Sampling and reconstruction of finite rate of innovation signals with applications in neuroscience

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Outline

1. FRI signals

2. Sampling and perfect reconstruction

3. Extension to other signals and application in neuroscience

4. Finite dimensional FRI
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4. Finite dimensional FRI
FRI signals

- Signals with a finite number of degrees of freedom:
  \[ x(t) = \sum_{k \in \mathbb{Z}} \sum_{r=0}^{R-1} a_{k,r} g_r(t - t_k) \]

  If the set of functions \( \{g_r(t)\}_{r=0,1,...,R-1} \) is known, the signal \( x(t) \) is perfectly determined by the parameters of amplitudes and temporal shifts \((a_{k,r}, t_k)\).

- Let us consider the example of a stream of Diracs in a temporal interval \( \tau \):
  \[ x(t) = \sum_{k=1}^{K} a_k \delta(t - t_k), \]  where \( t_k \in [0, \tau] \).
  - This signal has \( 2K \) degrees of freedom in the temporal interval \( \tau \)
  - Local rate of innovation: \( \rho = \frac{2K}{\tau} \)

- The sampling process of a continuous-time signal can be modelled by a filtering stage followed by a sampling stage at instants of time \( t = nT \)

- GOAL: reconstruct \( x(t) \) from samples \( y_n \).
The FRI model can be applied to a wide range of signals:

- Piecewise sinusoidals: estimation of frequencies
- Stream of pulses: application in radar systems or ultrasound imaging
- Stream of decaying exponentials: application in neuroscience to monitor neural activity

These are examples of band unlimited signals ⇒ it seems impossible to achieve perfect reconstruction from a set of samples obtained with a finite sampling frequency.

**Sampling theorem (Shannon 1949)**

If a function \( x(t) \) contains no frequencies higher than \( \omega_{\text{max}} \) [rad \( \cdot \) s\(^{-1} \)], it is completely determined by giving its ordinates at a series of points spaced \( T = \pi / \omega_{\text{max}} \) seconds apart.

- Shannon-Nyquist sampling theorem gives a sufficient condition (not necessary):
  - Under some conditions we can achieve perfect reconstruction of FRI signals (Vetterli et al. 2002).
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Sampling and perfect reconstruction

We are going to sample the analogue signal with a filter that satisfies a specific condition, it reproduces exponential functions:

\[ \sum_{n \in \mathbb{Z}} c_{m,n} \varphi(t - n) = e^{\alpha_m t}, \quad m = 0, 1, \ldots, P \]

(a) \( \varphi_1(t) \)
(b) \( e^{\alpha_m t}, \alpha_m = -1/2 \)
(c) \( e^{\alpha_m t}, \alpha_m = 1/2 \)
(d) \( \varphi_2(t) \)
(e) \( \Re\{e^{\alpha_m t}\}, \alpha_m = i\pi/6 \)
(f) \( \Re\{e^{\alpha_m t}\}, \alpha_m = i\pi/3 \)

**Figure:** The function \( \varphi_1(t) \) reproduces the exponentials in (b) and (c) and the function \( \varphi_2(t) \) reproduces the exponentials in (e) and (f). The dotted lines show the true exponentials, the thin continuous lines show \( \varphi(t) \) shifted and weighted by the coefficients \( c_{m,n} \), and the thick lines show the sum \( \sum_{n \in \mathbb{Z}} c_{m,n} \varphi(t - n) \).
Exponential reproduction: \[ \sum_{n \in \mathbb{Z}} c_{m,n} \varphi(t - n) = e^{i\omega_m t}, \quad m = 0, 1, \ldots, P \]

### Generalised Strang-Fix conditions (Strang & Fix 1971, Unser & Blu 2005)

A function \( \varphi(t) \) together with a linear combination of its shifted versions, \( \{ \varphi(t - n) \}_{n \in \mathbb{Z}} \), can reproduce exponentials of the form \( \{ e^{i\omega_m t} \}_{m=0}^P \) if and only if its Fourier transform satisfies

\[ \hat{\varphi}(\omega_m) \neq 0 \quad \text{and} \quad \hat{\varphi}(\omega_m + 2\pi \ell) = 0, \]

where \( m = 0, 1, \ldots, P \) and \( \ell \in \mathbb{Z} \setminus \{0\} \).

- A family of functions that satisfy these conditions are the E-splines (\( \Omega = (\omega_0, \omega_1, \ldots, \omega_P) \) is a design parameter):

\[ \hat{\varphi}_\Omega(\omega) = \prod_{m=0}^{P} \left( \frac{1 - e^{-i(\omega - \omega_m)}}{- i (\omega - \omega_m)} \right) \]

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(a) \( |\hat{\varphi}_\Omega(\omega)|, \Omega = (0) \)

(b) \( |\hat{\varphi}_\Omega(\omega)|, \Omega = (\omega_0) \)

(c) \( |\hat{\varphi}_\Omega(\omega)|, \Omega = (\omega_m)_{m=0}^P \)
If $\varphi(t)$ reproduces exponentials ($\sum_{n\in\mathbb{Z}} c_{m,n} \varphi(t - n) = e^{i\omega_m t}$), we can combine samples $y_n$ with the coefficients $c_{m,n}$ and we obtain:

$$s_m = \sum_{n=1}^{N} c_{m,n} \langle x(t), \varphi(t/T - n) \rangle = \int_{-\infty}^{+\infty} x(t) \sum_{n=1}^{N} c_{m,n} \varphi(t/T - n) dt$$

$$= \int_{-\infty}^{+\infty} x(t) e^{i\omega_m t/T} dt = \hat{x}(-\omega_m/T), \quad m = 0, 1, \ldots, P$$

The samples $s_m$ correspond to the Fourier transform of $x(t)$ at $\omega = -\omega_m/T$. 

(a) $x(t)$  
(b) Sampling kernel $h(t)$  
(c) $y(t)$ and $y_n$  
(d) $s_m = \sum_n c_{m,n} y_n$
If \( x(t) \) is a stream of \( K \) Diracs

\[
x(t) = \sum_{k=1}^{K} a_k \delta(t - t_k)
\]

it is possible to recover the parameters \( \{(a_k, t_k)\}_{k=1}^{K} \) from only \( 2K \) evenly spaced samples of its Fourier transform.

The frequencies \( \omega_m \) are a design parameter, we impose the following form: \( \omega_m = \omega_0 + \lambda m, \quad m = 0, 1, \ldots, P \).

The values \( s_m = \sum_n c_{m,n} y_n \) are given by:

\[
s_m = \sum_{k=0}^{K} a_k e^{i\omega_0 t_k/T} \left( e^{i\lambda t_k/T} \right)^m = \sum_{k=0}^{K} b_k u_k^m \quad m = 0, 1, \ldots, P
\]

The recovery of the parameters \( \{(a_k, t_k)\}_{k=1}^{K} \) is equivalent to estimating \( \{(b_k, u_k)\}_{k=1}^{K} \) from the sequence \( s_m \):

- It is a classic problem in spectral estimation and can be exactly solved from only \( 2K \) samples (\( \Rightarrow \) there are various methods, e.g. **annihilating filter method**).
Input signal: $x(t) = \sum_{k=1}^{K} a_k \delta(t - t_k)$

Samples: $y_n = \langle x(t), \varphi(t/T - n) \rangle$

New sequence: $s_m = \sum_n c_{m,n} y_n = \sum_{k=1}^{K} b_k u_k^m$, $m = 0, 1, \ldots, P$

- We can search a filter that annihilates the sequence $s_m$. FIR filter with zeros at $z = u_k$:
  \[
  H(z) = \prod_{k=1}^{K} (1 - u_k z^{-1}) = 1 + \sum_{m=1}^{K} h_m z^{-m}
  \]

- $H(z)$ annihilates the sequence $s_m$: $h_m \ast s_m = 0$. The filter is still unknown, but we can compute it by writing the annihilation equation in matricial form:
  \[
  \begin{pmatrix}
  S_K & s_{K-1} & \cdots & s_0 \\
  s_{K+1} & S_K & \cdots & s_1 \\
  \vdots & \vdots & \ddots & \vdots \\
  s_{2K} & s_{2K-1} & \cdots & S_K \\
  \end{pmatrix}
  \begin{pmatrix}
  1 \\
  h_1 \\
  \vdots \\
  h_K \\
  \end{pmatrix}
  =
  \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0 \\
  \end{pmatrix}
  \]

- If all $t_k$ are distinct and $a_k \neq 0$, then the matrix $S$ has rank $K$: we can drop the last row and obtain the unique solution from only $2K$ values $s_m$:
  - We first retrieve $h$ from $s_m$
  - We obtain $u_k$ from the roots of the polynomial $H(z)$
  - The problem is then linear in $b_k$
  - We can then recover the parameters $\{(a_k, t_k)\}_{k=1}^{K}$
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The example of the stream of Diracs may seem limited, but many phenomena can be modeled as a stream of Diracs convolved with a pulse:

\[ x(t) = \sum_k a_k p(t - t_k) = p(t) * \sum_k a_k \delta(t - t_k) := s(t) \]

If the sampling kernel, \( \varphi(t) \), satisfies the generalised Strang-Fix conditions the function \( \varphi(t) * p(t) \) also satisfies them (we also require \( \int_{-\infty}^{\infty} p(t) e^{-i\omega m t} dt \neq 0 \)).

- The samples obtained by filtering \( x(t) \) with \( \varphi(t) \), are equivalent to the samples obtained by sampling the stream of Diracs \( s(t) \) with \( \varphi(t) * p(t) \).

In the case of a stream of decaying exponentials,

\[ x(t) = \sum_k a_k e^{-\alpha(t - t_k)} u(t - t_k), \]

we can extend this approach and apply an additional post-processing stage to obtain an equivalent sampling scheme (Oñatívía et al. 2013):

- Compute weighted finite differences of the samples: \( z_n = y_n - y_{n-1} e^{-\alpha T} \)
- Samples \( z_n \) are equivalent to sampling \( s(t) \) with \( \psi(t) := \beta_{\alpha T}(t) * \varphi(t) \)
(a) Original sampling scheme (exp. reproducing kernel \( \sum_n c_{m,n} \varphi(t - n) = e^{j\omega m t} \))

(b) Equivalent problem (exp. reproducing kernel \( \sum_n d_{m,n} \psi(t - n) = e^{j\omega m t} \))

**Figure:** Filtering and sampling of a train of decaying exponentials. Sampling a train of decaying exponentials with an exponential reproducing kernel and computing weighted differences (a) is equivalent to sampling the stream of Diracs that generates the train of decaying exponentials with a different kernel which is also able to reproduce exponentials (b).

- The locations and amplitudes of the stream of decaying exponentials are obtained from the sequence \( z_n \) combined with the coefficients \( d_{m,n} \):

\[
    s_m = \sum_n d_{m,n} z_n = \sum_k b_k u^m_k
\]
Figure: Fluorescence signal processing with a sliding window. For each time interval, the number of spikes within that interval is first estimated and then the locations of this specific number of spikes is retrieved.

(a) Retrieved locations  
(b) Histogram

Figure: (a) Plot of the sequentially estimated locations, the horizontal axis indicates the index of the sliding window and the vertical axis the location in time. (b) Histogram of the locations shown in (a). Horizontally aligned dots in (a) lead to peaks in the histogram in (b).
Application to neuroscience to monitor the spiking activity of neurons from two-photon imaging of calcium images:

- we obtain a sample of Ca$^{2+}$ concentration per frame and neuron (ROI in image)
- monitoring of tens of neurons in parallel and real-time
- very low temporal sampling rate due to limitations of the scanning laser
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Finite dimensional FRI

- Let $x \in \mathbb{C}^N$ be a finite dimensional signal formed by a stream of $K$ Diracs:

$$ x[n] = \sum_{k=1}^{K} a_k \delta[n-n_k] \quad \overset{DFT_N}{\longrightarrow} \quad \hat{x}[m] = \frac{1}{\sqrt{N}} \sum_{k=1}^{K} a_k e^{-j\omega_k m}, \quad \text{where} \quad \omega_k := \frac{2\pi n_k}{N} $$

- Let us also assume that we have access to $M < N$ coefficients of the DFT. We can express the sampling process in matricial form as follows:

$$ y = D x, $$

where $y \in \mathbb{C}^M$ are the available samples and $D \in \mathbb{C}^{M \times N}$ is a partial Fourier matrix: $(D)_{m,n} = (1/\sqrt{N}) \exp\left(-i \frac{2\pi}{N} m n\right)$, with $m = 0, \ldots, M-1$ and $n = 0, \ldots, N-1$.

- The acquisition process is modelled with a multiplication by a fat matrix:

$$ y = D x $$

![Diagram showing matrix multiplication]

**Sampling and reconstruction of FRI signals**
Input signal: \[ x[n] = \sum_{k=1}^{K} a_k \delta[n - n_k] \]

DFT\(_N\{x[n]\}\): \[ \hat{x}[m] = \sum_{k=1}^{K} \frac{a_k}{\sqrt{N}} e^{-j\omega_km} \]

- If \( M \geq 2K \) we can apply classical approaches such as the annihilating filter method or spectral estimation methods.
  - These methods require nonlinear steps based on solving some eigenproblem.
  - They are suitable for the continuous-time case, we want to exploit the fact that the dimension of the problem is finite.

- We can try to solve the underdetermined system \( y = Dx \)
  - We can estimate \( x \) by applying the pseudoinverse \( D^\dagger \), up to the null space:

\[
x^* = D^\dagger y + \sum_{\ell=1}^{L} \beta_\ell n_\ell
\]

where \( \beta_\ell \) are unknown coefficients, \( L = N - M \) and \( n_\ell \) are \( L \) orthonormal vectors that span the null space of \( D \), \( i.e. \) \( D n_\ell = 0 \).

- Nonunique solution \( \Rightarrow \) sparsity constrains on \( x^* \) in order to determine \( \beta_\ell \).
Available samples \( y = D \mathbf{x} \)

Estimate of \( \mathbf{x} \): \( \mathbf{x} = D^H y + \sum_{\ell=1}^{L} \beta_{\ell} \mathbf{n}_{\ell} \)

Estimate of \( \hat{\mathbf{x}} \): \( \hat{\mathbf{x}} = F \mathbf{x} = z + \sum_{\ell=1}^{L} \beta_{\ell} \mathbf{e}_{M+\ell} \)

Annihilating filter: \( S \mathbf{h} = 0 \)

- \( S \in \mathbb{C}^{(N-K) \times (K+1)} \) is the Toeplitz matrix constructed with samples \( \hat{\mathbf{x}} \), thus
  \[
  S = Z + \sum_{\ell=1}^{L} \beta_{\ell} E_{M+\ell}
  \]
  where \( Z \) and \( E_{M+\ell} \) are Toeplitz matrices constructed from \( z \) and \( e_{M+\ell} \).

- If we replace this expression of \( S \) in the annihilating filter system we obtain
  \[
  \begin{bmatrix}
  E_{M+1} \mathbf{h} & \cdots & E_{N} \mathbf{h}
  \end{bmatrix}
  \begin{bmatrix}
  \beta_{1} \\
  \vdots \\
  \beta_{L}
  \end{bmatrix} = -Z \mathbf{h}
  \]

- Last \( L \) equations form a determined system and therefore has a unique solution.

- Annihilating filter \( \mathbf{h} \) can be obtained from \( 2K \) consecutive samples of \( y \).

- The annihilating filter imposes the sparsity constraint and leads to a unique solution.
Figure: $N = 256$, $M = 32$ and $K = 16$. (a) Signal with $K = 16$ Diracs. (b) Perfect reconstruction of the signal, in red the original signal and in blue the reconstruction. (c) Real part of the Fourier coefficients in black and available samples in red. (d) Real part of the Fourier coefficients in black and estimated coefficients in red.
In the presence of noise we have:

\[ \tilde{y} = Dx + \epsilon \]

We obtain the solution by solving two Total Least Squares problems: one to obtain the annihilating filter and another one to find the coefficients \( \beta_\ell \).

The CS reconstruction is obtained by applying basis pursuit denoising:

\[
\min_{x \in \mathbb{C}^N} \|x\|_1 + \frac{1}{2\rho} \|y - Dx\|_2^2
\]
Background material:


Original contributions: