$.

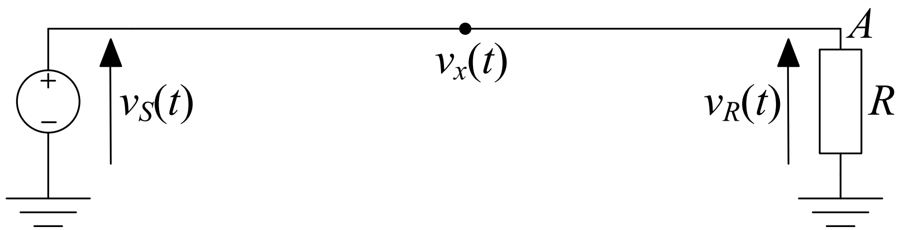
- Audio ($< 20\,\text{kHz}$) never matters.
- Computers ($1\,\text{GHz}$) usually matters.
- Radio/TV usually matters.

[Transmission Line Grounds]

For long coaxial or twisted pair cables, the “ground” wire has significant inductance and so its two ends are not necessarily at the same voltage. This means that $v_x(t)$, $f_x(t)$ and $g_x(t)$ are measured relative to the “ground” at position x as shown. It follows that potential differences like $v_R(t) = v_A(t) - v_B(t)$ make sense but talking about $v_A(t)$ on its own is meaningless.



Integrated circuits and printed circuit boards normally have a low impedance “ground plane” covering the entire circuit; in a multilayer printed circuit board this typically forms one entire layer. In this case we have a single ground reference for the whole circuit and it now makes sense to talk about the voltage “at” a node and to say $v_R(t) = v_A(t)$.



- Signals travel at around $u \approx \frac{1}{2}c = 15\,\text{cm/ns}$.
  Only matters for high frequencies or long cables.
- Forward and backward waves travel along the line:
  $$f_x(t) = f_0\left(t - \frac{x}{u}\right) \quad \text{and} \quad g_x(t) = g_0\left(t + \frac{x}{u}\right)$$
  - Knowing f_x and g_x at any single x position tells you everything
- Voltage and current are: $v_x = f_x + g_x$ and $i_x = \frac{f_x - g_x}{Z_0}$
- Terminating line with R at $x = L$ links the forward and backward waves:
  - backward wave is $g_L = \rho_L f_L$ where $\rho_L = \frac{R - Z_0}{R + Z_0}$
  - the reflection coefficient, $\rho_L \in \{-1, +1\}$ and increases with R
  - $R = Z_0$ avoids reflections: *matched* termination.
  - Reflections go on for ever unless one or both ends are matched.
  - f is infinite sum of copies of the input signal delayed successively by the round-trip delay, $\frac{2L}{u}$, and multiplied by $\rho_L \rho_0$.

18: Phasors and Transmission Lines

For a transmission line: $v(t,x) = f\left(t - \frac{x}{u}\right) + g\left(t + \frac{x}{u}\right)$ and
$i(t,x) = Z_0^{-1}\left(f(t - \frac{x}{u}) - g(t + \frac{x}{u})\right)$

We can use phasors to eliminate t from the equations if $f()$ and $g()$ are sinusoidal with the same ω: $f(t) = A\cos(\omega t + \phi) \Rightarrow F = Ae^{j\phi}$.

Then $f_x(t) = f(t - \frac{x}{u}) = A\cos\left(\omega\left(t - \frac{x}{u}\right) + \phi\right)$

$$\Rightarrow \; F_x = Ae^{j\left(-\frac{\omega}{u}x + \phi\right)} = Ae^{j\phi}e^{-j\frac{\omega}{u}x} = F_0 e^{-jkx}$$

where the *wavenumber* is $k \triangleq \frac{\omega}{u}$.
Units: ω is "radians per second", k is "radians per metre" (note $k \propto \omega$).

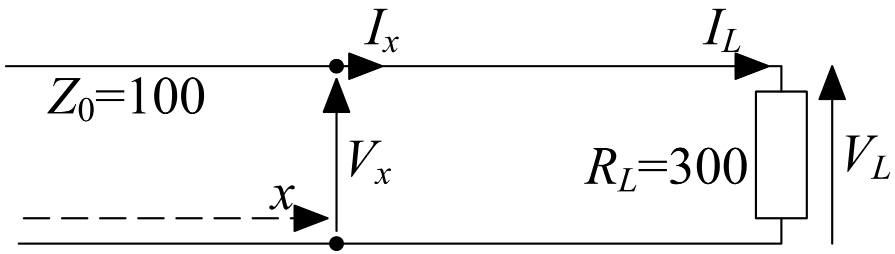
Similarly $G_x = G_0 e^{+jkx}$.

Everything is time-invariant: phasors do not depend on t.

Nice things about sine waves:
(1) a time delay is just a phase shift
(2) sum of delayed sine waves is another sine wave

Phasor Relationships

Time Domain	Phasor	Notes
$f(t) = A\cos(\omega t + \phi)$	$F = Ae^{j\phi}$	F indep of t
$f_x(t) = f\left(t - \frac{x}{u}\right)$ $= A\cos\left(\omega t + \phi - \frac{\omega}{u}x\right)$	$F_x = Ae^{j\left(\phi - \frac{\omega}{u}x\right)}$ $= Fe^{-jkx}$	$\|F_x\| \equiv \|F\|$ indep of x
$f_y(t) = f_x\left(t - \frac{(y-x)}{u}\right)$	$F_y = F_x e^{-jk(y-x)}$	Delayed by $\frac{y-x}{u}$
$g_y(t) = g_x\left(t + \frac{(y-x)}{u}\right)$	$G_y = G_x e^{+jk(y-x)}$	Advanced by $\frac{y-x}{u}$
$v_x(t) = f_x(t) + g_x(t)$	$V_x = F_x + G_x$	
$i_x(t) = \frac{f_x(t) - g_x(t)}{Z_0}$	$I_x = \frac{F_x - G_x}{Z_0}$	



Phasors obey Ohm's law: $\frac{V_L}{I_L} = R_L = \frac{F_L+G_L}{Z_0^{-1}(F_L-G_L)}$

So $G_L = \rho_L F_L$ where $\rho_L = \frac{R_L-Z_0}{R_L+Z_0}$

At any x, $\frac{G_x}{F_x} = \frac{G_L e^{-jk(L-x)}}{F_L e^{+jk(L-x)}} = \rho_L e^{-2jk(L-x)}$

Ohm's law at the load determines the ratio $\frac{G_x}{F_x}$ everywhere on the line.

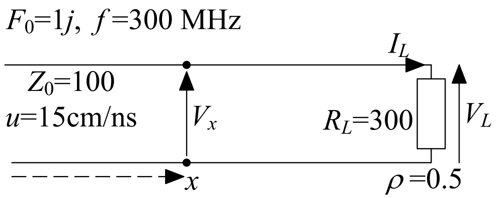
Note that $\left|\frac{G_x}{F_x}\right| \equiv |\rho_L|$ has the same value for all x.

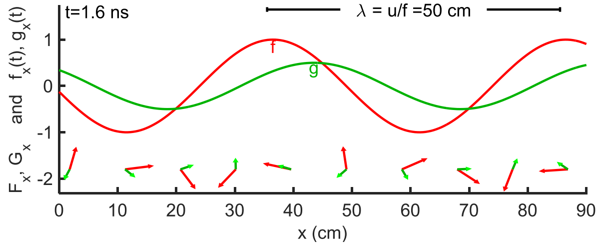
$$V_x = F_x + G_x = F_x\left(1 + \rho_L e^{-2jk(L-x)}\right)$$

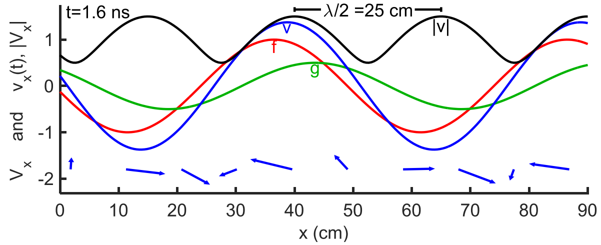
$$I_x = Z_0^{-1}(F_x - G_x) = Z_0^{-1} F_x\left(1 - \rho_L e^{-2jk(L-x)}\right)$$

The exponent $-2jk(L-x)$ is the phase delay from travelling from x to L and back again (hence the factor 2).

Standing Waves







Forward wave phasor: $F_x = Fe^{-jkx}$
Backward wave phasor: $G_x = \rho_L F_x e^{-2jk(L-x)} = \rho_L F e^{-2jkL} e^{+jkx}$

Line Voltage phasor: $V_x = F_x + G_x = Fe^{-jkx}\left(1 + \rho_L e^{-2jk(L-x)}\right)$
Line Voltage Amplitude: $|V_x| = |F|\left|1 + \rho_L e^{-2jk(L-x)}\right|$ varies with x but not t

Max amplitude equals $1 + |\rho_L|$ at values of x where F_x and G_x are in phase. This occurs every $\frac{\lambda}{2}$ away from L where λ is the *wavelength*, $\lambda = \frac{2\pi}{k} = \frac{u}{f}$.

Min amplitude equals $1 - |\rho_L|$ at values of x where F_x and G_x are out of phase.

Standing waves arise whenever a periodic wave meets its reflection: e.g. ponds, musical instruments, microwave ovens.

- Use phasors if forward and backward waves are sinusoidal with the same ω.
  - $f_x(t) = f\left(t - \frac{x}{u}\right) \quad \rightarrow \quad F_x = F_0 e^{-jkx}$
  - $g_x(t) = g\left(t + \frac{x}{u}\right) \quad \rightarrow \quad G_x = G_0 e^{+jkx}$
    - ▷ $k = \frac{\omega}{u}$ is the wavenumber in "radians per metre"
- Time delays $\simeq$ phase shifts: $F_y = F_x e^{-jk(y-x)}$
- When a periodic wave meets its reflection you get a standing wave:
  - Oscillation amplitude varies with x: $\propto \left|1 + \rho_L e^{-2jk(L-x)}\right|$
  - Max amplitude of $(1 + |\rho_L|)$ occurs every $\frac{\lambda}{2}$

Merry Xmas